ON THE DIMENSION OF MODULES AND ALGEBRAS IX

DIRECT LIMITS

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Let J be a directed set and let $\{\Lambda_j, \varphi_{ij}\}$ be a direct system of rings indexed by J and with limit Λ. Let $\{A_j, \psi_{ij}\}$ be a direct system of groups indexed by J. Assume that each A_j is a left Λ_j-module and that $\psi_{ij}(\lambda a) = \varphi_{ij}(\lambda)\psi_{ij}(a)$ for each $\lambda \in \Lambda_j$, $a \in A_j$. Then the limit A of $\{A_j, \psi_{ij}\}$ is a left Λ-module.

THEOREM. *If J is countable, then*

$$\text{l.}\dim_\Lambda A \leqq 1 + \sup_J \text{l.}\dim_{\Lambda_j} A_j.$$

COROLLARY 1. *If J is countable, then*

$$\text{l.gl.}\dim \Lambda \leqq 1 + \sup_J \text{l.gl.}\dim \Lambda_j.$$

COROLLARY 2. *Let $\{K_j, \nu_{ij}\}$ be a direct system of commutative rings indexed by J and with limit K. Assume that each Λ_j is a K_j-algebra and that $\varphi_{ij}(k\lambda) = \nu_{ij}(k)\varphi_{ij}(\lambda)$ for $k \in K_j$, $\lambda \in \Lambda_j$. Then Λ is a K-algebra. If J is countable, then*

$$K\text{-}\dim \Lambda \leqq 1 + \sup. K_j\text{-}\dim \Lambda_j.$$

To derive Cor. 2 we note that

$$K\text{-}\dim \Lambda = \text{l.}\dim_{\Lambda^e} \Lambda, \quad \text{where} \quad \Lambda^e = \Lambda \otimes_K \Lambda^*,$$

and that Λ^e is the direct limit of $\{\Lambda_j^e\}$. Cor. 2 is a generalization of a theorem by Kuročkin [1] (see also [2], p. 92).

Proof of the Theorem. We consider the exact sequences

$$0 \longrightarrow R_j \longrightarrow F_j \longrightarrow A_j \longrightarrow 0, \qquad j \in J$$
$$0 \longrightarrow R \longrightarrow F \longrightarrow A \longrightarrow 0$$

where F_j is the free Λ_j-module with the elements of A_j as Λ_j-basis and

Received December 19, 1957.

$F_j \longrightarrow A_j$ is the identity on the basis of F_j. Similarly for F. It is then easy to see that R may be regarded as the direct limit of the R_j's. If $n = \sup. \mathrm{l.dim}_{\Lambda_j} A_j$, $n > 0$ then

$$\sup_j . \mathrm{l.dim}_{\Lambda_j} R_j \leqq n-1.$$

Since also $\mathrm{l.dim}_\Lambda A \leqq 1 + \mathrm{l.dim}_\Lambda R$, it suffices to prove that $\mathrm{l.dim}_\Lambda R \leqq 1 + (n-1)$. This reduces the theorem dy induction to the case $n = 0$ i.e. to the case when each A_j is Λ_j-projective.

If each A_j is Λ_j-projective, then $B_j = \Lambda \otimes_{\Lambda_j} A_j$ is Λ-projective. The B_j's form a direct system of Λ-modules with A as limit. This further reduces the theorem to the case when the direct system $\{\Lambda_j, \varphi_{ij}\}$ is constant, i.e. $\Lambda_j = \Lambda$, $\varphi_{ij} =$ identity.

The above two reductions are valid for any indexing set J. Now we assume that J is countable. There exists then a sequence $a_1 < a_2 < \ldots < a_k < \ldots$ in J which is cofinal with J. Thus we may assume that $J = (1, 2, \ldots, k, \ldots)$.

Thus the direct system $\{A_j, \psi_{ij}\}$ is a direct sequence

$$A_1 \xrightarrow{\psi_{2,1}} A_2 \xrightarrow{\psi_{3,2}} \cdots \xrightarrow{\psi_{k,k-1}} A_k \xrightarrow{\psi_{k+1,k}} \cdots$$

where each A_k is a projective Λ-module. By definition, the limit A is the quotient of the direct sum $B = \sum_{k=1}^{\infty} A_k$ by the submodule generated by all the elements $a - \psi_{k+1,k}(a)$, $a \in A_k$. There results the exact sequence

$$0 \longrightarrow B \xrightarrow{\gamma} B \longrightarrow A \longrightarrow 0$$

where $\gamma(a) = a - \psi_{k+1,k}(a)$ for $a \in A_k$. Since B is Λ-projective, it follows that $\mathrm{l.dim}_\Lambda A \leqq 1$.

Bibliography

[1] V. M. Kuročkin, The decomposition of algebras into a semi-direct sum of the radical and a semi-simple subalgebra, C. R. (Doklady) Acad. Sci., URSS (N.S.) vol. **36** (1942), pp. 42-45.

[2] A. Rosenberg and D. Zelinsky, Cohomology of infinite algebras, Trans. Amer. Math. Soc., vol. **82** (1956), pp. 85-98.

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