FIXED POINTS OF ISOMETRIES

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1. Statement of Theorem

The purpose of this paper is to prove the following

THEOREM. *Let M be a Riemannian manifold of dimension n and let ξ be a Killing vector field* (*i.e., infinitesimal isometry*) *of M. Let F be the set of points x of M where ξ vanishes and let $F = \cup V_i$, where the V_i's are the connected components of F. Then* (*assuming F to be non-empty*)

(1) *Each V_i is a totally geodesic closed submanifold* (*without singularities*) *of M and the co-dimension of V_i* (*i.e.,* $\dim M - \dim V_i$) *is even.*

(2) *The structure group of the normal bundle over V_i can be reduced to $GL(r, C)$, where $2r$ is the co-dimension of V_i.*

(3) *If $x \in V_i$ and $y \in V_j$ and $i \neq j$, then there is a 1-parameter family of geodesics joining x and y provided M is complete; hence x and y are conjugate to each other.*

(4) *If M is, moreover, compact, then the Euler number of M is the sum of Euler numbers of V_i's:*

$$\chi(M) = \Sigma\chi(V_i),$$

(*the summation is well defined, as the number of connected components V_i is finite*).

Remarks. (2) implies that if M is orientable, then V_i is orientable.

If F consists of only isolated points, then (4) is a particular case of the Index Theorem, as the index of a Killing vector field at an isolated zero point is 1.

COROLLARY 1. *Let L be an abelian Lie algebra of Killing vector fields of M. Let F be the set of points x of M where every element of L vanishes. Then the same statements as in Theorem hold.*

Received September 12, 1957.
Revised November 11, 1957.
* This paper was sponsored in part by the National Science Foundation under Grant NSF G-3462, which the author held at the University of Chicago in the summer of 1957.

Remark. Let G be a torus group acting on a manifold M as a differentiable transformation group. Then we take a Riemannian metric on M invariant under G and apply Corollary 1, thus obtaining the same results as in Theorem.

COROLLARY 2. *Under the same assumption as in Corollary* 1, *if M is a symmetric space in the sense of E. Cartan, so is each V_i.*

COROLLARY 3. *Under the same assumption as in Corollary* 1, *if the sectional curvature of a complete Riemannian manifold M is non-positive, then F is either empty or connected.*

COROLLARY 4. *Let M be a compact manifold of dimension* $2m$. *Suppose that a torus group of dimension m acts on M (differentiably and effectively). Then the Euler number of M is zero or positive according as the fixed point set F is empty or not. If M is orientable and F is non-empty, then the Euler number of M is greater than or equal to* 2.

2. Proof of Theorem

(1) Let x be any point of F and let $T_x(M)$ be the tangent space to M at x. Then ξ induces an endomorphism of $T_x(M)$ and it is a skew-symmetric matrix with respect to an orthonormal basis of $T_x(M)$. (In classical terminologies, it is an endomorphism defined by the covariant derivatives of ξ.) If we choose a proper basis $e_1, \dots, e_n$, then this matrix can be reduced to the following

$$\begin{pmatrix} 0 & a_1 & & & & & & \\ -a_1 & 0 & & & & & & \\ & & \ddots & & & & & \\ & & & 0 & a_r & & & \\ & & & -a_r & 0 & & & \\ & & & & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{pmatrix}$$

The 1-parameter group of local isometries generated by ξ induces on $T_x(M)$ rotations of the form

$$\begin{pmatrix} \cos ta_1 & \sin ta_1 & & & & \\ -\sin ta_1 & \cos ta_1 & & & & \\ & & \ddots & & & \\ & & & \cos ta_r & \sin ta_r & \\ & & & -\sin ta_r & \cos ta_r & \\ & & & & & I_{n-2r} \end{pmatrix}$$

where t is the parameter. If $n-2r=0$, then x is an isolated zero of ξ and we are done. Suppose $n-2r>0$. If v is a vector spanned by $e_{2r+1}, \ldots, e_n$, then v is invariant under this 1-parameter group. Hence the geodesic issued from x to the direction of v is also left fixed (pointwise) by the group. In a certain neighborhood U of x the set of these geodesics forms an $(n-2r)$-dimensional submanifold U' of U. (Take, for instance, U to be a neighborhood of x such that for every y of U there exists a unique geodesic in U joining x and y.) Now we shall show that the zeros of ξ in U are exactly U'. If y is a zero of ξ in U, then we take a geodesic in U joining y and x. Since both x and y are left fixed by the 1-parameter group, this geodesic is also left fixed by the group. Hence the tangent vector to this geodesic must be spanned by $e_{2r+1}, \ldots, e_n$. This shows that y is in U'. Hence each V_i is a submanifold of M and its co-dimension is even. The fact that V_i is totally geodesic follows immediately. In fact, let x and y be any points of V_i sufficiently close to each other so that there is a unique shortest geodesic from x to y. Then this geodesic is left fixed pointwise by the group. Hence the geodesic is contained in V_i.

Remark. As it can be seen from the proof, the statement that V_i is a totally geodesic submanifold of M is true not only for 1-parameter group of isometries but also for any group of isometries.

(2)[1] Let A be a non-singular linear transformation of the $2r$-dimensional vector space R^{2r} with a positive definite inner product. By the inner product we can identify A with a bilinear form on R^{2r}. Assume that this bilinear form is skew-symmetric. Then there is a unique decomposition of R^{2r} into subspaces $S_1, \ldots, S_k$ such that

i) Each S_i is invariant by the transformation A and if $i \neq j$ then S_i and S_j are orthogonal to each other.

[1] The result of (2) is due to A. Dold and R. Thom. The proof presented here is a modification of theirs.

ii) Restricted to S_i, A^2 is equal to $-b_i^2 I$, where I is the identity transformation and b_i is a positive real number. If $i \neq j$, then b_i is different from b_j.

Let $c_i = 1/\sqrt{b_i}$. Let C be a non-singular linear transformation of R^{2r} defined by the following two properties: (i) C maps each S_i into itself, (ii) Restricted on S_i, C is equal to $c_i I$. Let J be the transformation CAC. Then $J^2 = -I$.

We showed in (1) that the endomorphism of $T_x(M)$ induced by ξ induces a non-singular linear transformation, denoted by A_x, of the normal space to V_i at x. Since A_x is skew symmetric with respect to the inner product on $T_x(M)$ defined by the Riemannian metric, we define, by the above argument, a linear transformation J_x of $\Gamma_x(M)$ such that $J_x^2 = -I$. It can be easily shown that J_x is a differentiable field of linear transformations. Now, J_x defines a complex structure on each normal space to V_i; hence the structure group of the normal bundle over V_i can be rebuced to $GL(r, C)$.

(3) Let $x \in V_i$, $y \in V_j$ and $i \neq j$. Let g be any geodesic from x to y. This geodesic can not be left fixed by the group generated by ξ. If it were left fixed, then V_i and V_j would be the same connected component.

(4) Let ε be a small positive number. We define S_x to be the set of points y in M such that there is a geodesic from x to y of the length not greater than ε and normal to V_i at x. Thus, to every point x of V_i, we attach a solid sphere S_x with center x and radius ε which is normal to V_i and has the dimension $2r$ (= codimension of V_i). Let $N_i = \bigcup_{x \in V_i} S_x$. Taking ε very small, we may assume that $N_i \cap N_j$ is empty if $i \neq j$ and that every point in N_i is exactly in one S_x. Let $N = \bigcup N_i$. Let K be the closure of $M - N$. Then $N \cap K$ is the boundary dN of N.

LEMMA. $\chi(M) = \chi(N) + \chi(K) - \chi(dN)$.

Proof. Consider an exact sequence of vector spaces:

$$\to A_k \to B_k \to C_k \to A_{k-1} \to B_{k-1} \to \ldots$$

Then it can be shown easily that

$$\sum(-1)^k \dim A_k - \sum(-1)^k \dim B_k + \sum(-1)^k \dim C_k = 0.$$

We apply this formula to the exact sequences of homology groups induced by

$$K \to M \to (M, K) \quad \text{and} \quad dN \to N \to (N, dN)$$

and we obtain

$$\chi(K)-\chi(M)+\chi(M,\ K)=0 \quad \text{and} \quad \chi(dN)-\chi(N)+\chi(N,\ dN)=0.$$

By Excision Axiom, (M, K) and (N, dN) have the same relative homology. Hence

$$\chi(M,\ K)=\chi(N,\ dN).$$

This completes the proof of Lemma.

The 1-parameter group generated by ξ has no fixed point in K nor dN. By Lefschetz Theorem, $\chi(K)=\chi(dN)=0$. Hence $\chi(M)=\chi(N)$. As N_i is a fibre bundle over V_i with solid sphere S as fibre, we have

$$\chi(N_i)=\chi(V_i)\chi(S)=\chi(V_i).$$

Finally we obtain

$$\chi(M)=\sum\chi(N_i)=\sum\chi(V_i).$$

3. Proof of Corollaries

Let ξ and η be Killing vector fields on M commuting with each other. Let $F=\cup V_i$ be the zeros of ξ as before. Since the group generated by η commutes with the group generated by ξ, it maps F into itself. Since it is a connected group, it transforms each V_i into itself. Hence η can be considered as a Killing vector field on V_i. Let F_i be the zeros of η on V_i and let $F_i=\cup_j W_{ij}$ be the decomposition into the connected components. We apply Theorem to each V_i and repeat this process and obtain Corollary 1.

Now, Corollary 2 follows from the fact that every totally geodesic submanifold of a symmetric space is a symmetric space. Note that if M is locally symmetric in the sense that the curvature tensor is parallel, then a simple calculation shows that every totally geodesic submanifold of M is also locally symmetric. Suppose M is globally symmetric. A symmetry of M around any point of a totally geodesic submanifold of M maps the submanifold into itself and induces a symmetry of the submanifold. Hence the submanifold is globally symmetric.

Remark.[2] It is not known whether the homogeneity of M implies the homogeneity of V_i.

[2] (Added in proof) We shall prove elsewhere that every totally geodesic submanifold of a homogeneous Riemannian manifold is homogeneous Riemannian.

Corollary 3 follows from (3) and the well known fact that a Riemannian manifold of non-positive curvature has no conjugate points.

Before going into the proof of Corollary 4, we shall make the following

Remark. Suppose that a torus group of dimension m acts on a manifold M of dimension n. Assume that the fixed point set F is non-empty. If $2r$ is the co-dimension of V_i, then $m \leqq r$.

To prove this, take any Riemannian metric on M invariant by the torus group G. Let $x \in V_i$. Every element of G induces an orthogonal transformation of $T_x(M)$ which is trivial on $T_x(V_i)$. Hence G can be considered as a group of orthogonal transformations of the normal space to V_i at x. G being abelian, $\dim G$ can not be greater than the rank of $0(2r)$, which is r.

The above remark shows than $m \leqq n/2$. It is therefore of interest to consider the extremal case $2m = n$. The above argument shows that in this case F consists of only isolated points, thus proving the first half of Corollary 4.

Suppose M is orientable and F consists of a single point x. If we take a proper basis of $T_x(M)$, the group G, considered as a group of orthogonal transformations of $T_x(M)$, can be written as follows.

$$\begin{pmatrix} \cos t_1 & \sin t_1 & & & \\ -\sin t_1 & \cos t_1 & & & \\ & & \ddots & & \\ & & & \cos t_m & \sin t_m \\ & & & -\sin t_m & \cos t_m \end{pmatrix}$$

where $(t_1, \ldots, t_m)$ is a parameter of G. Let G' be a torus group of dimension $m-1$ depending on $t_1, \ldots, t_{m-1}$. Let F' be the fixed point set of G' and let V be the connected component of F' containing x. Then V is a manifold of dimension 2 and is orientable by (2) of Theorem. The 1-parameter group depending on t_m maps V into itself. The fixed points of this 1-parameter group on V are in $F = \{x\}$. Hence $\chi(V)$ is equal to 1. On the other hand, the Euler number of a compact orientable surface is always even. This shows that F is either empty or contains more than 1 point.

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