AN INVESTIGATION ON THE LOGICAL STRUCTURE OF MATHEMATICS (III)[0]

FUNDAMENTAL DEDUCTIONS

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In this Part (III) the proofs in UL are given for fundamental formulas concerning the following dependent variables:

Elementary set	$\{a_1, \dots , a_n\}$;
Ordered pair[1]	$\langle a, b\rangle$;
Image[2] by σ of a	$\sigma\text{‘}a$;
Image[2] by σ of elements of a	$\sigma\text{“}a$;
Domain of operator	D_σ ;
Range of operator	W_σ ;
Uniqueness	Un ;
Bi-uniqueness	Un_2 ;
Inverse operator	σ^{-1} ;
One-to-one mapping	$\mathrm{Map}_2^{a,b}$;
Composition of operators	$\sigma\circ\tau$;
Restriction of operator	$\sigma\upharpoonright a$;
Identical mapping	ι.

The following defining formulas are used for these dependent variables:

$$u\in\{a_1, \dots , a_n\}\equiv u=a_1\vee \dots \vee u=a_n,$$

$$u\in\langle ab\rangle\equiv u=\{a\}\vee u=\{ab\},$$

$$u\in\sigma\text{‘}a\equiv\exists x.\ \langle ax\rangle\in\sigma\wedge u\in x,$$

$$u\in\sigma\text{“}a\equiv\exists x.\ x\in a\wedge\langle xu\rangle\in\sigma,$$

$$u\in\mathrm{D}_\sigma\equiv\exists x.\ \langle ux\rangle\in\sigma,$$

$$u\in\mathrm{W}_\sigma\equiv\exists x.\ \langle xu\rangle\in\sigma,$$

$$u\in\mathrm{Un}\equiv\forall xyz.\ \langle xy\rangle\in u\wedge\langle xz\rangle\in u\rightarrow y=z,$$

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[0] See foot note [0] in Part (IV) published in this same volume.

[1] $\langle a, b\rangle$ is also written as $\langle ab\rangle$.

[2] $\sigma\text{‘}a$ as well $\sigma\text{“}a$ are also written in the same way as a^σ, when the distinction is clear by the context.

$$u \in \mathrm{Un}_2 \equiv u \in \mathrm{Un} \wedge u^{-1} \in \mathrm{Un},$$
$$u \in \sigma^{-1} \equiv \exists xy.\ u = \langle xy \rangle \wedge \langle yx \rangle \in \sigma,$$
$$u \in \mathrm{Map}_2^{a,b} \equiv u \in \mathrm{Un}_2 \wedge \mathrm{D}_u = a \wedge \mathrm{W}_u = b,$$
$$u \in \sigma \circ \tau \equiv \exists xyz.\ u = \langle xz \rangle \wedge \langle xy \rangle \in \tau \wedge \langle yz \rangle \in \sigma,$$
$$u \in \sigma \restriction a \equiv \exists xy.\ u = \langle xy \rangle \wedge u \in \sigma \wedge x \in a,$$
$$u \in \iota \equiv \exists x.\ u = \langle xx \rangle.$$

Besides these dependent variables, we use the universal constant V and $a \cap b$. All the dependent variables used in the following deductions are the variables listed above, or those which are results of successive substitutions of the above listed dependent variables, such as $\langle a a^{\sigma} \rangle$, $(\sigma \circ \tau)`a$, $\mathrm{D}_\sigma \restriction a$ etc.

Since the variables defined above belong all to the consistent V-system,[3] all the formulas proved in this Part (III) are not only theorems of UL, but also theorems of the consistent V-system of UL. Specifically, the variables Un, Un_2, $\mathrm{Map}_2^{a,b}$, and also D_m, W_m with independent or dependent variable m are not used as sets but only as concepts in the following deductions.[4]

The formulas proved in this Part (III) are not an arbitrary collection of formulas but are those which are needed as ordinary cuts in the deduction in UL of some branches of mathematics. Hence, most of the fundamental properties, though not all, of these dependent variables are deduced in this part.

Our purpose of these deductions is to construct mathematics, as fas as possible, consistently in UL. On the one hand, we know some part of abstract mathematics is consistent. On the other hand, we know also some part of concrete mathematics is consistent. The most essential part of the consistency proof consists in those subsystems of UL in which concrete and abstract mathematics, or our formal and intuitive knowledge, are combined. In virtue of the strong theorem of Gödel on the impossibility[5] of the proof of consistency

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[3] Cf. Part (VI).

[4] What dependent variables are used as sets (see Part (X)) can be seen from each proof, although we do not list up in each case. Such a list is needed, for instance, in the observation of Burali-Forti's contradiction.

[5] Impossible in the sense precisely formulated and proved by Gödel. Moreover, if there were a criterion for any premise σ of UL to be consistent or inconsistent, then either the proof for the criterion could be, after Gödel mapping, formalized in UL and the premise τ of the formalized proof for the criterion would turn out to be inconsistent by the criterion itself, or UL would be still narrow enough to formalize our logical thought. This would mean almost absolute impossibility of such criterion; or else, the criterion or the proof for it would be expressed by means of such part of intuitive logic or mathematics that would be susceptible of no formalization.

as well as in virtue of the difficulty of the problem shown by the development in these decades, it seems almost hopeless to solve the consistency problem by a "once-and-for-all" method as was originally planned by Hilbert. However, it seems there remains yet a "step-by-step" method by which we gradually incease the parts of mathematics which are proved to be consistent. Hence we should construct *actually* even a small part of mathematics consistently. The deductions in this Part (III) will serve as preparation for this purpose. Another purpose of this Part (III) is to show some examples and the elegance of the actual deductions in UL.[6]

Equality (=)

$=*1 \qquad a \subseteq a$

$$
\begin{array}{lll}
 & - & =*1 \\
\hline
(w) & - & \forall x.\ x\in a \rightarrow x\in a \\
\hline
 & - & w\in a \rightarrow w\in a \\
\hline
 & 1 & w\notin a \\
 & & \underset{(1)}{w\in a}
\end{array}
$$

$=*2 \qquad a\subseteq b \wedge b\subseteq c \rightarrow a\subseteq c$

$$
\begin{array}{llll}
 & - & =*2 & \\
\hline
 & 1 & a\nsubseteq b & \\
 & 2 & b\nsubseteq c & \\
 & - & a\subseteq c & \\
\hline
(w) & - & \forall x.\ x\in a\rightarrow x\in c & \\
\hline
 & 3 & w\notin a & \\
 & 4 & w\in c & \\
\hline
(1) & - & \neg\forall x.\ x\in a\rightarrow x\in b & \\
\hline
 & - & \neg.\ w\in a\rightarrow w\in b & \\
\hline
 & & \underset{(3)}{w\in a} & \overset{5}{\underset{(2)}{}}\ w\notin b \\
 & & & -\ \neg\forall x.\ x\in b\rightarrow x\in c \\
 & & & -\ \neg.\ w\in b\rightarrow w\in c \\
\hline
 & & \underset{(5)}{w\in b} & \underset{(4)}{w\notin c}
\end{array}
$$

$=*3 \qquad a=a$

$$
\begin{array}{llll}
 & - & =*3 & \\
(w) & - & \forall x.\ x\in a\equiv x\in a & \\
 & - & w\in a\equiv w\in a & \\
 & & 1\ w\in a & 1\ w\in a \\
 & & \underset{(1)}{w\notin a} & \underset{(1)}{w\notin a}
\end{array}
$$

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[6] See Part (IV); Compendium for deductions, contained in this same volume.

$=*4$ $\quad a=b\rightarrow b=a$

$$\begin{array}{ll}
- & =*4 \\ \hline
1 & a\neq b \\
- & b=a \\ \hline
(w)\ - & \forall x.\ x\in b\equiv x\in a \\ \hline
(1)\ 2 & w\in b\equiv w\in a \\ \hline
- & \neg\forall x.\ x\in a\equiv x\in b \\ \hline
(2)\ 3 & w\in a\not\equiv w\in b \\ \hline
\multicolumn{2}{l}{
\begin{array}{ll}
4 & w\in a \\
(3)\ 5 & w\notin b \\ \hline
\multicolumn{2}{c}{\begin{array}{cc} w\in b & w\notin a \\ (5) & (4) \end{array}}
\end{array}
\qquad
\begin{array}{ll}
4 & w\in b \\
(3)\ 5 & w\notin a \\ \hline
\multicolumn{2}{c}{\begin{array}{cc} w\notin b & w\in a \\ (4) & (5) \end{array}}
\end{array}}
\end{array}$$

$=*5$ $\quad a=b\wedge b=c\rightarrow a=c$

$$\begin{array}{ll}
- & =*5 \\ \hline
- & \neg\forall x.\ x\in a\equiv x\in b \\
- & \neg\forall x.\ x\in b\equiv x\in c \\
(w)\ - & \forall x.\ x\in a\equiv x\in c \\ \hline
- & w\in a\equiv w\in c \\
1 & w\in a\not\equiv w\in b \\
2 & w\in b\not\equiv w\in c \\ \hline
\multicolumn{2}{l}{
\begin{array}{ll}
3 & w\in a \\
(1)\ 4 & w\notin c \\ \hline
\multicolumn{2}{l}{\begin{array}{c} w\notin a \\ (3) \end{array} \quad
\begin{array}{ll}
(2)\ 5 & w\in b \\ \hline
\multicolumn{2}{c}{\begin{array}{cc} w\notin b & w\in c \\ (5) & (4) \end{array}}
\end{array}}
\end{array}
\qquad
\begin{array}{ll}
3 & w\in c \\
(1)\ 4 & w\notin a \\ \hline
\multicolumn{2}{l}{\begin{array}{c} w\in a \\ (4) \end{array} \quad
\begin{array}{ll}
(2)\ 5 & w\notin b \\ \hline
\multicolumn{2}{c}{\begin{array}{cc} w\in b & w\notin c \\ (5) & (3) \end{array}}
\end{array}}
\end{array}}
\end{array}$$

$=*6$ $\quad a=b\rightarrow a\subseteq b\wedge b\subseteq a$

$$\begin{array}{ll}
- & =*6 \\ \hline
- & a\neq b \\
1 & a\subseteq b\wedge b\subseteq a \\ \hline
(1)\ 2 & \neg\forall x.\ x\in a\equiv x\in b \\ \hline
\multicolumn{2}{l}{
\begin{array}{ll}
- & a\subseteq b \\ \hline
(w)\ - & \forall x.\ x\in a\rightarrow x\in b \\ \hline
- & w\in a\rightarrow w\in b \\ \hline
3 & w\notin a \\
(2)\ 4 & w\in b \\ \hline
- & w\in a\not\equiv w\in b \\ \hline
\multicolumn{2}{c}{\begin{array}{cc} w\in a & w\notin b \\ (3) & (4) \end{array}}
\end{array}
\qquad
\begin{array}{ll}
- & b\subseteq a \\ \hline
(w)\ - & \forall x.\ x\in b\rightarrow x\in a \\ \hline
- & w\in b\rightarrow w\in a \\ \hline
3 & w\notin b \\
(2)\ 4 & w\in a \\ \hline
- & w\in a\not\equiv w\in b \\ \hline
\multicolumn{2}{c}{\begin{array}{cc} w\notin a & w\in b \\ (4) & (3) \end{array}}
\end{array}}
\end{array}$$

=*7 $a \subseteq b \wedge b \subseteq a \rightarrow a = b$

$-\quad \equiv *7$

$1\quad a \not\subseteq b$

$2\quad b \not\subseteq a$

$-\quad a = b$

$-\quad \forall x.\ x \in a \equiv x \in b$

(w)

$-\quad w \in a \equiv w \in b$

$3\quad w \in a$	$3\quad w \in b$
$4\quad w \notin b$	$4\quad w \notin a$
(2)	(1)
$-\quad \neg \forall x.\ x \in b \rightarrow x \in a$	$-\quad \neg \forall x.\ x \in a \rightarrow x \in b$
$-\quad \neg .\ w \in b \rightarrow w \in a$	$-\quad \neg .\ w \in a \rightarrow w \in b$
$w \in b$ (4) $\quad w \notin a$ (3)	$w \in a$ (4) $\quad w \notin b$ (3)

=*8 $a = b \equiv a \subseteq b \wedge b \subseteq a$ (From =*6 and =*7.)

=*9 $b \subseteq a \wedge p \in b \rightarrow p \in a$

Elementary set (El)

El*1 $\{a_1, \ldots, a_m\} \subseteq \{b_1, \ldots, b_n\} \rightarrow \overset{m}{\underset{i=1}{\forall}}\ \overset{n}{\underset{j=1}{\exists}}\ a_i = b_j$

$-\quad$ El*1

$-\quad \{a_1, \ldots, a_m\} \not\subseteq \{b_1, \ldots, b_n\}$

$1\quad \overset{m}{\underset{i=1}{\forall}}\ \overset{n}{\underset{j=1}{\exists}}\ a_i = b_j$

$-\quad a_i \in \{a_1, \ldots, a_m\}$	$2\quad a_1 \notin \{b_1, \ldots, b_n\}$
$a_i = a_i$	$\vdots$
$=*3$	$a_m \notin \{b_1, \ldots, b_n\}$
(1)	
$-\quad \overset{n}{\underset{j=1}{\exists}}\ a_1 = b_j$	$\ldots\ \overset{n}{\underset{j=1}{\exists}}\ a_m = b_j$
$3\quad a_1 = b_1$	Symmetric to the left
$\vdots$	
$4\quad a_1 = b_n$	
(2)	
$a_1 \neq b_1$ (3)	$\ldots\quad a_1 \neq b_n$ (4)

El*2 $\overset{m}{\underset{i=1}{\forall}}\ \overset{n}{\underset{j=1}{\exists}}\ a_i = b_j \rightarrow \{a_1, \ldots, a_m\} \subseteq \{b_1, \ldots, b_n\}$

$-\quad$ El*2

$1\quad \neg \overset{m}{\underset{i=1}{\forall}}\ \overset{n}{\underset{j=1}{\exists}}\ a_i = b_j$

$-\quad \{a_1, \ldots, a_m\} \subseteq \{b_1, \ldots, b_n\}$

$- \ \forall x.\ x \in \{a_1, \ldots, a_m\} \rightarrow x \in \{b_1, \ldots, b_n\}$

(w)

2 $\quad w \notin \{a_1, \ldots, a_m\}$

$- \quad w \in \{b_1, \ldots, b_n\}$

3 $\quad w = b_1$

$\vdots$

4 $\quad w = b_n$

(1)

5 $\quad \neg \overset{n}{\underset{j=1}{\exists}} a_1 = b_j$

$\vdots$

(2)

6 $\quad w \neq a_1 \quad \ldots \quad w \neq a_n$

(5)

$\underset{(3,\,6)}{a_1 \neq b_1} \ldots \underset{(1,\,6)}{a_1 \neq b_n}$ Symmetric to the left

Cut $= *5$ Cut $= *5$

El$*3$ $\quad \{a_1, \ldots, a_m\} = \{b_1, \ldots, b_n\} \equiv \overset{m}{\underset{i=1}{\forall}} \overset{n}{\underset{j=1}{\exists}} a_i = b_j \wedge \overset{n}{\underset{j=1}{\forall}} \overset{m}{\underset{i=1}{\exists}} a_i = b_j$

From El$*1$. El$*2$ and $= *8$.

El$*4$ $\quad \{a, b\} = \{c, d\} \rightarrow [a = c \wedge b = d] \vee [a = d \wedge b = c]$

$-$ El$*4$

1 $\quad \{a, b\} \neq \{c, d\}$

2 $\quad a = c \wedge b = d$

3 $\quad a = d \wedge b = c$ $\qquad$ Cut $= *8$

(1)

4 $\quad \{a, b\} \nsubseteq \{c, d\}$

5 $\quad \{c, d\} \nsubseteq \{a, b\}$ $\qquad$ Cut El$*1$

$- \quad \neg.\ \{a, b\} \subseteq \{c, d\} \rightarrow [a = c \vee a = d] \wedge [b = c \vee b = d]$

$\underset{(1)}{\{a, b\} \subseteq \{c, d\}}$ $\qquad$ 6 $\ \neg.\ a = c \vee a = d$

7 $\ \neg.\ b = c \vee b = d$ $\qquad$ Cut El$*1$

$- \quad \neg.\ \{c, d\} \subseteq \{a, b\} \rightarrow [c = a \vee c = b] \wedge [d = a \vee d = b]$

$\underset{(5)}{\{c, d\} \subseteq \{a, b\}}$ $\qquad$ 8 $\ \neg.\ c = a \vee c = b$

9 $\ \neg.\ d = a \vee d = b$

(2)

10 $\quad a = c$ $\qquad\qquad$ 10 $\quad b = d$

(3) $\qquad\qquad$ (3)

11 $\ a = d$ $\qquad$ 11 $\ b = c$ $\qquad$ 11 $\ a = d$ $\qquad$ 11 $\ b = c$

(6) $\qquad$ (8) $\qquad$ (9) $\qquad$ (7)

$\underset{(10)}{a \neq c} \quad \underset{(11)}{a \neq d} \qquad \underset{(10)}{c \neq a} \quad \underset{(11)}{c \neq b} \qquad \underset{(11)}{d \neq a} \quad \underset{(10)}{d \neq b} \qquad \underset{(11)}{b \neq c} \quad \underset{(10)}{b \neq d}$

$= *4 \quad = *4 \qquad = *4 \quad = *4$

El$*5$ $\quad \langle ab \rangle = \{\{a\}, \{a, b\}\}$

$-$ El$*5$

$- \quad \forall x.\ x \in \langle ab \rangle \equiv x \in \{\{a\}, \{a, b\}\}$

(w)

$- \quad w \in \langle ab \rangle \equiv w \in \{\{a\}, \{a, b\}\}$

$$\begin{array}{rll}
- & w\in\langle ab\rangle & \\
1 & w\notin\{\{a\},\ \{a,\ b\}\} & \\ \hline
- & w=\{a\}\vee w=\{a,\ b\} & \\ \hline
2 & w=\{a\} & \\
3 & w=\{a,\ b\} & \\
(1) & & \\ \hline
- & \daleth .\ w=\{a\}\vee w=\{a,\ b\} & \\ \hline
 & \underset{(2)}{w\neq\{a\}} & \underset{(3)}{w\neq\{a,\ b\}}
\end{array}
\qquad
\begin{array}{rll}
- & w\in\{\{a\},\ \{a,\ b\}\} & \\
1 & w\notin\langle ab\rangle & \\ \hline
- & w=\{a\}\vee w=\{a,\ b\} & \\ \hline
2 & w=\{a\} & \\
3 & w=\{a,\ b\} & \\
(1) & & \\ \hline
- & \daleth .\ w=\{a\}\vee w=\{a,\ b\} & \\ \hline
 & \underset{(2)}{w\neq\{a\}} & \underset{(3)}{w\neq\{a,\ b\}}
\end{array}$$

*El*6 $\langle ab\rangle\subseteq\langle cd\rangle\rightarrow a=c\wedge b=d$

$$\begin{array}{rlll}
- & \text{El}*6 & & \\ \hline
1 & \langle ab\rangle\nsubseteq\langle cd\rangle & & \\
2 & a=c\wedge b=d & \text{Cut El}*5 & \\ \hline
3 & \langle ab\rangle\neq\{\{a\},\ \{a,\ b\}\} & & \\
4 & \langle cd\rangle\neq\{\{c\},\ \{c,\ d\}\} & & \\
(1,3,4,=) & & & \\ \hline
5 & \{\{a\},\ \{a,\ b\}\}\nsubseteq\{\{c\},\ \{c,\ d\}\} & \text{Cut El}*1 & \\
6 & \daleth .\ \{a\}\subseteq\{c\}\rightarrow a=c & \text{Cut El}*1 & \\ \hline
7 & \daleth .\ \{c,\ d\}\subseteq\{a\}\rightarrow a=c\wedge a=d & & \\
8 & \daleth .\ \{a,\ b\}\subseteq\{c\}\rightarrow c=a\wedge c=b & & \text{Cut El}*4 \\
(5) & & & \\ \hline
9 & \daleth .\ \{a,\ b\}\subseteq\{c,\ d\}\rightarrow[a=c\wedge b=d]\vee[a=d\wedge b=c] & & \\
10 & \daleth .\ [\{a\}=\{c\}\wedge\{a,\ b\}=\{c,\ d\}]\vee[\{a\}=\{c,\ d\}\wedge\{a,\ b\}=\{c\}] & & \\
(2) & & & \\ \hline
11 & \underset{(*)}{a=c} & 11 & \underset{(**)}{b=d}
\end{array}$$

$$(*)$$

$$\begin{array}{rll}
(10) & & \\ \hline
12 & \{a\}\neq\{c\} & \\
\text{Spf.} & \quad\cdots\cdot & \\
(6) & & \\ \hline
 & \underset{(12)}{\{a\}\subseteq\{c\}} & \underset{(11)}{a\neq c}
\end{array}
\qquad
\begin{array}{rll}
 & & \\ \hline
12 & \{a\}\neq\{c,\ d\} & \\
\text{Spf.} & \quad\cdots\cdot & \\
(7) & & \\ \hline
 & \underset{(12)}{\{c,\ d\}\subseteq\{a\}} & a\neq c \\
 & & \text{Spf. } \underset{(11)}{a\neq d}
\end{array}$$

$$(**)$$

$$\begin{array}{rllll}
(10) & & & & \\ \hline
12 & \{a\}\neq\{c\} & & & \\
13 & \{a,\ b\}\neq\{c,\ d\} & & & \\
(9,\ 13) & & & & \\ \hline
- & \daleth .\ [a=c\wedge b=d]\vee[a=d\wedge b=c] & & & \\ \hline
*\text{Spf.} & a\neq c & 14 & a\neq d & \\
 & \underset{(11)}{b\neq d} & 15 & b\neq c & \\
 & & (6) & & \\ \cline{4-5}
 & & & \underset{(12)}{\{a\}\subseteq\{c\}} & \underset{(11,\ 11,\ 15,\ =)}{a\neq c}
\end{array}
\qquad
\begin{array}{rlll}
 & & & \\ \hline
12 & \{a\}\neq\{c,\ d\} & & \\
13 & \{a,\ b\}\neq\{c\} & & \\
(7) & & & \\ \hline
 & \underset{(12)}{\{c,\ d\}\subseteq\{a\}} & \text{Spf.} & a\neq c \\
 & & 14 & a\neq d \\
(8) & & & \\ \hline
 & \underset{(13)}{\{a,\ b\}\subseteq\{c\}} & & c\neq a \\
 & & & \underset{(11,\ 11,\ =)}{c\neq b}
\end{array}$$

Remark: Cut $=*6$ is used at every place indicated by (12) and (13) in the proof of El*6.

*El*7 $\langle ab\rangle=\langle cd\rangle\rightarrow a=c\wedge b=d$ (From *El*6)

El*8 $a=c\wedge b=d\to\langle ab\rangle\subseteq\langle cd\rangle$

- El*8

1 $a\neq c$

2 $b\neq d$

\- $\langle ab\rangle\subseteq\langle cd\rangle$

(w) - $\forall x.\ x\in\langle ab\rangle\to x\in\langle cd\rangle$

\- $w\in\langle ab\rangle\to w\in\langle cd\rangle$

\- $w\notin\langle ab\rangle$

\- $w\in\langle cd\rangle$

3 $\neg.\ w=\{a\}\vee w=\{a,b\}$

\- $w=\{c\}\vee w=\{c,d\}$

\- $w=\{c\}$

\- $w=\{c,d\}$

4 $\forall x.\ x\in w\equiv x\in\{c\}$

5 $\forall x.\ x\in w\equiv x\in\{c,d\}$

(3)

6 $w\neq\{a\}$ (*)	6 $w\neq\{a,b\}$ (**)

6 (*)

7 $\neg\forall x.\ x\in w\equiv x\in\{a\}$

(r, 4)

8 $r\in w\equiv r\in\{c\}$

(7)

9 $r\in w\not\equiv r\in\{a\}$

(8)

10 $r\in w$	- $r\in\{c\}$
- $r\notin\{c\}$	10 $r\notin w$
11 $r\neq c$	11 $r=c$
(9)	(9)
$r\notin w$ (10) - $r\in\{a\}$	$r\in w$ (10) - $r\notin\{a\}$
$r=a$ (1, 11)	$r\neq a$ (1, 11)
Cut =*4 Cut =*5	Cut =*5

6 (**)

7 $\neg\forall x.\ x\in w\equiv x\in\{a,b\}$

(r, 5)

8 $r\in w\equiv r\in\{c,d\}$

(7)

9 $r\in w\not\equiv r\in\{a,b\}$

(8)

10 $r\in w$	- $r\in\{c,d\}$
- $r\notin\{c,d\}$	10 $r\notin w$
11 $\neg.\ r=c\vee r=d$	- $r=c\vee r=d$
(9)	11 $r=c$
$r\notin w$ (10) - $r\in\{a,b\}$	12 $r=d$
- $r=a\vee r=b$	(9)
12 $r=a$	$r\in w$ (10) - $r\notin\{a,b\}$
13 $r=b$	- $\neg.\ r=a\vee r=b$
(11)	
$r\neq c$ (1, 12) $r\neq d$ (2, 13)	$r\neq a$ (1, 11) $r\neq b$ (2, 12)
Cut =*4 Cut =*4 Cut =*5 Cut =*5	Cut =*5 Cut =*5

El*9 $a=c\wedge b=d\to\langle ab\rangle=\langle cd\rangle$ (From El*8, =*8)

El*10 $a\in c\wedge b\in c\to\{a,\ b\}\subseteq c$

$$\begin{array}{rc}
- & \mathrm{El}*10 \\ \hline
1 & a\notin c \\
2 & b\notin c \\
- & \{a,\ b\}\subseteq c \\ \hline
(w)\ - & \forall x.\ x\in\{a,\ b\}\to x\in c \\ \hline
- & w\in\{a,\ b\}\to w\in c \\ \hline
- & w\notin\{a,\ b\} \\
3 & w\in c \\ \hline
- & \neg.\ w=a\vee w=b \\ \hline
 & \begin{array}{cc} w\neq a & w\neq b \\ {\scriptstyle (1,\ 3,\ 1)} & {\scriptstyle (2,\ 3,\ I)} \end{array}
\end{array}$$

El*11 $\forall xy[x\in a\wedge y\in a\to x=y]\wedge p\in a\to a=\{p\}$

$$\begin{array}{rc}
- & \mathrm{El}*11 \\ \hline
1 & \neg\forall xy.\ x\in a\wedge y\in a\to x=y \\
2 & p\notin a \\
- & a=\{p\} \\ \hline
(w)\ - & \forall x.\ x\in a\equiv x\in\{p\} \\ \hline
- & w\in a\equiv w\in\{p\} \\ \hline
 & \begin{array}{ll}
\begin{array}{rl} 3 & w\in a \\ - & w\notin\{p\} \\ \hline & w\neq p \\ & {\scriptstyle (2,\ 3,\ I)} \end{array}
&
\begin{array}{rl} - & w\in\{p\} \\ 3 & w\notin a \\ \hline 4 & w=p \\ (1)\ \ & \\ \hline - & \neg.\ w\in a\wedge p\in a\to w=p \\ \hline & \begin{array}{ccc} w\in a & p\in a & w\neq p \\ {\scriptstyle (3)} & {\scriptstyle (2)} & {\scriptstyle (1)} \end{array} \end{array}
\end{array}
\end{array}$$

Image of Operator (Im)

Im*1 $\sigma\in\mathrm{Un}\wedge\tau\subseteq\sigma\to\tau\in\mathrm{Un}$

$$\begin{array}{rc}
- & \mathrm{Im}*1 \\ \hline
- & \sigma\notin\mathrm{Un} \\
- & \tau\nsubseteq\sigma \\
- & \tau\in\mathrm{Un} \\ \hline
1 & \neg\forall xyz.\ \langle xy\rangle\in\sigma\wedge\langle xz\rangle\in\sigma\to y=z \\
2 & \neg\forall x.\ x\in\tau\to x\in\sigma \\
- & \forall xyz.\ \langle xy\rangle\in\tau\wedge\langle xz\rangle\in\tau\to y=z \\ \hline
(r,\ s,\ t)\ - & \langle rs\rangle\in\tau\wedge\langle rt\rangle\in\tau\to s=t \\ \hline
3 & \langle rs\rangle\notin\tau \\
4 & \langle rt\rangle\notin\tau \\
5 & s=t
\end{array}$$

(2) ---

$-\quad \neg.\ \langle rs\rangle\in\tau\rightarrow\langle rs\rangle\in\sigma$

$6\quad \neg.\ \langle rt\rangle\in\tau\rightarrow\langle rt\rangle\in\sigma$

$\underset{(3)}{\langle rs\rangle\in\tau}\qquad 7\ (6)\quad \langle rs\rangle\notin\sigma$

$\underset{(4)}{\langle rt\rangle\in\tau}\qquad 8\ (1)\quad \langle rt\rangle\notin\sigma$

$-\quad \neg.\ \langle rs\rangle\in\sigma\wedge\langle rt\rangle\in\sigma\rightarrow s=t$

$\underset{(7)}{\langle rs\rangle\in\sigma}\qquad \underset{(8)}{\langle rt\rangle\in\sigma}\qquad \underset{(5)}{s\neq t}$

Im*2 $\sigma\in\mathrm{Un}\wedge\langle ab\rangle\in\sigma\rightarrow a^{\sigma}=b$

$-\quad \mathrm{Im}*2$

$1\quad \sigma\notin\mathrm{Un}$

$2\quad \langle ab\rangle\notin\sigma$

$-\quad a^{\sigma}=b$

$(w)\ -\quad \forall x.\ x\in a^{\sigma}\equiv x\in b$

$-\quad w\in a^{\sigma}$

$3\quad w\notin b$

$-\quad \exists x.\ \langle ax\rangle\in\sigma\wedge w\in x$

$-\quad \langle ab\rangle\in\sigma\wedge w\in b$

$\underset{(2)}{\langle ab\rangle\in\sigma}\qquad \underset{(3)}{w\in b}$

$3\quad w\in b$

$-\quad w\notin a^{\sigma}$

$-\quad \neg\exists x.\ \langle ax\rangle\in\sigma\wedge w\in x$

$(b_1)\ -\quad \neg.\ \langle ab_1\rangle\in\sigma\wedge w\in b_1$

$4\quad \langle ab_1\rangle\notin\sigma$

$5\quad w\notin b_1$

(1) ---

$-\quad \neg\forall xyz.\ \langle xy\rangle\in\sigma\wedge\langle xz\rangle\in\sigma\rightarrow y=z$

$-\quad \neg.\ \langle ab\rangle\in\sigma\wedge\langle ab_1\rangle\in\sigma\rightarrow b=b_1$

$\underset{(2)}{\langle ab\rangle\in\sigma}\qquad \underset{(4)}{\langle ab_1\rangle\in\sigma}\qquad \underset{(3,\,5,\,=)}{b\neq b_1}$

Im*3 $\sigma\in\mathrm{Un}\wedge a\in\mathrm{D}_{\sigma}\rightarrow\langle aa^{\sigma}\rangle\in\sigma$

$-\quad \mathrm{Im}*3$

$1\quad \sigma\notin\mathrm{Un}$

$-\quad a\notin D_{\sigma}$

$2\quad \langle aa^{\sigma}\rangle\in\sigma$

$-\quad \neg\exists x.\ \langle ax\rangle\in\sigma$

(r) ---

$3\quad \langle ar\rangle\notin\sigma\quad$ Cut Im*2

$-\quad \neg.\ \sigma\in\mathrm{Un}\wedge\langle ar\rangle\in\sigma\rightarrow a^{\sigma}=r$

$\underset{(1)}{\sigma\in\mathrm{Un}}\qquad \underset{(3)}{\langle ar\rangle\in\sigma}\qquad \underset{(2,\,3,\,=,\,\mathrm{I})}{a^{\sigma}\neq r}$

Im*4 $\sigma\in\mathrm{Un}\wedge a\in\mathrm{D}_{\sigma}\wedge a\in A\rightarrow a^{\sigma}\in A^{\sigma}$

$-\quad \mathrm{Im}*4$

$1\quad \sigma\notin\mathrm{Un}\qquad 3\quad a\notin A$

$2\quad a\notin\mathrm{D}_{\sigma}\qquad 4\quad a^{\sigma}\in A^{\sigma}$

(4) ---

$-\quad \exists x.\ x\in A\wedge\langle xa^{\sigma}\rangle\in\sigma$

$-\quad a\in A\wedge\langle aa^{\sigma}\rangle\in\sigma$

$a\in A$ (3) 5 $\langle aa^{\sigma}\rangle\in\sigma$ Cut Im*3

\- $\urcorner.\ \sigma\in\mathrm{Un}\wedge a\in\mathrm{D}_{\sigma}\rightarrow\langle aa^{\sigma}\rangle\in\sigma$

$\sigma\in\mathrm{Un}$ (1) $a\in\mathrm{D}_{\sigma}$ (2) $\langle aa^{\sigma}\rangle\notin\sigma$ (5)

Im*5 $A\subseteq B\rightarrow A^{\sigma}\subseteq B^{\sigma}$

\- Im*5

1 $A\not\subseteq B$

\- $A^{\sigma}\subseteq B^{\sigma}$

(s) - $\forall x.\ x\in A^{\sigma}\rightarrow x\in B^{\sigma}$

\- $s\in A^{\sigma}\rightarrow s\in B^{\sigma}$

\- $s\notin A^{\sigma}$

\- $s\in B^{\sigma}$

(r) - $\urcorner\exists x.\ x\in A\wedge\langle xs\rangle\in\sigma$

2 $\exists x.\ x\in B\wedge\langle xs\rangle\in\sigma$

\- $\urcorner.\ r\in A\wedge\langle rs\rangle\in\sigma$

3 $r\notin A$

4 $\langle rs\rangle\notin\sigma$

(2) - $r\in B\wedge\langle rs\rangle\in\sigma$

5 $r\in B$ $\langle rs\rangle\in\sigma$ (4)

(1) - $\urcorner\forall x.\ x\in A\rightarrow x\in B$

\- $\urcorner.\ r\in A\rightarrow r\in B$

$r\in A$ (3) $r\notin B$ (5)

Im*6 $\sigma\in\mathrm{Un}\wedge a\in\mathrm{D}_{\sigma}\rightarrow a^{\sigma}\in\mathrm{W}_{\sigma}$

\- Im*6

1 $\sigma\notin\mathrm{Un}$

\- $a\notin\mathrm{D}_{\sigma}$

\- $a^{\sigma}\in\mathrm{W}_{\sigma}$

(r) - $\urcorner\exists x.\ \langle ax\rangle\in\sigma$

2 $\exists x.\ \langle xa^{\sigma}\rangle\in\sigma$

3 $\langle ar\rangle\notin\sigma$

(2) 4 $\langle aa^{\sigma}\rangle\in\sigma$ Cut Im*2

\- $\urcorner.\ \sigma\in\mathrm{Un}\wedge\langle ar\rangle\in\sigma\rightarrow a^{\sigma}=r$

$\sigma\in\mathrm{Un}$ (1) $\langle ar\rangle\in\sigma$ (3) $a^{\sigma}\neq r$ (3, 4, =, I)

Im*7 $\langle ab\rangle\notin\sigma\rightarrow a\in\mathrm{D}_{\sigma}$

Im*8 $\langle ab\rangle\in\sigma\rightarrow b\in\mathrm{W}_{\sigma}$

Im*9 $W_\sigma=(D_\sigma)^\sigma$

- Im*9

(s) - $\forall x.\ x\in W_\sigma\equiv x\in(D_\sigma)^\sigma$

- $s\in W_\sigma\equiv s\in(D_\sigma)^\sigma$

- $s\in W_\sigma$
- $s\notin(D_\sigma)^\sigma$

1 $\exists x.\ \langle xs\rangle\in\sigma$

(r) - $\neg\exists x.\ x\in D_\sigma\wedge\langle xs\rangle\in\sigma$

- $\neg.\ r\in D_\sigma\wedge\langle rs\rangle\in\sigma$

Spf $r\notin D_\sigma$

(1) 2 $\langle rs\rangle\notin\sigma$

$\langle rs\rangle\in\sigma$ (2)

- $s\in(D_\sigma)^\sigma$
- $s\notin W_\sigma$

1 $\exists x.\ x\in D_\sigma\wedge\langle xs\rangle\in\sigma$

(r) - $\neg\exists x.\ \langle xs\rangle\in\sigma$

(1) 2 $\langle rs\rangle\notin\sigma$

- $r\in D_\sigma\wedge\langle rs\rangle\in\sigma$

- $r\in D_\sigma$ $\langle rs\rangle\in\sigma$ (2)

- $\exists x.\ \langle rx\rangle\in\sigma$

$\langle rs\rangle\in\sigma$ (2)

Inverse Operator (− 1)

− 1*1 $(\sigma^{-1})^{-1}\subseteq\sigma$

- − 1*1

- $a\notin(\sigma^{-1})^{-1}$

1 $a\in\sigma$

- $\neg\exists xy.\ a=\langle xy\rangle\wedge\langle yx\rangle\in\sigma^{-1}$

(r, s)

- $\neg.\ a=\langle rs\rangle\wedge\langle sr\rangle\in\sigma^{-1}$

2 $a\neq\langle rs\rangle$

- $\langle sr\rangle\notin\sigma^{-1}$

- $\neg\exists xy.\ \langle sr\rangle=\langle xy\rangle\wedge\langle yx\rangle\in\sigma$

(r_1, s_1)

- $\neg.\ \langle sr\rangle=\langle s_1r_1\rangle\wedge\langle r_1s_1\rangle\in\sigma$

$\langle sr\rangle\neq\langle s_1r_1\rangle$

$\langle r_1s_1\rangle\notin\sigma$

(1, 2, =, I)

− 1*2 $\sigma\in\mathrm{Un}\rightarrow(\sigma^{-1})^{-1}\in\mathrm{Un}$

- − 1*2

1 $\sigma\notin\mathrm{Un}$

2 $(\sigma^{-1})^{-1}\in\mathrm{Un}$ Cut Im*1

- $\neg.\ \sigma\in\mathrm{Un}\wedge(\sigma^{-1})^{-1}\subseteq\sigma\rightarrow(\sigma^{-1})^{-1}\in\mathrm{Un}$

$\sigma\in\mathrm{Un}$ (1) $(\sigma^{-1})^{-1}\subseteq\sigma$ − 1*1 $(\sigma^{-1})^{-1}\notin\mathrm{Un}$ (2)

$-1*3$ $\sigma\in\mathrm{Un}_2\to\sigma^{-1}\in\mathrm{Un}_2$,

$$\begin{array}{ll}
- & -1*3\\ \hline
- & \sigma\notin\mathrm{Un}_2\\
- & \sigma^{-1}\in\mathrm{Un}_2\\ \hline
- & \neg.\ \sigma\in\mathrm{Un}\wedge\sigma^{-1}\in\mathrm{Un}\\
1 & \sigma^{-1}\in\mathrm{Un}\wedge(\sigma^{-1})^{-1}\in\mathrm{Un}\\ \hline
2 & \sigma\notin\mathrm{Un}\\
3 & \sigma^{-1}\notin\mathrm{Un}\\ \hline
(1) & \begin{array}{lll}
\underset{(3)}{\sigma^{-1}\in\mathrm{Un}} & 4 & (\sigma^{-1})^{-1}\in\mathrm{Un}\quad \text{Cut}\ -1*2\\ \cline{2-3}
 & - & \neg.\ \sigma\in\mathrm{Un}\to(\sigma^{-1})^{-1}\in\mathrm{Un}\\ \cline{2-3}
 & & \underset{(2)}{\sigma\in\mathrm{Un}}\qquad \underset{(4)}{(\sigma^{-1})^{-1}\notin\mathrm{Un}}
\end{array}
\end{array}$$

$-1*4$ $\mathrm{D}_\sigma\subseteq\mathrm{W}_{\sigma^{-1}}$

$$\begin{array}{ll}
- & -1*4\\ \hline
- & a\notin\mathrm{D}_\sigma\\
- & a\in\mathrm{W}_{\sigma^{-1}}\\ \hline
(r)- & \neg\exists x.\ \langle ax\rangle\in\sigma\\
1 & \exists x.\ \langle xa\rangle\in\sigma^{-1}\\ \hline
2 & \langle ar\rangle\notin\sigma\\ \hline
(1)- & \langle ra\rangle\in\sigma^{-1}\\ \hline
- & \exists xy.\ \langle ra\rangle=\langle xy\rangle\wedge\langle yx\rangle\in\sigma\\ \hline
- & \langle ra\rangle=\langle ra\rangle\wedge\langle ar\rangle\in\sigma\\ \hline
 & \underset{(=)}{\langle ra\rangle=\langle ra\rangle}\qquad \underset{(2)}{\langle ar\rangle\in\sigma}
\end{array}$$

$-1*5$ $\mathrm{W}_{\sigma^{-1}}\subseteq\mathrm{D}_\sigma$

$$\begin{array}{ll}
- & -1*5\\ \hline
- & a\notin\mathrm{W}_{\sigma^{-1}}\\
- & a\in\mathrm{D}_\sigma\\ \hline
(r)- & \neg\exists x.\ \langle xa\rangle\in\sigma^{-1}\\
1 & \exists x.\ \langle ax\rangle\in\sigma\\ \hline
- & \langle ra\rangle\in\sigma^{-1}\\ \hline
- & \neg\exists xy.\ \langle ra\rangle=\langle xy\rangle\wedge\langle yx\rangle\in\sigma\\ \hline
(r_1,a_1)- & \neg.\ \langle ra\rangle=\langle r_1a_1\rangle\wedge\langle a_1r_1\rangle\in\sigma\\ \hline
2 & \langle ra\rangle\neq\langle r_1a_1\rangle\\
3 & \langle a_1r_1\rangle\notin\sigma\\ \hline
(1) & \underset{(2,3,=,1)}{\langle ar\rangle\in\sigma}
\end{array}$$

$-1*6$ $\mathrm{D}_\sigma=\mathrm{W}_{\sigma^{-1}}$ (From $-1*4$, $-1*5$ and $=*7$)

$-1*7$ $\mathrm{W}_\sigma\subseteq\mathrm{D}_{\sigma^{-1}}$ (Similar to $-1*4$)

$-1*8$ $\mathrm{D}_{\sigma^{-1}}\subseteq\mathrm{W}_\sigma$ (Similar to $-1*5$)

$-1*9$ $\quad \mathrm{W}_\sigma = \mathrm{D}_{\sigma^{-1}}$ $\quad$ (From $-1*7$, $-1*8$ and $=*7$)

$-1*10$ $\quad \mathrm{D}_\sigma = (\mathrm{W}_\sigma)^{\sigma^{-1}}$

```
-                                   -1*10
──────────────────────────────────────────────────────────────────────
-          r∈D_σ                      -          r∈(W_σ)^{σ^-1}
-          r∉(W_σ)^{σ^-1}             -          r∉D_σ
─────────────────────────────────     ────────────────────────────────
1      ∃x. ⟨rx⟩∈σ                     1      ∃x. x∈W_σ∧⟨xr⟩∈σ^-1
-    ¬∃x. x∈W_σ∧⟨xr⟩∈σ^-1             -    ¬∃x. ⟨rx⟩∈σ
(s)- - - - - - - - - - - - - - - -    (s)- - - - - - - - - - - - - - -
-    ¬. s∈W_σ∧⟨sr⟩∈σ^-1               2          ⟨rs⟩∉σ
─────────────────────────────────     (1)─────────────────────────────
Spf.          s∉W_σ                   -      s∈W_σ∧⟨sr⟩∈σ^-1
-            ⟨sr⟩∉σ^-1                ────────────────────────────────────────────────────
- - - - - - - - - - - - - - - - -     -    s∈W_σ             -        ⟨sr⟩∈σ^-1
-  ¬∃xy. ⟨sr⟩=⟨xy⟩∧⟨yx⟩∈σ             - - - - - - - - - -    - - - - - - - - - - - - - - - - - -
(s₁r₁)───────────────────────────     - ∃x. ⟨xs⟩∈σ          - ∃xy. ⟨sr⟩=⟨xy⟩∧⟨yx⟩∈σ
-  ¬. ⟨sr⟩=⟨s₁r₁⟩∧⟨r₁s₁⟩∈σ            ──────────────────    - - - - - - - - - - - - - - - - - -
─────────────────────────────────         ⟨rs⟩∈σ            -   ⟨sr⟩=⟨sr⟩∧⟨rs⟩∈σ
2          ⟨sr⟩≠⟨s₁r₁⟩                      (2)             ──────────────────────────────────
3          ⟨r₁s₁⟩∉σ                                            ⟨sr⟩=⟨sr⟩        ⟨rs⟩∈σ
(1)- - - - - - - - - - - - - - - -                                (=)              (2)
           ⟨rs⟩∈σ
        (2, 3, =, I)
```

$-1*11$ $\quad \sigma\in\mathrm{Map}_2^{a,b} \rightarrow \sigma^{-1}\in\mathrm{Map}_2^{b,a}$

```
-                          -1*11
──────────────────────────────────────────────────────
-                        σ∉Map_2^{a,b}
-                        σ^-1∈Map_2^{b,a}
- - - - - - - - - - - - - - - - - - - - - - - - - - -
                  ¬. σ∈Un_2∧D_σ=a∧W_σ=b
1               σ^-1∈Un_2∧D_{σ^-1}=b∧W_{σ^-1}=a
──────────────────────────────────────────────────────
2                          σ∉Un_2
3                          D_σ≠a
4                          W_σ≠b
- - - - - - - - - - - - - - - - - - - - - - - - - - -
      σ^-1∈Un_2         D_{σ^-1}=b        W_{σ^-1}=a
         (2)              (4, =)            (3, =)
      Cut  -1*3         Cut  -1*9         Cut  -1*6
```

$-1*12$ $\quad \sigma\in\mathrm{Un}_2\wedge a\in\mathrm{D}_\sigma \rightarrow a^{\sigma\sigma^{-1}} = a$

```
-                 -1*12
- - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
1       σ∉Un           3     a∉D_σ
2     σ^-1∉Un          4     a^{σσ^-1}=a              Cut ∘*2
──────────────────────────────────────────────────────────────
-           ¬. σ∈Un∧a∈D_σ→a^{σσ^-1}=a^{σ^-1∘σ}
- - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
   σ∈Un    a∈D_σ       5      a^{σσ^-1}≠a^{σ^-1∘σ}      Cut ι*12
    (1)     (3)            - - - - - - - - - - - - - - - - - - -
                           - ¬. σ^-1∈Un→σ^-1∘σ=ι↾D_σ
──────────────────────────────────────────────────────────────
    σ^-1∈Un            6      σ^-1∘σ≠ι↾D_σ       Cut ↾*8
      (2)                  ───────────────────────────────────
                           -     ¬. a∈D_σ→a^{ι↾D_σ}=a^ι
- - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
     a∈D_σ             7            a^{ι↾D_σ}≠a^ι     Cut ι*2
      (3)                  - - - - - - - - - - - - - - - - - -
                                       a^ι≠a
                                  (4, 5, 6, 7, =)
```

$-1*13$ $\sigma \in \mathrm{Un} \wedge a \in \mathrm{D}_\sigma \rightarrow a^{(\sigma^{-1})^{-1}} = a^\sigma$

$-$ $-1*13$

1 $\sigma \notin \mathrm{Un}$

2 $a \notin \mathrm{D}_\sigma$

$-$ $a^{(\sigma^{-1})^{-1}} = a^\sigma$

(w) $-$ $\forall x.\ x \in a^{(\sigma^{-1})^{-1}} \equiv x \in a^\sigma$

$-$ $w \in a^{(\sigma^{-1})^{-1}} \equiv w \in a^\sigma$

$-$ $w \in a^{(\sigma^{-1})^{-1}}$

3 $w \notin a^\sigma$

$-$ $\exists x.\ \langle ax \rangle \in (\sigma^{-1})^{-1} \wedge w \in x$

$-$ $\langle aa^\sigma \rangle \in (\sigma^{-1})^{-1} \wedge w \in a^\sigma$

$-$ $\langle aa^\sigma \rangle \in (\sigma^{-1})^{-1}$ $w \in a^\sigma$ (3)

$-$ $\langle a^\sigma a \rangle \in \sigma^{-1}$

4 $\langle aa^\sigma \rangle \in \sigma$ Cut Im$*3$

$-$ $\neg.\ \sigma \in \mathrm{Un} \wedge a \in \mathrm{D}_\sigma \rightarrow \langle aa^\sigma \rangle \in \sigma$

$\sigma \in \mathrm{Un}$ (1) $a \in \mathrm{D}_\sigma$ (2) $\langle aa^\sigma \rangle \notin \sigma$ (4)

3 $w \in a^\sigma$

$-$ $w \notin a^{(\sigma^{-1})^{-1}}$

(b) $-$ $\neg \exists x.\ \langle ax \rangle \in (\sigma^{-1})^{-1} \wedge w \in x$

$-$ $\neg.\ \langle ab \rangle \in (\sigma^{-1})^{-1} \wedge w \in b$

$-$ $\langle ab \rangle \notin (\sigma^{-1})^{-1}$

(a, b) 4 $w \notin b$

(a, b) $-$ $\langle ba \rangle \notin \sigma^{-1}$

5 $\langle ab \rangle \notin \sigma$ Cut Im$*2$

$-$ $\neg.\ \sigma \in \mathrm{Un} \wedge \langle ab \rangle \in \sigma \rightarrow a^\sigma = b$

$\sigma \in \mathrm{Un}$ (1) $\langle ab \rangle \in \sigma$ (5, =, 1) $a^\sigma \neq b$ (3, 4, =)

$-1*14$ $\sigma \in \mathrm{Un}_2 \wedge a \in \mathrm{W}_\sigma \rightarrow a^{\sigma^{-1}\sigma} = a$

$-$ $-1*14$

1 $\sigma \notin \mathrm{Un}_2$

2 $a \notin \mathrm{W}_\sigma$

3 $a^{\sigma^{-1}\sigma} = a$ Cut $-1*12$

$-$ $\neg.\ \sigma^{-1} \in \mathrm{Un}_2 \wedge a \in \mathrm{D}_{\sigma^{-1}} \rightarrow a^{\sigma^{-1}(\sigma^{-1})^{-1}} = a$

$\sigma^{-1} \in \mathrm{Un}_2$ (1) Cut $-1*3$ $a \in \mathrm{D}_{\sigma^{-1}}$ (2, =) Cut $-1*9$

4 $a^{\sigma^{-1}(\sigma^{-1})^{-1}} \neq a$ Cut $-1*13$

$-$ $\neg.\ \sigma \in \mathrm{Un} \wedge a^{\sigma^{-1}} \in \mathrm{D}_\sigma \rightarrow a^{\sigma^{-1}(\sigma^{-1})^{-1}} = a^{\sigma^{-1}\sigma}$

$\sigma \in \mathrm{Un}$ (1) 5 $a^{\sigma^{-1}} \in \mathrm{D}_\sigma$ Cut $-1*7$ $a^{\sigma^{-1}(\sigma^{-1})^{-1}} \neq a^{\sigma^{-1}\sigma}$ (3, 4, =)

$-$ $\neg.\ a \in \mathrm{W}_\sigma \rightarrow a \in \mathrm{D}_{\sigma^{-1}}$

$a \in \mathrm{W}_\sigma$ (2) 6 $a \notin \mathrm{D}_{\sigma^{-1}}$ Cut Im$*6$

$-$ $\neg.\ \sigma^{-1} \in \mathrm{Un} \wedge a \in \mathrm{D}_{\sigma^{-1}} \rightarrow a^{\sigma^{-1}} \in \mathrm{W}_{\sigma^{-1}}$

$\sigma^{-1} \in \mathrm{Un}$ (1) $a \in \mathrm{D}_{\sigma^{-1}}$ (6) $a^{\sigma^{-1}} \notin \mathrm{W}_{\sigma^{-1}}$ (5, =) Cut $-1*6$

Composition of Operators $(\circ)$

$\circ *1$ $A^{\sigma\tau} = A^{\tau\circ\sigma}$ $(A^{\sigma\tau} = (A^{\sigma})^{\tau})$

$-$ $\circ *1$

1 $t \in A^{\sigma\tau}$
$-$ $t \notin A^{\tau\circ\sigma}$
(r) $-$ $\neg\exists x.\ x \in A \wedge \langle xt\rangle \in \tau\circ\sigma$
$-$ $\neg.\ r \in A \wedge \langle rt\rangle \in \tau\circ\sigma$
2 $r \notin A$
$-$ $\langle rt\rangle \notin \tau\circ\sigma$
(r_1, s, t_1) $-$ $\neg\exists xyz.\ \langle rt\rangle = \langle xz\rangle \wedge \langle xy\rangle \in \sigma \wedge \langle yz\rangle \in \tau$
$-$ $\neg.\ \langle rt\rangle = \langle r_1t_1\rangle \wedge \langle r_1s\rangle \in \sigma \wedge \langle st_1\rangle \in \tau$
3 $\langle rt\rangle \neq \langle r_1t_1\rangle$
4 $\langle r_1s\rangle \notin \sigma$
5 $\langle st_1\rangle \notin \tau$
(1) $-$ $\exists x.\ x \in A^{\sigma} \wedge \langle xt\rangle \in \tau$
$-$ $s \in A^{\sigma} \wedge \langle st\rangle \in \tau$
$-$ $s \in A^{\sigma}$ $\quad \underset{(3,\,5,\,=,\,\mathrm{I})}{\langle st\rangle \in \tau}$
$-$ $\exists x.\ x \in A \wedge \langle xs\rangle \in \sigma$
$-$ $r \in A \wedge \langle rs\rangle \in \sigma$
$\underset{(2)}{r \in A}$ $\quad \underset{(3,\,4,\,=,\,\mathrm{I})}{\langle rs\rangle \in \sigma}$

1 $t \in A^{\tau\circ\sigma}$
$-$ $t \notin A^{\sigma\tau}$
(s) $-$ $\neg\exists x.\ x \in A^{\sigma} \wedge \langle xt\rangle \in \tau$
$-$ $\neg.\ s \in A^{\sigma} \wedge \langle st\rangle \in \tau$
$-$ $s \notin A^{\sigma}$
2 $\langle st\rangle \notin \tau$
(r) $-$ $\neg\exists x.\ x \in A \wedge \langle xs\rangle \in \sigma$
$-$ $\neg.\ r \in A \wedge \langle rs\rangle \in \sigma$
3 $r \notin A$
4 $\langle rs\rangle \notin \sigma$
(1) $-$ $\exists x.\ x \in A \wedge \langle xt\rangle \in \tau\circ\sigma$
$-$ $r \in A \wedge \langle rt\rangle \in \tau\circ\sigma$
$\underset{(3)}{r \in A}$ $-$ $\langle rt\rangle \in \tau\circ\sigma$
$-$ $\exists xyz.\ \langle rt\rangle = \langle xz\rangle \wedge \langle xy\rangle \in \sigma \wedge \langle yz\rangle \in \tau$
$-$ $\langle rt\rangle = \langle rt\rangle \wedge \langle rs\rangle \in \sigma \wedge \langle st\rangle \in \tau$
$\underset{(=)}{\langle rt\rangle = \langle rt\rangle}$ $\quad \underset{(4)}{\langle rs\rangle \in \sigma}$ $\quad \underset{(2)}{\langle st\rangle \in \tau}$

$\circ *2$ $\sigma \notin \mathrm{Un} \wedge a \in \mathrm{D}_{\sigma} \rightarrow a^{\sigma\tau} = a^{\tau\circ\sigma}$ $(a^{\sigma\tau} = (a^{\sigma})^{\tau})$

$-$ $\circ *2$
1 $\sigma \notin \mathrm{Un}$
2 $a \notin \mathrm{D}_{\sigma}$
$-$ $a^{\sigma\tau} = a^{\tau\circ\sigma}$
(w) $-$ $\forall x.\ x \in a^{\sigma\tau} \equiv x \in a^{\tau\circ\sigma}$
$-$ $w \in a^{\sigma\tau} \equiv w \in a^{\tau\circ\sigma}$

3 $w \in a^{\sigma\tau}$
$-$ $w \notin a^{\tau\circ\sigma}$
(c) $-$ $\neg\exists x.\ \langle ax\rangle \in \tau\circ\sigma \wedge w \in x$
$-$ $\neg.\ \langle ac\rangle \in \tau\circ\sigma \wedge w \in c$
$-$ $\langle ac\rangle \notin \tau\circ\sigma$
4 $w \notin c$
(a, b, c) $-$ $\neg\exists xyz.\ \langle ac\rangle = \langle xz\rangle \wedge \langle xy\rangle \in \sigma \wedge \langle yz\rangle \in \tau$
$-$ $\neg.\ \langle ac\rangle = \langle ac\rangle \wedge \langle ab\rangle \in \sigma \wedge \langle bc\rangle \in \tau$
5 $\langle ab\rangle \notin \sigma$
6 $\langle bc\rangle \notin \tau$
(3) $-$ $\exists x.\ \langle a^{\sigma}x\rangle \in \tau \wedge w \in x$

3 $w \in a^{\tau\circ\sigma}$
$-$ $w \notin a^{\sigma\tau}$
(c) $-$ $\neg\exists x.\ \langle a^{\sigma}x\rangle \in \tau \wedge w \in x$
$-$ $\neg.\ \langle a^{\sigma}c\rangle \in \tau \wedge w \in c$
4 $\langle a^{\sigma}c\rangle \notin \tau$
5 $w \notin c$
(3) $-$ $\exists x.\ \langle ax\rangle \in \tau\circ\sigma \wedge w \in x$
$-$ $\langle ac\rangle \in \tau\circ\sigma \wedge w \in c$
$-$ $\langle ac\rangle \in \tau\circ\sigma$ $\quad \underset{(5)}{w \in c}$
$-$ $\exists xyz.\ \langle ac\rangle = \langle xz\rangle \wedge \langle xy\rangle \in \sigma \wedge \langle yz\rangle \in \tau$
$-$ $\langle aa^{\sigma}\rangle \in \sigma \wedge \langle a^{\sigma}c\rangle \in \tau$

- $\langle a^{\sigma}c\rangle\in\tau\wedge w\in c$

7 $\langle a^{\sigma}c\rangle\in\tau$ Cut Im*2 $\underset{(4)}{w\in c}$

- $\neg.\ \sigma\in\mathrm{Un}\wedge\langle ab\rangle\in\sigma\rightarrow a^{\sigma}=b$

$\underset{(1)}{\sigma\in\mathrm{Un}}$ $\underset{(5,\ =,\ 1)}{\langle ab\rangle\in\sigma}$ $\underset{(6,\ 7,\ =,\ 1)}{a^{\sigma}\neq b}$

6 $\langle aa^{\sigma}\rangle\in\sigma$ Cut Im*3 $\underset{(4)}{\langle a^{\sigma}c\rangle\in\tau}$

- $\neg.\ \sigma\in\mathrm{Un}\wedge a\in\mathrm{D}_{\sigma}\rightarrow\langle aa^{\sigma}\rangle\in\sigma$

$\underset{(1)}{\sigma\in\mathrm{Un}}$ $\underset{(2)}{a\in\mathrm{D}_{\sigma}}$ $\underset{(6)}{\langle aa^{\sigma}\rangle\notin\sigma}$

$\circ$*3 $\sigma\subseteq\tau\rightarrow\sigma\circ\rho\subseteq\tau\circ\rho$

- $\circ$*3

1 $\sigma\nsubseteq\tau$

- $\sigma\circ\rho\subseteq\tau\circ\rho$

- $\forall x.\ x\in\sigma\circ\rho\rightarrow x\in\tau\circ\rho$

(w)

- $w\in\sigma\circ\rho\rightarrow w\in\tau\circ\rho$

- $w\notin\sigma\circ\rho$

2 $w\in\tau\circ\rho$

(r, s, t) - $\neg\exists xyz.\ w=\langle xz\rangle\wedge\langle xy\rangle\in\rho\wedge\langle yz\rangle\in\sigma$

- $\neg.\ w=\langle rt\rangle\wedge\langle rs\rangle\in\rho\wedge\langle st\rangle\in\sigma$

3 $w\neq\langle rt\rangle$

4 $\langle rs\rangle\notin\rho$

5 $\langle st\rangle\notin\sigma$

(1)

- $\neg\forall x.\ x\in\sigma\rightarrow x\in\tau$

- $\neg.\ \langle st\rangle\in\sigma\rightarrow\langle st\rangle\in\tau$

$\underset{(5)}{\langle st\rangle\in\sigma}$

6 $\langle st\rangle\notin\tau$

(2)

- $\exists xyz.\ w=\langle xz\rangle\wedge\langle xy\rangle\in\rho\wedge\langle yz\rangle\in\tau$

- $w=\langle rt\rangle\wedge\langle rs\rangle\in\rho\wedge\langle st\rangle\in\tau$

$\underset{(3)}{w=\langle rt\rangle}$ $\underset{(4)}{\langle rs\rangle\in\rho}$ $\underset{(6)}{\langle st\rangle\in\tau}$

$\circ$*4 $\sigma=\tau\rightarrow\sigma\circ\rho=\tau\circ\rho$ (From $\circ$*3 and $=$*6, 7)

$\circ$*5 $\sigma=\tau\rightarrow\rho\circ\sigma=\rho\circ\tau$ (Similar to $\circ$*4)

$\circ$*6 $\rho\circ(\tau\circ\sigma)\subseteq(\rho\circ\tau)\circ\sigma$

- $\circ$*6

- $a\notin\rho\circ(\tau\circ\sigma)$

1 $a\in(\rho\circ\tau)\circ\sigma$

(r, t, u) - $\neg\exists xyz.\ a=\langle xz\rangle\wedge\langle xy\rangle\in\tau\circ\sigma\wedge\langle yz\rangle\in\rho$

- $\neg.\ a=\langle ru\rangle\wedge\langle rt\rangle\in\tau\circ\sigma\wedge\langle tu\rangle\in\rho$

2 $a\neq\langle ru\rangle$

- $\langle rt\rangle\notin\tau\circ\sigma$

3 $\langle tu\rangle\notin\rho$

(r, s, t) - $\neg\exists xyz.\ \langle rt\rangle=\langle xz\rangle\wedge\langle xy\rangle\in\sigma\wedge\langle yz\rangle\in\tau$

- $\neg.\ \langle rt\rangle=\langle rt\rangle\wedge\langle rs\rangle\in\sigma\wedge\langle st\rangle\in\tau$

4 $\langle rs\rangle \notin \sigma$

5 $\langle st\rangle \notin \tau$

(1)

$-\quad \exists xyz.\ a=\langle xz\rangle \wedge \langle xy\rangle \in \sigma \wedge \langle yz\rangle \in \rho\circ\tau$

$-\quad a=\langle ru\rangle \wedge \langle rs\rangle \in \sigma \wedge \langle su\rangle \in \rho\circ\tau$

$a=\langle ru\rangle$ (2) $\qquad \langle rs\rangle \in \sigma$ (4, =, I) $\qquad -\quad \langle su\rangle \in \rho\circ\tau$

$-\quad \exists xyz.\ \langle su\rangle=\langle xz\rangle \wedge \langle xy\rangle \in \tau \wedge \langle yz\rangle \in \rho$

$-\quad \langle su\rangle=\langle su\rangle \wedge \langle st\rangle \in \tau \wedge \langle tu\rangle \in \rho$

$\langle st\rangle \in \tau$ (5, =, I) $\qquad \langle tu\rangle \in \rho$ (3)

∘*7 $\quad (\rho\circ\tau)\circ\sigma \subseteq \rho\circ(\tau\circ\sigma)$ $\qquad$ (Similar to ∘*6)

∘*8 $\quad (\rho\circ\tau)\circ\sigma = \rho\circ(\tau\circ\sigma)$ $\qquad$ (From ∘*6, ∘*7 and =*7)

∘*9 $\quad (\tau\circ\sigma)^{-1}=\sigma^{-1}\circ\tau^{-1}$

$-\quad$ ∘*9

1 $a\in(\tau\circ\sigma)^{-1}$

$-\quad a\notin\sigma^{-1}\circ\tau^{-1}$

(t, s, r) $-\quad \neg\exists xyz.\ a=\langle xz\rangle \wedge \langle xy\rangle \in \tau^{-1} \wedge \langle yz\rangle \in \sigma^{-1}$

$-\quad \neg.\ a=\langle tr\rangle \wedge \langle ts\rangle \in \tau^{-1} \wedge \langle sr\rangle \in \sigma^{-1}$

2 $a\neq\langle tr\rangle$

$-\quad \langle ts\rangle \notin \tau^{-1}$

$-\quad \langle sr\rangle \notin \sigma^{-1}$

(t, s) $-\quad \neg\exists xy.\ \langle ts\rangle=\langle xy\rangle \wedge \langle yx\rangle \in \tau$

(s, r) $-\quad \neg\exists xy.\ \langle sr\rangle=\langle xy\rangle \wedge \langle yx\rangle \in \sigma$

$-\quad \neg.\ \langle ts\rangle=\langle ts\rangle \wedge \langle st\rangle \in \tau$

$-\quad \neg.\ \langle sr\rangle=\langle sr\rangle \wedge \langle rs\rangle \in \sigma$

3 $\langle st\rangle \notin \tau$

4 $\langle rs\rangle \notin \sigma$

(1)

$-\quad \exists xy.\ a=\langle xy\rangle \wedge \langle yx\rangle \in \tau\circ\sigma$

$-\quad a=\langle tr\rangle \wedge \langle rt\rangle \in \tau\circ\sigma$

$a=\langle tr\rangle$ (2) $\qquad -\quad \langle rt\rangle \in \tau\circ\sigma$

$-\quad \exists xyz.\ \langle rt\rangle=\langle xz\rangle \wedge \langle xy\rangle \in \sigma \wedge \langle yz\rangle \in \tau$

$-\quad \langle rt\rangle=\langle rt\rangle \wedge \langle rs\rangle \in \sigma \wedge \langle st\rangle \in \tau$

$\langle rs\rangle \in \sigma$ (4, =, I) $\qquad \langle st\rangle \in \tau$ (3, =, I)

1 $a\in\sigma^{-1}\circ\tau^{-1}$

$a\notin(\tau\circ\sigma)^{-1}$

(r, t) $-\quad \neg\exists xy.\ a=\langle xy\rangle \wedge \langle yx\rangle \in \tau\circ\sigma$

$-\quad \neg.\ a=\langle tr\rangle \wedge \langle rt\rangle \in \tau\circ\sigma$

2 $a\neq\langle tr\rangle$

$-\quad \langle rt\rangle \notin \tau\circ\sigma$

(r, s, t) $-\quad \neg\exists xyz.\ \langle rt\rangle=\langle xz\rangle \wedge \langle xy\rangle \in \sigma \wedge \langle yz\rangle \in \tau$

$-\quad \neg.\ \langle rt\rangle=\langle rt\rangle \wedge \langle rs\rangle \in \sigma \wedge \langle st\rangle \in \tau$

3 $\langle rs\rangle \notin \sigma$

4 $\langle st\rangle \notin \tau$

(1)

$-\quad \exists xyz.\ a=\langle xz\rangle \wedge \langle xy\rangle \in \tau^{-1} \wedge \langle yz\rangle \in \sigma^{-1}$

$-\quad a=\langle tr\rangle \wedge \langle ts\rangle \in \tau^{-1} \wedge \langle sr\rangle \in \sigma^{-1}$

$a=\langle tr\rangle$ (2) $\qquad -\quad \langle ts\rangle \in \tau^{-1}$ $\qquad \langle sr\rangle \in \sigma^{-1}$

$-\quad \exists xy.\ \langle ts\rangle=\langle xy\rangle \wedge \langle yx\rangle \in \tau$

$-\quad \langle ts\rangle=\langle ts\rangle \wedge \langle st\rangle \in \tau$

$\langle st\rangle \in \tau$ (4, =, I)

Similar to the left (3, =, I)

∘*10 $\quad \sigma\in\mathrm{Un}\wedge\tau\in\mathrm{Un}\rightarrow\tau\circ\sigma\in\mathrm{Un}$

$-\quad$ ∘*10

1 $\sigma\notin\mathrm{Un}$

2 $\tau\notin\mathrm{Un}$

$-\quad \tau\circ\sigma\in\mathrm{Un}$

(r, t, t') – $\forall xyz.\ \langle xy\rangle\in\tau\circ\sigma\wedge\langle xz\rangle\in\tau\circ\sigma\rightarrow y=z$

– $\langle rt\rangle\in\tau\circ\sigma\wedge\langle rt'\rangle\in\tau\circ\sigma\rightarrow t=t'$

– $\langle rt\rangle\notin\tau\circ\sigma$

– $\langle rt'\rangle\notin\tau\circ\sigma$

3 $t=t'$

(r, s_1, t) – $\neg\exists xyz.\ \langle rt\rangle=\langle xz\rangle\wedge\langle xy\rangle\in\sigma\wedge\langle yz\rangle\in\tau$

(r, s_2, t') – $\neg\exists xyz.\ \langle rt'\rangle=\langle xz\rangle\wedge\langle xy\rangle\in\sigma\wedge\langle yz\rangle\in\tau$

– $\neg.\ \langle rt\rangle=\langle rt\rangle\wedge\langle rs_1\rangle\in\sigma\wedge\langle s_1t\rangle\in\tau$

– $\neg.\ \langle rt'\rangle=\langle rt'\rangle\wedge\langle rs_2\rangle\in\sigma\wedge\langle s_2t'\rangle\in\tau$

---

4 $\langle rs_1\rangle\notin\sigma$

5 $\langle s_1t\rangle\notin\tau$

6 $\langle rs_2\rangle\notin\sigma$

7 $\langle s_2t'\rangle\notin\tau$

(1, 2)

– $\neg\forall xyz.\ \langle xy\rangle\in\sigma\wedge\langle xz\rangle\in\sigma\rightarrow y=z$

– $\neg\forall xyz.\ \langle xy\rangle\in\tau\wedge\langle xz\rangle\in\tau\rightarrow y=z$

– $\neg.\ \langle rs_1\rangle\in\sigma\wedge\langle rs_2\rangle\in\sigma\rightarrow s_1=s_2$

8 $\neg.\ \langle s_1t\rangle\in\tau\wedge\langle s_1t'\rangle\in\tau\rightarrow t=t'$

$\langle rs_1\rangle\in\sigma$ (4, =, I) $\langle rs_2\rangle\in\sigma$ (6, =, I) 9 $s_1\neq s_2$

(8)

$\langle s_1t\rangle\in\tau$ (5, =, I) $\langle s_1t'\rangle\in\tau$ (7, 9, =. 1) $t\neq t'$ (3)

○*11 $\sigma\in\mathrm{Un}_2\wedge\tau\in\mathrm{Un}_2\rightarrow\tau\circ\sigma\in\mathrm{Un}_2$

– ○*11

– $\sigma\notin\mathrm{Un}_2$

– $\tau\notin\mathrm{Un}_2$

1 $\tau\circ\sigma\in\mathrm{Un}_2$

---

2 $\sigma\notin\mathrm{Un}$ 4 $\tau\notin\mathrm{Un}$

3 $\sigma^{-1}\notin\mathrm{Un}$ 5 $\tau^{-1}\notin\mathrm{Un}$

(1)

$\tau\circ\sigma\in\mathrm{Un}$ (2, 4) 6 $(\tau\circ\sigma)^{-1}\in\mathrm{Un}$ Cut ○*10

Cut ○*10 –$\neg.\ \tau^{-1}\in\mathrm{Un}\wedge\sigma^{-1}\in\mathrm{Un}\rightarrow\sigma^{-1}\circ\tau^{-1}\in\mathrm{Un}$

$\tau^{-1}\in\mathrm{Un}$ (5) $\sigma^{-1}\in\mathrm{Un}$ (3) 7 $\sigma^{-1}\circ\tau^{-1}\notin\mathrm{Un}$ Cut ○*9

$(\tau\circ\sigma)^{-1}\neq\sigma^{-1}\circ\tau^{-1}$ (6, 7, =, I)

○*12 $\mathrm{W}_\sigma\subseteq\mathrm{D}_\tau\rightarrow\mathrm{D}_{\tau\circ\sigma}=\mathrm{D}_\sigma$

– ○*12

– $\mathrm{W}_\sigma\not\subseteq\mathrm{D}_\tau$

– $\mathrm{D}_{\tau\circ\sigma}=\mathrm{D}_\sigma$

---

1 $\neg\forall x.\ x\in\mathrm{W}_\sigma\rightarrow x\in\mathrm{D}_\tau$

(r) – $\forall x.\ x\in\mathrm{D}_{\tau\circ\sigma}\equiv x\in\mathrm{D}_\sigma$

– $r\in\mathrm{D}_{\tau\circ\sigma}\equiv r\in\mathrm{D}_\sigma$

$$
\begin{array}{lll}
\hline
 & 2 & r\in \mathrm{D}_{\tau\circ\sigma} \\
 & - & r\notin \mathrm{D}_{\sigma} \\
\hline
(s) & - & \neg\exists x.\ \langle rx\rangle\in\sigma \\
\hline
 & 3 & \langle rs\rangle\notin\sigma \\
\hline
(1) & - & \neg.\ s\in \mathrm{W}_{\sigma}\rightarrow s\in \mathrm{D}_{\tau} \\
\hline
\end{array}
$$

$$
\begin{array}{ll}
- & s\in \mathrm{W}_{\sigma} \\
\hline
- & \exists x.\ \langle xs\rangle\in\sigma \\
\hline
 & \underset{(3)}{\langle rs\rangle\in\sigma}
\end{array}
\qquad
\begin{array}{lll}
 & - & s\notin \mathrm{D}_{\tau} \\
\hline
(t) & - & \neg\exists x.\ \langle sx\rangle\in\tau \\
\hline
 & 4 & \langle st\rangle\notin\tau \\
\hline
(2) & - & \exists x.\ \langle rx\rangle\in\tau\circ\sigma \\
\hline
 & - & \langle rt\rangle\in\tau\circ\sigma
\end{array}
$$

$$
\begin{array}{ll}
\hline
- & \exists xyz.\ \langle rt\rangle=\langle xz\rangle\wedge\langle xy\rangle\in\sigma\wedge\langle yz\rangle\in\tau \\
\hline
- & \langle rt\rangle=\langle rt\rangle\wedge\langle rs\rangle\in\sigma\wedge\langle st\rangle\in\tau \\
\hline
 & \underset{(3)}{\langle rs\rangle\in\sigma}\qquad\qquad \underset{(4)}{\langle st\rangle\in\tau}
\end{array}
$$

$$
\begin{array}{lll}
\hline
 & 2 & r\in \mathrm{D}_{\sigma} \\
 & - & r\notin \mathrm{D}_{\tau\circ\sigma} \\
\hline
(t) & - & \neg\exists x.\ \langle rx\rangle\in\tau\circ\sigma \\
\hline
 & - & \langle rt\rangle\notin\tau\circ\sigma \\
\hline
(r,s,t) & - & \neg\exists xyz.\ \langle rt\rangle=\langle xz\rangle\wedge\langle xy\rangle\in\sigma\wedge\langle yz\rangle\in\tau \\
\hline
 & - & \neg.\ \langle rt\rangle=\langle rt\rangle\wedge\langle rs\rangle\in\sigma\wedge\langle st\rangle\in\tau \\
\hline
 & 3 & \langle rs\rangle\notin\sigma \\
 & \text{Spf.} & \langle st\rangle\notin\tau \\
\hline
(2) & - & \exists x.\ \langle rx\rangle\in\sigma \\
\hline
 & & \underset{(3,\ =,\ \mathrm{I})}{\langle rs\rangle\in\sigma}
\end{array}
$$

∘*13 $\mathrm{D}_{\tau}\subseteq \mathrm{W}_{\sigma}\rightarrow \mathrm{W}_{\tau\circ\sigma}=\mathrm{W}_{\tau}$ (Similar to ∘*12)

∘*14 $\mathrm{W}_{\sigma}=\mathrm{D}_{\tau}\rightarrow \mathrm{D}_{\tau\circ\sigma}=\mathrm{D}_{\sigma}\wedge \mathrm{W}_{\tau\circ\sigma}=\mathrm{W}_{\tau}$ (From ∘*12, ∘*13 and =*6)

∘*15 $\sigma\in \mathrm{Map}_2^{a,b}\wedge\tau\in \mathrm{Map}_2^{b,c}\rightarrow\tau\circ\sigma\in \mathrm{Map}_2^{a,c}$

$$
\begin{array}{llllllll}
 & - & \multicolumn{6}{c}{\circ *15} \\
\hline
 & - & \multicolumn{6}{c}{\sigma\notin \mathrm{Map}_2^{a,b}} \\
 & - & \multicolumn{6}{c}{\tau\notin \mathrm{Map}_2^{b,c}} \\
 & - & \multicolumn{6}{c}{\tau\circ\sigma\in \mathrm{Map}_2^{a,c}} \\
\hline
 & - & \multicolumn{6}{l}{\neg.\ \sigma\in \mathrm{Un}_2\wedge \mathrm{D}_{\sigma}=a\wedge \mathrm{W}_{\sigma}=b} \\
 & - & \multicolumn{6}{l}{\neg.\ \tau\in \mathrm{Un}_2\wedge \mathrm{D}_{\tau}=b\wedge \mathrm{W}_{\tau}=c} \\
 & 1 & \multicolumn{6}{l}{\tau\circ\sigma\in \mathrm{Un}_2\wedge \mathrm{D}_{\tau\circ\sigma}=a\wedge \mathrm{W}_{\tau\circ\sigma}=c} \\
\hline
 & 2 & \sigma\notin \mathrm{Un}_2 & 4 & \mathrm{D}_{\sigma}\neq a & 6 & \mathrm{W}_{\sigma}\neq b & \\
 & 3 & \tau\notin \mathrm{Un}_2 & 5 & \mathrm{D}_{\tau}\neq b & 7 & \mathrm{W}_{\tau}\neq c & \text{Cut }\circ *14 \\
\hline
 & - & \multicolumn{6}{l}{\neg.\ \mathrm{W}_{\sigma}=\mathrm{D}_{\tau}\rightarrow \mathrm{D}_{\tau\circ\sigma}=\mathrm{D}_{\sigma}\wedge \mathrm{W}_{\tau\circ\sigma}=\mathrm{W}_{\tau}} \\
\hline
 & & \multicolumn{2}{l}{\underset{(5,\,6,\,=)}{\mathrm{W}_{\sigma}=\mathrm{D}_{\tau}}} & 8 & \mathrm{D}_{\tau\circ\sigma}\neq \mathrm{D}_{\sigma} & & \\
 & & & & 9 & \mathrm{W}_{\tau\circ\sigma}\neq \mathrm{W}_{\tau} & & \\
\hline
(1) & 10 & \multicolumn{3}{l}{\tau\circ\sigma\in \mathrm{Un}_2\quad \text{Cut }\circ *11} & \underset{(4,\,8,\,=)}{\mathrm{D}_{\tau\circ\sigma}=a} & \underset{(7,\,9,\,=)}{\mathrm{W}_{\tau\circ\sigma}=c} & \\
\hline
 & - & \multicolumn{3}{l}{\neg.\ \sigma\in \mathrm{Un}_2\wedge\tau\in \mathrm{Un}_2\rightarrow\tau\circ\sigma\in \mathrm{Un}_2} & & & \\
\hline
 & & \underset{(2)}{\sigma\in \mathrm{Un}_2} & \underset{(3)}{\tau\in \mathrm{Un}_2} & \underset{(10)}{\tau\circ\sigma\notin \mathrm{Un}_2} & & &
\end{array}
$$

Restriction of Operator ($\upharpoonright$)

*↾*1 $\sigma\upharpoonright a\subseteq\sigma$

$$
\begin{array}{lll}
 & - & *\upharpoonright *1 \\
\hline
(w) & - & \forall x.\ x\in\sigma\upharpoonright a\rightarrow x\in\sigma
\end{array}
$$

$-\quad w\notin\sigma\upharpoonright a$

$1\quad w\in\sigma$

$(r,s)\ -\quad \neg\exists xy.\ w=\langle xy\rangle\wedge w\in\sigma\wedge x\in a$

*Spf. $\quad w\neq\langle rs\rangle$

$w\notin\sigma$

*Spf. $\quad \underset{(1)}{r\notin a}$

$\upharpoonright *2\qquad a\subseteq b\rightarrow\sigma\upharpoonright a\subseteq\sigma\upharpoonright b$

$-\quad \upharpoonright *2$

$-\quad a\nsubseteq b$

$-\quad \sigma\upharpoonright a\subseteq\sigma\upharpoonright b$

$1\quad \neg\forall x.\ x\in a\rightarrow x\in b$

$(w)\ -\quad \forall x.\ x\in\sigma\upharpoonright a\rightarrow x\in\sigma\upharpoonright b$

$-\quad w\notin\sigma\upharpoonright a$

$-\quad w\in\sigma\upharpoonright b$

$(r,s)\ -\quad \neg\exists xy.\ w=\langle xy\rangle\wedge w\in\sigma\wedge x\in a$

$2\quad \exists xy.\ w=\langle xy\rangle\wedge w\in\sigma\wedge x\in b$

$3\quad w\neq\langle rs\rangle$

$4\quad w\notin\sigma$

$(2)\ 5\quad r\notin a$

$\underset{(3)}{w=\langle rs\rangle}\qquad \underset{(4)}{w\in\sigma}\qquad (1)\ 6\ \dfrac{r\in b}{\underset{(5)}{r\in a}\qquad\underset{(6)}{r\notin b}}$

$\upharpoonright *3\qquad \langle pq\rangle\in\sigma\wedge p\in a\rightarrow\langle pq\rangle\in\sigma\upharpoonright a$

$*\upharpoonright *4\qquad \sigma\in\mathrm{Un}\rightarrow\sigma\upharpoonright a\in\mathrm{Un}$

$-\quad *\upharpoonright *4$

$1\quad \sigma\notin\mathrm{Un}$

$2\quad \sigma\upharpoonright a\in\mathrm{Un}$ Cut Im$*1$

$-\quad \neg.\ \sigma\in\mathrm{Un}\wedge\sigma\upharpoonright a\subseteq\sigma\rightarrow\sigma\upharpoonright a\in\mathrm{Un}$

$\underset{(1)}{\sigma\in\mathrm{Un}}\qquad \underset{*\upharpoonright *1}{\sigma\upharpoonright a\subseteq a}\qquad \underset{(2)}{\sigma\upharpoonright a\notin\mathrm{Un}}$

$*\upharpoonright *5\qquad \sigma^{-1}\in\mathrm{Un}\rightarrow(\sigma\upharpoonright a)^{-1}\in\mathrm{Un}$

$-\quad *\upharpoonright *5$

$-\quad \sigma^{-1}\notin\mathrm{Un}$

$-\quad (\sigma\upharpoonright a)^{-1}\in\mathrm{Un}$

$1\quad \neg\forall xyz.\ \langle xy\rangle\in\sigma^{-1}\wedge\langle xz\rangle\in\sigma^{-1}\rightarrow y=z$

$(r,s,t)\ -\quad \forall xyz.\ \langle xy\rangle\in(\sigma\upharpoonright a)^{-1}\wedge\langle xz\rangle\in(\sigma\upharpoonright a)^{-1}\rightarrow y=z$

$-\quad \langle rs\rangle\in(\sigma\upharpoonright a)^{-1}\wedge\langle rt\rangle\in(\sigma\upharpoonright a)^{-1}\rightarrow s=t$

$-\quad \langle rs\rangle\notin\langle\sigma\upharpoonright a)^{-1}$

$-\quad \langle rt\rangle\notin(\sigma\upharpoonright a)^{-1}$

$2\quad s=t$

(r, s) $-\quad \neg\exists xy.\ \langle rs\rangle=\langle xy\rangle\wedge\langle yx\rangle\in\sigma\upharpoonright a$

(r, t) $-\quad \neg\exists xy.\ \langle rt\rangle=\langle xy\rangle\wedge\langle yx\rangle\in\sigma\upharpoonright a$

$-\quad \langle sr\rangle\notin\sigma\upharpoonright a$

$-\quad \langle tr\rangle\notin\sigma\upharpoonright a$

$-\quad \neg.\ \langle sr\rangle\in\sigma\wedge s\in a$

$-\quad \neg.\ \langle tr\rangle\in\sigma\wedge t\in a$

3 $\langle sr\rangle\notin\sigma$

4 $\langle tr\rangle\notin\sigma$

*Spf. $s\notin a$

*Spf. $t\notin a$

(1)

$-\quad \neg.\ \langle rs\rangle\in\sigma^{-1}\wedge\langle rt\rangle\in\sigma^{-1}\to s=t$

$-\quad \langle rs\rangle\in\sigma^{-1}$	$-\quad \langle rt\rangle\in\sigma^{-1}$	$s\neq t$ (2)
$\langle sr\rangle\in\sigma$ (3, =, I)	$\langle tr\rangle\in\sigma$ (4, =, I)	

$*\upharpoonright*6$ $\quad \sigma\in\mathrm{Un}_2\to\sigma\upharpoonright a\in\mathrm{Un}_2$

$-\quad *\upharpoonright*6$

$-\quad \sigma\notin\mathrm{Un}_2$

1 $\sigma\upharpoonright a\in\mathrm{Un}_2$

$-\quad \neg.\ \sigma\in\mathrm{Un}\wedge\sigma^{-1}\in\mathrm{Un}$

2 $\sigma\notin\mathrm{Un}$

3 $\sigma^{-1}\notin\mathrm{Un}$

(1)

$-\quad \sigma\upharpoonright a\in\mathrm{Un}\wedge(\sigma\upharpoonright a)^{-1}\in\mathrm{Un}$

4 $\sigma\upharpoonright a\in\mathrm{Un}$ Cut $*\upharpoonright*4$	4 $(\sigma\upharpoonright a)^{-1}\in\mathrm{Un}$ Cut $*\upharpoonright*5$
$-\quad \neg.\ \sigma\in\mathrm{Un}\to\sigma\upharpoonright a\in\mathrm{Un}$	$-\quad \neg.\ \sigma^{-1}\in\mathrm{Un}\to(\sigma\upharpoonright a)^{-1}\in\mathrm{Un}$
$\sigma\in\mathrm{Un}$ (2) $\qquad \sigma\upharpoonright a\notin\mathrm{Un}$ (4)	$\sigma^{-1}\in\mathrm{Un}$ (3) $\qquad (\sigma\upharpoonright a)^{-1}\notin\mathrm{Un}$ (4)

$\upharpoonright*7$ $\quad b\subseteq a\to(\sigma\upharpoonright a)\upharpoonright b=\sigma\upharpoonright b$

$-\quad \upharpoonright*7$

1 $\neg\forall x.\ x\in b\to x\in a$

$-\quad (\sigma\upharpoonright a)\upharpoonright b=\sigma\upharpoonright b$

$-\quad \forall x.\ x\in(\sigma\upharpoonright a)\upharpoonright b\equiv x\in\sigma\upharpoonright b$

(w)

$-\quad w\in(\sigma\upharpoonright a)\upharpoonright b\equiv w\in\sigma\upharpoonright b$

$-\quad w\in(\sigma\upharpoonright a)\upharpoonright b$	$-\quad w\in\sigma\upharpoonright b$
$-\quad w\notin\sigma\upharpoonright b$	$-\quad w\notin(\sigma\upharpoonright a)\upharpoonright b$
2 $\exists xy.\ w=\langle xy\rangle\wedge w\in\sigma\upharpoonright a\wedge x\in b$	2 $\exists xy.\ w=\langle xy\rangle\wedge w\in\sigma\wedge x\in b$
$-\neg\exists xy.\ w=\langle xy\rangle\wedge w\in\sigma\wedge x\in b$	$-\neg\exists xy.\ w=\langle xy\rangle\wedge w\in\sigma\upharpoonright a\wedge x\in b$
(r, s)	(r, s)
$-\quad \neg.\ w=\langle rs\rangle\wedge w\in\sigma\wedge r\in b$	$-\quad \neg.\ w=\langle rs\rangle\wedge w\in\sigma\upharpoonright a\wedge r\in b$
3 $w\neq\langle rs\rangle$	3 $w\neq\langle rs\rangle$
4 $w\notin\sigma$	$-\quad w\notin\sigma\upharpoonright a$
5 $r\notin b$	4 $r\notin b$

(2)
$-\quad w=\langle rs\rangle\wedge w\in\sigma\restriction a\wedge r\in b$

$\underset{(3)}{w=\langle rs\rangle}\quad -\quad w\in\sigma\restriction a\quad \underset{(5)}{r\in b}$

$-\quad \exists xy.\ w=\langle xy\rangle\wedge w\in\sigma\wedge x\in a$

$-\quad w=\langle rs\rangle\wedge w\in\sigma\wedge r\in a$

$\underset{(3)}{w=\langle rs\rangle}\quad \underset{(4)}{w\in\sigma}\quad {}^{6}\ r\in a$

(1) $-\neg.\ r\in b\rightarrow r\in a$

$\underset{(5)}{r\in b}\quad \underset{(6)}{r\notin a}$

$-\quad \neg\exists xy.\ w=\langle xy\rangle\wedge w\in\sigma\wedge x\in a$

(r_1,s_1) $-\quad \neg.\ w=\langle r_1s_1\rangle\wedge w\in\sigma\wedge r_1\in a$

Spf. $\quad w\neq\langle r_1s_1\rangle$

5 $\quad w\notin\sigma$

Spf. $\quad r_1\notin a$

(2) $-\quad w=\langle rs\rangle\wedge w\in\sigma\wedge r\in b$

$\underset{(3)}{w=\langle rs\rangle}\quad \underset{(5)}{w\in\sigma}\quad \underset{(4)}{r\in b}$

$\vdash*8\qquad p\in a\rightarrow p^{\sigma\restriction a}=p^{\sigma}$

$-\quad \vdash*8$

1 $\quad p\notin a$

$-\quad p^{\sigma\restriction a}=p^{\sigma}$

(w) $-\quad \forall x.\ x\in p^{\sigma\restriction a}\equiv x\in p^{\sigma}$

$-\quad w\in p^{\sigma\restriction a}\equiv w\in p^{\sigma}$

$-\quad w\in p^{\sigma\restriction a}$

$-\quad w\notin p^{\sigma}$

2 $\quad \exists x.\ \langle px\rangle\in\sigma\restriction a\wedge w\in x$

(q) $-\quad \neg\exists x.\ \langle px\rangle\in\sigma\wedge w\in x$

$-\quad \neg.\ \langle pq\rangle\in\sigma\wedge w\in q$

3 $\quad \langle pq\rangle\notin\sigma$

4 $\quad w\notin q$

(2) $-\quad \langle pq\rangle\in\sigma\restriction a\wedge w\in q$

$-\quad \langle pq\rangle\in\sigma\restriction a\qquad \underset{(4)}{w\in q}$

$-\quad \langle pq\rangle\in\sigma\wedge p\in a$

$\underset{(3)}{\langle pq\rangle\in\sigma}\quad \underset{(1)}{p\in a}$

$-\quad w\in p^{\sigma}$

$-\quad w\notin p^{\sigma\restriction a}$

2 $\quad \exists x.\ \langle px\rangle\in\sigma\wedge w\in x$

(q) $-\quad \neg\exists x.\ \langle px\rangle\in\sigma\restriction a\wedge w\in x$

$-\quad \neg.\ \langle pq\rangle\in\sigma\restriction a\wedge w\in q$

$-\quad \langle pq\rangle\notin\sigma\restriction a$

3 $\quad w\notin q$

$-\quad \langle pq\rangle\in\sigma\wedge p\in a$

4 $\quad \langle pq\rangle\in\sigma\qquad \underset{(1)}{p\in a}$

(2) $-\quad \langle pq\rangle\in\sigma\wedge w\in q$

$\underset{(4)}{\langle pq\rangle\in\sigma}\quad \underset{(3)}{w\in q}$

$\vdash*9\qquad b\subseteq a\rightarrow b^{\sigma\restriction a}=b^{\sigma}$

$-\quad \vdash*9$

1 $\quad b\nsubseteq a$

$-\quad b^{\sigma\restriction a}=b^{\sigma}$

$-\quad \forall x.\ x\in b^{\sigma\restriction a}\equiv x\in b^{\sigma}$

$-\quad s\in b^{\sigma\restriction a}\equiv s\in b^{\sigma}$

$-\quad s\in b^{\sigma\restriction a}$

$-\quad s\notin b^{\sigma}$

2 $\quad \exists x.\ x\in b\wedge\langle xs\rangle\in\sigma\restriction a$

(r) $-\quad \neg\exists x.\ x\in b\wedge\langle xs\rangle\in\sigma$

$-\quad \neg.\ r\in b\wedge\langle rs\rangle\in\sigma$

$-\quad s\in b^{\sigma}$

$-\quad s\notin b^{\sigma\restriction a}$

2 $\quad \exists x.\ x\in b\wedge\langle xs\rangle\in\sigma$

(r) $-\quad \neg\exists x.\ x\in b\wedge\langle xs\rangle\in\sigma\restriction a$

$-\quad \neg.\ r\in b\wedge\langle rs\rangle\in\sigma\restriction a$

3 $r \notin b$

4 $\langle rs \rangle \notin \sigma$

(2)

$-$ $r \in b \wedge \langle rs \rangle \in \sigma \restriction a$

$r \in b$ (3) $-$ $\langle rs \rangle \in \sigma \restriction a$

$-$ $\langle rs \rangle \in \sigma \wedge r \in a$

$\langle rs \rangle \in \sigma$ (1) 5 $r \in a$

(1) $-$ $\neg \forall x.\ x \in b \rightarrow x \in a$

$-$ $\neg.\ r \in b \rightarrow r \in a$

$r \in b$ (3) $r \notin a$ (5)

3 $r \notin b$

$-$ $\langle rs \rangle \notin \sigma \restriction a$

$-$ $\neg.\ \langle rs \rangle \in \sigma \wedge r \in a$

4 $\langle rs \rangle \notin \sigma$

(2) Spf. $r \notin a$

$-$ $r \in b \wedge \langle rs \rangle \in \sigma$

$r \in b$ (3) $\langle rs \rangle \in \sigma$ (4)

$\restriction * 10$ $\sigma \in \mathrm{Un} \wedge a \subseteq \mathrm{D}_\sigma \rightarrow \mathrm{D}_{\sigma \restriction a} = a$

$-$ $\restriction * 10$

1 $\sigma \notin \mathrm{Un}$

2 $a \nsubseteq \mathrm{D}_\sigma$

$-$ $\mathrm{D}_{\sigma \restriction a} = a$

(r) $-$ $\forall x.\ x \in \mathrm{D}_{\sigma \restriction a} \equiv x \in a$

$-$ $r \in \mathrm{D}_{\sigma \restriction a} \equiv r \in a$

$-$ $r \in \mathrm{D}_{\sigma \restriction a}$

3 $r \notin a$

$-$ $\exists x.\ \langle rx \rangle \in \sigma \restriction a$

$-$ $\langle r r^\sigma \rangle \in \sigma \restriction a$

$-$ $\langle r r^\sigma \rangle \in \sigma \wedge r \in a$

4 $\langle r r^\sigma \rangle \in \sigma$ $r \in a$ (3)

(2) $-$ $\neg \forall x.\ x \in a \rightarrow x \in \mathrm{D}_\sigma$

$-$ $\neg.\ r \in a \rightarrow r \in \mathrm{D}_\sigma$

$r \in a$ (3) 5 $r \notin \mathrm{D}_\sigma$ Cut Im$*3$

$-$ $\neg.\ \sigma \in \mathrm{Un} \wedge r \in \mathrm{D}_\sigma \rightarrow \langle r r^\sigma \rangle \in \sigma$

$\sigma \in \mathrm{Un}$ (1) $r \in \mathrm{D}_\sigma$ (5) $\langle r r^\sigma \rangle \notin \sigma$ (4)

3 $r \in a$

$-$ $r \notin \mathrm{D}_{\sigma \restriction a}$

(s) $-$ $\neg \exists x.\ \langle rx \rangle \in \sigma \restriction a$

$-$ $\langle rs \rangle \notin \sigma \restriction a$

$-$ $\neg.\ \langle rs \rangle \in \sigma \wedge r \in a$

Spf. $\langle rs \rangle \notin \sigma$

$r \notin a$ (3)

$\restriction * 11$ $\sigma \in \mathrm{Un} \wedge a \subseteq \mathrm{D}_\sigma \rightarrow \mathrm{W}_{\sigma \restriction a} = a^\sigma$

$-$ $\restriction * 11$

1 $\sigma \notin \mathrm{Un}$

2 $a \nsubseteq \mathrm{D}_\sigma$

3 $\mathrm{W}_{\sigma \restriction a} = a^\sigma$ Cut Im$*9$

4 $\mathrm{W}_{\sigma \restriction a} \neq (\mathrm{D}_{\sigma \restriction a})^{\sigma \restriction a}$ Cut $\restriction * 10$

$-$ $\neg.\ \sigma \in \mathrm{Un} \wedge a \subseteq \mathrm{D}_\sigma \rightarrow \mathrm{D}_{\sigma \restriction a} = a$

$\sigma \in \mathrm{Un}$ (1) $a \subseteq \mathrm{D}_\sigma$ (2) 5 $\mathrm{D}_{\sigma \restriction a} \neq a$ Cut $\restriction * 9$

$-$ $\neg.\ a \subseteq a \rightarrow a^{\sigma \restriction a} = a^\sigma$

$a \subseteq a$ $= * 1$ $a^{\sigma \restriction a} \neq a^\sigma$ (3, 4, 5, =, I)

$\upharpoonright *12$ $\quad \sigma\in\mathrm{Un}_2\wedge a\subseteq\mathrm{D}_\sigma\rightarrow\sigma\upharpoonright a\in\mathrm{Map}_2^{a,\,a^\sigma}$

$$\begin{array}{ll}
- & \upharpoonright *12 \\ \hline
1 & \sigma\notin\mathrm{Un}_2 \\
2 & a\not\subseteq\mathrm{D}_\sigma \\
- & \sigma\upharpoonright a\in\mathrm{Map}_2^{a,\,a^\sigma} \\ \hline
- & \sigma\upharpoonright a\in\mathrm{Un}_2\wedge\mathrm{D}_{\sigma\upharpoonright a}=a\wedge\mathrm{W}_{\sigma\upharpoonright a}=a^\sigma \\ \hline
\end{array}$$

$$\underset{(1)}{\sigma\upharpoonright a\in\mathrm{Un}_2}\qquad \underset{(1,\,2)}{\mathrm{D}_{\sigma\upharpoonright a}=a}\qquad \underset{(1,\,2)}{\mathrm{W}_{\sigma\upharpoonright a}=a^\sigma}$$

Cut $*\upharpoonright *6$ Cut $\upharpoonright *10$ Cut $\upharpoonright *11$

$\upharpoonright *13$ $\quad \sigma\in\mathrm{Un}\wedge\tau\in\mathrm{Un}\wedge a\subseteq\mathrm{D}_\sigma\wedge a\subseteq\mathrm{D}_\tau\wedge\overset{a}{\forall}x[x^\sigma=x^\tau]\rightarrow\sigma\upharpoonright a=\tau\upharpoonright a$

$$\begin{array}{ll}
- & \upharpoonright *13 \\ \hline
1 & \sigma\notin\mathrm{Un}\quad 3\ \ a\not\subseteq\mathrm{D}_\sigma\quad 5\ \ \neg\overset{a}{\forall}x.\ x^\sigma=x^\tau \\
2 & \tau\notin\mathrm{Un}\quad 4\ \ a\not\subseteq\mathrm{D}_\tau\quad 6\ \ \sigma\upharpoonright a=\tau\upharpoonright a \\ \hline
(6)\ - & \forall x.\ x\in\sigma\upharpoonright a\equiv x\in\tau\upharpoonright a \\ \hline
(w)\ - & w\in\sigma\upharpoonright a\equiv w\in\tau\upharpoonright a \\ \hline
\end{array}$$

$$\begin{array}{lll}
- & w\notin\sigma\upharpoonright a & w\notin\tau\upharpoonright a \\
- & w\in\tau\upharpoonright a & w\in\sigma\upharpoonright a \\ \hline
(r,s)\ - & \neg\exists xy.\ w=\langle xy\rangle\wedge\langle xy\rangle\in\sigma\wedge x\in a & \text{Symmetric to} \\
7 & \exists xy.\ w=\langle xy\rangle\wedge\langle xy\rangle\in\tau\wedge x\in a & \text{the left.} \\ \hline
- & \neg.\ w=\langle rs\rangle\wedge\langle rs\rangle\in\sigma\wedge r\in a & \text{(3) is used} \\ \hline
 & & \text{here.} \\
8 & w\neq\langle rs\rangle & \\
9 & \langle rs\rangle\notin\sigma & \\
10 & r\notin a & \\ \hline
(7)\ - & w=\langle rs\rangle\wedge\langle rs\rangle\in\tau\wedge r\in a & \\ \hline
\end{array}$$

$$\underset{(8)}{w=\langle rs\rangle}\qquad \begin{array}{ll} 11 & \langle rs\rangle\in\tau \\ \hline (5)\ - & \neg.\ r\in a\rightarrow r^\sigma=r^\tau \\ \hline \end{array}\qquad \underset{(10)}{r\in a}$$

$$\underset{(10)}{r\in a}\qquad \begin{array}{ll} 12 & r^\sigma\neq r^\tau \qquad\qquad \text{Cut Im}*2 \\ \hline - & \neg.\ \sigma\in\mathrm{Un}\wedge\langle rs\rangle\in\sigma\rightarrow r^\sigma=s \\ \hline \end{array}$$

$$\underset{(1)}{\sigma\in\mathrm{Un}}\qquad \underset{(9)}{\langle rs\rangle\in\sigma}\qquad \begin{array}{ll} 13 & r^\sigma\neq s\quad \text{Cut Im}*3 \\ \hline - & \neg.\ \tau\in\mathrm{Un}\wedge r\in\mathrm{D}_\tau\rightarrow\langle rr^\tau\rangle\in\tau \\ \hline \end{array}$$

$$\underset{(2)}{\tau\in\mathrm{Un}}\qquad \underset{(4,\,10)}{r\in\mathrm{D}_\tau}\qquad \underset{(11,\,12,\,13,\,=,\,\mathrm{I})}{\langle rr^\tau\rangle\notin\tau}$$

Cut $=*9$

$\upharpoonright *14$ $\quad \sigma\circ(\tau\upharpoonright a)=(\sigma\circ\tau)\upharpoonright a$

$$\begin{array}{ll}
- & \upharpoonright *14 \\ \hline
- & \forall x.\ x\in\sigma\circ(\tau\upharpoonright a)\equiv x\in(\sigma\circ\tau)\upharpoonright a \\ \hline
(w)\ - & w\in\sigma\circ(\tau\upharpoonright a)\equiv w\in(\sigma\circ\tau)\upharpoonright a \\
\end{array}$$

$$
\begin{array}{rc}
- & w\in\sigma\circ(\tau\upharpoonright a) \\
- & w\notin(\sigma\circ\tau)\upharpoonright a \\
\hdashline
1 & \exists xyz.\ w=\langle xz\rangle\land\langle xy\rangle\in\tau\upharpoonright a\land\langle yz\rangle\in\sigma \\
(r,t)\ - & \neg\exists xy.\ w=\langle xy\rangle\land\langle xy\rangle\in\sigma\circ\tau\land x\in a \\
\hline
- & \neg.\ w=\langle rt\rangle\land\langle rt\rangle\in\sigma\circ\tau\land r\in a \\
\hdashline
2 & w\neq\langle rt\rangle \\
- & \langle rt\rangle\notin\sigma\circ\tau \\
3 & r\notin a \\
\hdashline
(r,s,t)\ - & \neg\exists xyz.\ \langle rt\rangle=\langle xz\rangle\land\langle xy\rangle\in\tau\land\langle yz\rangle\in\sigma \\
\hline
4 & \langle rs\rangle\notin\tau \\
5 & \langle st\rangle\notin\sigma \\
(1) & \\
\hdashline
- & w=\langle rt\rangle\land\langle rs\rangle\in\tau\upharpoonright a\land\langle st\rangle\in\sigma \\
\hline
\end{array}
$$

$$
\begin{array}{ccc}
\underset{(2)}{w=\langle rt\rangle} &
\begin{array}{rc}
- & \langle rs\rangle\in\tau\upharpoonright a \\
\hdashline
- & \langle rs\rangle\in\tau\land r\in a \\
\hline
\multicolumn{2}{c}{\underset{(4,\,=,\,\mathrm{I})}{\langle rs\rangle\in\tau}\quad \underset{(3)}{r\in a}}
\end{array}
& \underset{(5,\,=,\,\mathrm{I})}{\langle st\rangle\in\sigma}
\end{array}
$$

$$
\begin{array}{rc}
- & w\in(\sigma\circ\tau)\upharpoonright a \\
- & w\notin\sigma\circ(\tau\upharpoonright a) \\
\hdashline
1 & \exists xy.\ w=\langle xy\rangle\land\langle xy\rangle\in\sigma\circ\tau\land x\in a \\
(r,s,t)\ - & \neg\exists xyz.\ w=\langle xz\rangle\land\langle xy\rangle\in\tau\upharpoonright a\land\langle yz\rangle\in\sigma \\
\hdashline
- & \neg.\ w=\langle rt\rangle\land\langle rs\rangle\in\tau\upharpoonright a\land\langle st\rangle\in\sigma \\
\hline
2 & w\neq\langle rt\rangle \\
- & \langle rs\rangle\notin\tau\upharpoonright a \\
3 & \langle st\rangle\notin\sigma \\
\hdashline
(r,s)\ - & \neg\exists xy.\ \langle rs\rangle=\langle xy\rangle\land\langle xy\rangle\in\tau\land x\in a \\
\hline
4 & \langle rs\rangle\notin\tau \\
5 & r\notin a \\
(1) & \\
\hdashline
- & w=\langle rt\rangle\land\langle rt\rangle\in\sigma\circ\tau\land r\in a \\
\hdashline
\end{array}
$$

$$
\begin{array}{ccc}
\underset{(2)}{w=\langle rt\rangle} &
\begin{array}{rc}
- & \langle rt\rangle\in\sigma\circ\tau \\
\hline
- & \langle rs\rangle\in\tau\land\langle st\rangle\in\sigma \\
\hline
\multicolumn{2}{c}{\underset{(4,\,=,\,\mathrm{I})}{\langle rs\rangle\in\tau}\quad \underset{(3,\,=,\,\mathrm{I})}{\langle st\rangle\in\sigma}}
\end{array}
& \underset{(5,\,=,\,\mathrm{I})}{r\in a}
\end{array}
$$

$\upharpoonright *15 \qquad \tau\in\mathrm{Un}\rightarrow(\sigma\upharpoonright a)\circ\tau=(\sigma\circ\tau)\upharpoonright a^{\tau^{-1}}$

$$
\begin{array}{rc}
- & \upharpoonright *15 \\
\hline
0 & \tau\notin\mathrm{Un} \\
- & (\sigma\upharpoonright a)\circ\tau=(\sigma\circ\tau)a^{\tau^{-1}} \\
\hline
(w)\ - & \forall x.\ x\in(\sigma\upharpoonright a)\circ\tau\equiv x\in(\sigma\circ\tau)a^{\tau^{-1}} \\
\hline
- & w\in(\sigma\upharpoonright a)\circ\tau\equiv w\in(\sigma\circ\tau)a^{\tau^{-1}} \\
\hline
\end{array}
$$

$$
\begin{array}{cc}
\begin{array}{rc}
- & w\in(\sigma\upharpoonright a)\circ\tau \\
- & w\notin(\sigma\circ\tau)\upharpoonright a^{\tau^{-1}} \\
\hdashline
\multicolumn{2}{c}{(*)}
\end{array}
&
\begin{array}{rc}
- & w\in(\sigma\circ\tau)\upharpoonright a^{\tau^{-1}} \\
- & w\notin(\sigma\upharpoonright a)\circ\tau \\
\hline
\multicolumn{2}{c}{(**)}
\end{array}
\end{array}
$$

(*)

$$
\begin{array}{rc}
1 & \exists xyz.\ w=\langle xz\rangle\land\langle xy\rangle\in\tau\land\langle yz\rangle\in\sigma\upharpoonright a \\
(r,t)\ - & \neg\exists xy.\ w=\langle xy\rangle\land\langle xy\rangle\in\sigma\circ\tau\land x\in a^{\tau^{-1}} \\
\hline
- & \neg.\ w=\langle rt\rangle\land\langle rt\rangle\in\sigma\circ\tau\land r\in a^{\tau^{-1}} \\
\hdashline
2 & w\neq\langle rt\rangle \\
- & \langle rt\rangle\notin\sigma\circ\tau \\
3 & r\notin a^{\tau^{-1}} \\
\hline
(r,s,t)\ - & \neg\exists xyz.\ \langle rt\rangle=\langle xz\rangle\land\langle xy\rangle\in\tau\land\langle yz\rangle\in\sigma \\
\hline
4 & \langle rs\rangle\notin\tau \\
5 & \langle st\rangle\notin\sigma \\
(1) & \\
\hline
- & w=\langle rt\rangle\land\langle rs\rangle\in\tau\land\langle st\rangle\in\sigma\upharpoonright a \\
\hline
\end{array}
$$

$$
\begin{array}{ccrc}
\underset{(2)}{w=\langle rt\rangle} & \underset{(4,\,=,\,\mathrm{I})}{\langle rs\rangle\in\tau} & - & \langle st\rangle\in\sigma\upharpoonright a \\
\cline{3-4}
 & & - & \exists xy.\ \langle st\rangle=\langle xy\rangle\land\langle xy\rangle\in\sigma\land x\in a
\end{array}
$$

$\langle st\rangle\in\sigma$ (5)

6 $s\in a$

(3)

$- \quad \neg\exists x.\ x\in a\wedge\langle xr\rangle\in\tau^{-1}$

(s_1)

$- \quad \neg.\ s_1\in a\wedge\langle s_1r\rangle\in\tau^{-1}$

7 $s_1\notin a$

$- \quad \langle s_1r\rangle\in\tau^{-1}$

8 $\langle rs_1\rangle\in\tau$

(0)

$- \quad \neg.\ \forall xyz.\ \langle xy\rangle\in\tau\wedge\langle xz\rangle\in\tau\rightarrow y=z$

$- \quad \neg.\ \langle rs\rangle\in\tau\wedge\langle rs_1\rangle\in\tau\rightarrow s=s_1$

$\langle rs\rangle\in\tau$ (4, =, I) $\qquad \langle rs_1\rangle\in\tau$ (8) $\qquad s\neq s_1$ (6, 7, =, I)

(**)

1 $\exists xy.\ w=\langle xy\rangle\wedge\langle xy\rangle\in\sigma\circ\tau\wedge x\in a^{\tau^{-1}}$

(r, s, t) $- \quad \neg\exists xyz.\ w=\langle xz\rangle\wedge\langle xy\rangle\in\tau\wedge\langle yz\rangle\in\sigma\upharpoonright a$

$- \quad \neg.\ w=\langle rt\rangle\wedge\langle rs\rangle\in\tau\wedge\langle st\rangle\in\sigma\upharpoonright a$

2 $w\neq\langle rt\rangle$

3 $\langle rs\rangle\notin\tau$

$- \quad \langle st\rangle\notin\sigma\upharpoonright a$

(s, t) $- \quad \neg\exists xy.\ \langle st\rangle=\langle xy\rangle\wedge\langle xy\rangle\in\sigma\wedge x\in a$

4 $\langle st\rangle\notin\sigma$

5 $s\notin a$

(1)

$- \quad w=\langle rt\rangle\wedge\langle rt\rangle\in\sigma\circ\tau\wedge r\in a^{\tau^{-1}}$

$w=\langle rt\rangle$ (2)

$- \quad \langle rt\rangle\in\sigma\circ\tau$

$- \quad \exists xyz.\ \langle rt\rangle=\langle xz\rangle\wedge\langle xy\rangle\in\tau\wedge\langle yz\rangle\in\sigma$

$- \quad \langle rt\rangle=\langle rt\rangle\wedge\langle rs\rangle\in\tau\wedge\langle st\rangle\in\sigma$

$\langle rs\rangle\in\tau$ (3) $\qquad \langle st\rangle\in\sigma$ (4, =, I)

$- \quad r\in a^{\tau^{-1}}$

$- \quad \exists x.\ x\in a\wedge\langle xr\rangle\in\tau^{-1}$

$- \quad s\in a\wedge\langle sr\rangle\in\tau^{-1}$

$s\in a$ (5) $\qquad - \quad \langle sr\rangle\in\tau^{-1}$

$\langle rs\rangle\in\tau$ (3, =, I)

$\upharpoonright *16 \qquad (\sigma\cap V^2)\upharpoonright a=\sigma\upharpoonright a$

$- \quad \upharpoonright *16$

(w) $- \quad \forall x.\ x\in(\sigma\cap V^2)\upharpoonright a\equiv x\in\sigma\upharpoonright a$

$- \quad w\in(\sigma\cap V^2)\upharpoonright a\equiv w\in\sigma\upharpoonright a$

$- \quad w\in(\sigma\cap V^2)\upharpoonright a$

$- \quad w\notin\sigma\upharpoonright a$

1 $\exists xy.\ w=\langle xy\rangle\wedge\langle xy\rangle\in\sigma\cap V^2\wedge x\in a$

(r, s) $- \quad \neg\exists xy.\ w=\langle xy\rangle\wedge\langle xy\rangle\in\sigma\wedge x\in a$

2 $w=\langle rs\rangle$

3 $\langle rs\rangle\notin\sigma$

4 $r\notin a$

$- \quad w\in\sigma\upharpoonright a$

$- \quad w\notin(\sigma\cap V^2)\upharpoonright a$

1 $\exists xy.\ w=\langle xy\rangle\wedge\langle xy\rangle\in\sigma\wedge x\in a$

(r, s) $- \quad \neg\exists xy.\ w=\langle xy\rangle\wedge\langle xy\rangle\in\sigma\cap V^2\wedge x\in a$

2 $w\neq\langle rs\rangle$

$- \quad \langle rs\rangle\notin\sigma\cap V^2$

3 $r\notin a$

$$\begin{array}{lcc}
{}^{(1)} & & \\ \hline
\underset{(3)}{w=\langle rs\rangle} & -\quad \langle rs\rangle\in\sigma\cap\mathrm{V}^2 & \underset{(4)}{r\in a}\\ \cline{2-2}
 & \underset{(2)}{\langle rs\rangle\in\sigma}\quad -\quad \langle rs\rangle\in\mathrm{V}^2 & \\ \cline{2-2}
 & -\ \exists xy.\ \langle rs\rangle=\langle xy\rangle & \\ \cline{2-2}
 & \underset{(=)}{\langle rs\rangle=\langle rs\rangle} &
\end{array}
\qquad
\begin{array}{lccc}
\hline
{}^{4} & & \langle rs\rangle\notin\sigma & \\
\mathrm{Spf.} & & \langle rs\rangle\notin\mathrm{V}^2 & \\ \hline
{}_{(1)} & \underset{(2)}{w=\langle rs\rangle} & \underset{(4)}{\langle rs\rangle\in\sigma} & \underset{(3)}{r\in a}
\end{array}$$

$\upharpoonright*17$ $\sigma\in\mathrm{Un}_2\wedge a\subseteq\mathrm{D}_\sigma\rightarrow\sigma\upharpoonright a\in\mathrm{Map}_2^{a,\,a^\sigma}$

$$\begin{array}{lccc}
- & & \upharpoonright*17 & \\ \hline
 & {}^{1}\ \sigma\notin\mathrm{Un} & {}^{2}\ \sigma^{-1}\notin\mathrm{Un} & {}^{3}\ a\not\subseteq\mathrm{D}_\sigma\\
- & & \sigma\upharpoonright a\in\mathrm{Map}_2^{a,\,a^\sigma} & \\ \hline
- & \multicolumn{3}{c}{\sigma\upharpoonright a\in\mathrm{Un}_2\wedge\mathrm{D}_{\sigma\upharpoonright a}=a\wedge\mathrm{W}_{\sigma\upharpoonright a}=a^\sigma}\\ \hline
 & \underset{(1,\,2)}{\sigma\upharpoonright a\in\mathrm{Un}_2} & \underset{(1,\,3)}{\mathrm{D}_{\sigma\upharpoonright a}=a} & \underset{(1,\,3)}{\mathrm{W}_{\sigma\upharpoonright a}=a^\sigma}\\
 & \mathrm{Cut}\ *\upharpoonright*6 & \mathrm{Cut}\ \upharpoonright*10 & \mathrm{Cut}\ \upharpoonright*11
\end{array}$$

Identical Operator (ι)

$\iota*1$ $\langle aa\rangle\in\iota$

$$\begin{array}{lc}
- & \iota*1\\ \hline
- & \exists x.\ \langle aa\rangle=\langle xx\rangle\\ \hline
 & \underset{(=)}{\langle aa\rangle=\langle aa\rangle}
\end{array}$$

$\iota*2$ $a^\iota=a$

$$\begin{array}{cc}
\begin{array}{lc}
- & \iota*2\\ \hline
{}_{(w)}- & \forall x.\ x\in a^\iota\equiv x\in a\\ \hline
- & w\in a^\iota\equiv w\in a
\end{array} & \\ \hline
\begin{array}{lc}
- & w\in a^\iota\\
1 & w\notin a\\ \hline
- & \exists x.\ \langle ax\rangle\in\iota\wedge w\in x\\ \hline
- & \langle aa\rangle\in\iota\wedge w\in a\\ \hline
 & \langle aa\rangle\in\iota\qquad \underset{(1)}{w\in a}\\
 & \iota*1\qquad\qquad
\end{array} &
\begin{array}{lc}
1 & w\in a\\
- & w\notin a^{i}\\ \hline
{}_{(b)}- & \neg\exists x.\ \langle ax\rangle\in\iota\wedge w\in x\\ \hline
- & \neg.\ \langle ab\rangle\in\iota\wedge w\in b\\ \hline
- & \langle ab\rangle\notin\iota\\
2 & w\notin b\\ \hline
{}_{(r)}- & \neg\exists x.\ \langle ab\rangle=\langle xx\rangle\\ \hline
 & \underset{(1,\,2,\,=)}{\langle ab\rangle\neq\langle rr\rangle}
\end{array}
\end{array}$$

$\iota*3$ $\iota\in\mathrm{Un}$

$$\begin{array}{lc}
- & \iota*3\\ \hline
{}_{(r,\,s,\,t)}- & \forall xyz.\ \langle xy\rangle\in\iota\wedge\langle xz\rangle\in\iota\rightarrow y=z\\ \hline
- & \langle rs\rangle\in\iota\wedge\langle rt\rangle\in\iota\rightarrow s=t
\end{array}$$

$$\begin{array}{ll}
\hdashline
- & \langle rs\rangle \notin \iota \\
- & \langle rt\rangle \notin \iota \\
1 & s=t \\
\hline
(a)\ - & \neg\exists x.\ \langle rs\rangle=\langle xx\rangle \\
(b)\ - & \neg\exists x.\ \langle rt\rangle=\langle xx\rangle \\
\hline
 & \langle rs\rangle\neq\langle aa\rangle \\
 & \langle rt\rangle\neq\langle bb\rangle \\
 & {\scriptstyle (1,\ =)}
\end{array}$$

$\iota{*}4$ $\iota^{-1}\in\mathrm{Un}$ (Similar to $\iota{*}3$)

$\iota{*}5$ $\iota\in\mathrm{Un}_2$ (From $\iota{*}3$ and $\iota{*}4$)

$*\iota{*}6$ $\iota\upharpoonright a\in\mathrm{Un}_2$ (From $*\upharpoonright{*}6$ and $\iota{*}5$)

$\iota{*}7$ $\mathrm{D}_\iota=\mathrm{V}$

$$\begin{array}{ll}
- & \iota{*}7 \\
\hdashline
(r)\ - & \forall x.\ x\in\mathrm{D}_\iota \\
\hdashline
- & r\in\mathrm{D}_\iota \\
\hdashline
- & \exists x.\ \langle rx\rangle\in\iota \\
\hdashline
- & \langle rr\rangle\in\iota \\
\hdashline
- & \exists x.\ \langle rr\rangle=\langle xx\rangle \\
\hdashline
 & \langle rr\rangle=\langle rr\rangle \\
 & {\scriptstyle (=)}
\end{array}$$

$\iota{*}8$ $\mathrm{W}_\iota=\mathrm{V}$

$\iota{*}9$ $\mathrm{D}_{\iota\upharpoonright a}=a$

$$\begin{array}{c}
\begin{array}{ll}
- & \iota{*}9 \\
\hline
(r)\ - & \forall x.\ x\in\mathrm{D}_{\iota\upharpoonright a}\equiv x\in a \\
\hdashline
- & r\in\mathrm{D}_{\iota\upharpoonright a}\equiv r\in a
\end{array} \\
\hdashline
\begin{array}{ll|ll}
- & r\in\mathrm{D}_{\iota\upharpoonright a} & 1 & r\in a \\
1 & r\notin a & - & r\notin\mathrm{D}_{\iota\upharpoonright a} \\
\hline
- & \exists x.\ \langle rx\rangle\in\iota\upharpoonright a & (s)\ - & \neg\exists x.\ \langle rx\rangle\in\iota\upharpoonright a \\
\hline
- & \langle rr\rangle\in\iota\upharpoonright a & - & \langle rs\rangle\notin\iota\upharpoonright a \\
\hline
- & \exists x.\ \langle rr\rangle=\langle xx\rangle\wedge x\in a & (t)\ - & \neg\exists x.\ \langle rs\rangle=\langle xx\rangle\wedge x\in a \\
\hline
- & \langle rr\rangle=\langle rr\rangle\wedge r\in a & - & \neg.\ \langle rs\rangle=\langle tt\rangle\wedge t\in a \\
\hline
 & \langle rr\rangle=\langle rr\rangle \qquad r\in a & & \langle rs\rangle\neq\langle tt\rangle \\
 & {\scriptstyle (=)} \qquad\qquad\quad {\scriptstyle (1)} & & t\notin a \\
 & & & {\scriptstyle (1,\ =,\ 1)}
\end{array}
\end{array}$$

$\iota{*}10$ $\mathrm{W}_{\iota\upharpoonright a}=a$ (Similar to $\iota{*}9$)

$\iota{*}11$ $\iota\upharpoonright a\in\mathrm{Map}_2^{a,\,a}$ (From $*\iota{*}6$, $\iota{*}9$ and $\iota{*}10$)

ι*12 $\quad \sigma^{-1}\in \mathrm{Un}\rightarrow \sigma^{-1}\circ\sigma=\iota\upharpoonright \mathrm{D}_\sigma$

$-\quad \iota$*12

$1\quad \sigma^{-1}\notin \mathrm{Un}$

$-\quad \sigma^{-1}\circ\sigma=\iota\upharpoonright \mathrm{D}_\sigma$

$(w)-\quad \forall x.\ x\in\sigma^{-1}\sigma\circ\equiv x\in\iota\upharpoonright \mathrm{D}_\sigma$

$-\quad w\in\sigma^{-1}\circ\sigma\equiv w\in\iota\upharpoonright \mathrm{D}_\sigma$

$2\quad w\in\sigma^{-1}\circ\sigma$

$-\quad w\notin\iota\upharpoonright \mathrm{D}_\sigma$

$(r,r_1)-\quad \neg\exists xy.\ w=\langle xy\rangle\land\langle xy\rangle\in\iota\land x\in \mathrm{D}_\sigma$

$-\quad \neg.\ w=\langle rr_1\rangle\land\langle rr_1\rangle\in\iota\land r\in \mathrm{D}_\sigma$

$3\quad w\neq\langle rr_1\rangle$

$-\quad \langle rr_1\rangle\notin\iota$

$-\quad r\notin \mathrm{D}_\sigma$

$(r_2)-\quad \neg\exists x.\ \langle rr_1\rangle=\langle xx\rangle$

$(s)-\quad \neg\exists x.\ \langle rx\rangle\in\sigma$

$4\quad \langle rr_1\rangle\neq\langle r_2r_2\rangle$

$5\quad \langle rs\rangle\notin\sigma$

(2)

$-\quad \exists yxz.\ w=\langle xz\rangle\land\langle xy\rangle\in\sigma\land\langle yz\rangle\in\sigma^{-1}$

$-\quad w=\langle rr_1\rangle\land\langle rs\rangle\in\sigma\land\langle sr_1\rangle\in\sigma^{-1}$

$w=\langle rr_1\rangle$ (3) $\qquad \langle rs\rangle\in\sigma$ (5) $\qquad -\ \langle sr_1\rangle\in\sigma^{-1}$

$\langle r_1s\rangle\in\sigma$ (4, 5, =, I)

$2\quad w\in\iota\upharpoonright \mathrm{D}_\sigma$

$-\quad w\notin\sigma^{-1}\circ\sigma$

$(r,s,t)-\quad \neg\exists xyz.\ w=\langle xz\rangle\land\langle xy\rangle\in\sigma\land\langle yz\rangle\in\sigma^{-1}$

$-\quad \neg.\ w=\langle rt\rangle\land\langle rs\rangle\in\sigma\land\langle st\rangle\in\sigma^{-1}$

$3\quad w\neq\langle rt\rangle$

$4\quad \langle rs\rangle\notin\sigma$

$5\quad \langle st\rangle\notin\sigma^{-1}$

(2)

$-\quad \exists xy.\ w=\langle xy\rangle\land\langle xy\rangle\in\iota\land x\in \mathrm{D}_\sigma$

$-\quad w=\langle rt\rangle\land\langle rt\rangle\in\iota\land r\in \mathrm{D}_\sigma$

$w=\langle rt\rangle$ (3) $\qquad -\ \langle rt\rangle\in\iota \qquad -\ r\in \mathrm{D}_\sigma$

$-\ \exists x.\ \langle rt\rangle=\langle xx\rangle \qquad -\ \exists x.\ \langle rx\rangle\in\sigma$

$6\quad \langle rt\rangle=\langle rr\rangle$ (*) $\qquad \langle rs\rangle\in\sigma$ (4)

(*)

(1)

$-\quad \neg\forall xyz.\ \langle xy\rangle\in\sigma^{-1}\land\langle xz\rangle\in\sigma^{-1}\rightarrow y=z$

$-\quad \neg.\ \langle st\rangle\in\sigma^{-1}\land\langle sr\rangle\in\sigma^{-1}\rightarrow t=r$

$\langle st\rangle\in\sigma^{-1}$ (5, =, I) $\qquad -\ \langle sr\rangle\in\sigma^{-1} \qquad 7\quad t\neq r$ (6, =)

$\langle rs\rangle\in\sigma$ (4)

ι*13 $\quad \sigma\in \mathrm{Un}\rightarrow\sigma\circ\sigma^{-1}=\iota\upharpoonright \mathrm{D}_{\sigma^{-1}}$ $\qquad$ (Similar to ι*12)

ι*14 $\quad \sigma\in \mathrm{Un}_2\rightarrow\sigma^{-1}\circ\sigma=\iota\upharpoonright \mathrm{D}_\sigma\land\sigma\circ\sigma^{-1}=\iota\upharpoonright \mathrm{D}_{\sigma^{-1}}$ $\qquad$ (From ι*12 and ι*13)

ι*15 $\quad \sigma\circ\iota=\sigma\cap \mathrm{V}^2$

$-\quad \iota$*15

$(w)-\quad \forall x.\ x\in\sigma\circ\iota\equiv x\in\sigma\cap \mathrm{V}^2$

$-\quad w\in\sigma\circ\iota\equiv w\in\sigma\cap \mathrm{V}^2$

$1\quad w\in\sigma\circ\iota$

$-\quad w\notin\sigma\cap \mathrm{V}^2$

$-\quad \neg.\ w\in\sigma\land w\in \mathrm{V}^2$

$1\quad w\in\sigma\cap \mathrm{V}^2$

$-\quad w\notin\sigma\circ\iota$

$(r,s,t)-\quad \neg\exists xyz.\ w=\langle xz\rangle\land\langle xy\rangle\in\iota\land\langle yz\rangle\in\sigma$

$$\begin{array}{lc}
\hline
2 & w\notin\sigma\\
- & w\notin\mathrm{V}^2\\
\hline
(r,s)\ - & \neg\exists xy.\ w=\langle xy\rangle\\
\hline
3 & w\neq\langle rs\rangle\\
\hline
(1)\ - & \exists xyz.\ w=\langle xz\rangle\wedge\langle xy\rangle\in\iota\wedge\langle yz\rangle\in\sigma\\
\hline
- & w=\langle rs\rangle\wedge\langle rr\rangle\in\iota\wedge\langle rs\rangle\in\sigma\\
\hline
\multicolumn{2}{c}{\begin{array}{ccc} w=\langle rs\rangle & \langle rr\rangle\in\iota & \langle rs\rangle\in\sigma\\ \scriptstyle(3) & \iota*1 & \scriptstyle(2,\,3,\,=,\,\mathrm{I})\end{array}}
\end{array}$$

$$\begin{array}{lc}
\hline
- & \neg.\ w=\langle rt\rangle\wedge\langle rs\rangle\in\iota\wedge\langle st\rangle\in\sigma\\
\hline
2 & w\neq\langle rt\rangle\\
- & \langle rs\rangle\notin\iota\\
3 & \langle st\rangle\notin\sigma\\
\hline
(r)\ - & \neg\exists x.\ \langle rs\rangle=\langle xx\rangle\\
\hline
4 & \langle rs\rangle\neq\langle rr\rangle\\
\hline
(1)\ - & w\in\sigma\wedge w\in\mathrm{V}^2\\
\hline
\multicolumn{2}{c}{\begin{array}{cc} \begin{array}{c} w\in\sigma\\ \scriptstyle(2,\,3,\,4,\,=,\,\mathrm{I})\end{array} & \begin{array}{lc} - & w\in\mathrm{V}^2\\ \hline - & \exists xy.\ w=\langle xy\rangle\\ \hline \multicolumn{2}{c}{w=\langle rt\rangle}\\ \multicolumn{2}{c}{\scriptstyle(2)}\end{array}\end{array}}
\end{array}$$

$\iota*16$ $\quad \sigma\circ\iota=\iota\circ\sigma=\sigma\cap\mathrm{V}^2$

We can prove the latter equality similarly as $\iota*15$.

$\iota*17$ $\quad \sigma\circ(\iota\upharpoonright a)=\sigma\upharpoonright a$

$$\begin{array}{lc}
- & \iota*17\\
\hline
(w)\ - & \forall x.\ x\in\sigma\circ(\iota\upharpoonright a)\equiv x\in\sigma\upharpoonright a\\
\hline
- & w\in\sigma\circ(\iota\upharpoonright a)\equiv w\notin\sigma\upharpoonright a
\end{array}$$

$$\begin{array}{lc}
\hline
- & w\in\sigma\circ(\iota\upharpoonright a)\\
- & w\notin\sigma\upharpoonright a\\
\hline
1 & \exists xyz.\ w=\langle xz\rangle\wedge\langle xy\rangle\in\iota\upharpoonright a\wedge\langle yz\rangle\in\sigma\\
(r,s)\ - & \neg\exists xy.\ w=\langle xy\rangle\wedge\langle xy\rangle\in\sigma\wedge x\in a\\
\hline
- & \neg.\ w=\langle rs\rangle\wedge\langle rs\rangle\in\sigma\wedge r\in a\\
\hline
2 & w\neq\langle rs\rangle\\
3 & \langle rs\rangle\notin\sigma\\
4 & r\notin a\\
\hline
(1)\ - & w=\langle rs\rangle\wedge\langle rr\rangle\in\iota\upharpoonright a\wedge\langle rs\rangle\in\sigma\\
\hline
\multicolumn{2}{c}{\begin{array}{ccc} \begin{array}{c} w=\langle rs\rangle\\ \scriptstyle(2)\end{array} & \begin{array}{lc} - & \langle rr\rangle\in\iota\upharpoonright a\\ \hline - & \langle rr\rangle\in\iota\wedge r\in a\\ \hline \multicolumn{2}{c}{\begin{array}{cc} \langle rr\rangle\in\iota & r\in a\\ \iota*1 & \scriptstyle(4)\end{array}}\end{array} & \begin{array}{c} \langle rs\rangle\in\sigma\\ \scriptstyle(3)\end{array}\end{array}}
\end{array}$$

$$\begin{array}{lc}
\hline
- & w\in\sigma\upharpoonright a\\
- & w\notin\sigma\circ(\iota\upharpoonright a)\\
\hline
1 & \exists xy.\ w=\langle xy\rangle\wedge\langle xy\rangle\in\sigma\wedge x\in a\\
(r,s,t)\ - & \neg\exists xyz.\ w=\langle xz\rangle\wedge\langle xy\rangle\in\iota\upharpoonright a\wedge\langle yz\rangle\in\sigma\\
\hline
- & \neg.\ w=\langle rt\rangle\wedge\langle rs\rangle\in\iota\upharpoonright a\wedge\langle st\rangle\in\sigma\\
\hline
2 & w\neq\langle rt\rangle\\
- & \langle rs\rangle\notin\iota\upharpoonright a\\
3 & \langle st\rangle\notin\sigma\\
\hline
(r,s)\ - & \neg\exists xy.\ \langle rs\rangle=\langle xy\rangle\wedge\langle xy\rangle\in\iota\wedge x\in a\\
\hline
- & \langle rs\rangle\notin\iota\\
4 & s\notin a\\
\hline
(r)\ - & \neg\exists x=\langle rs\rangle=\langle xx\rangle\\
\hline
5 & \langle rs\rangle\neq\langle rr\rangle\\
\hline
(1)\ - & w=\langle rt\rangle\wedge\langle rt\rangle\in\sigma\wedge r\in a\\
\hline
\multicolumn{2}{c}{\begin{array}{ccc} w=\langle rt\rangle & \langle rt\rangle\in\sigma & r\in a\\ \scriptstyle(2) & \scriptstyle(3,\,5,\,=,\,\mathrm{I}) & \scriptstyle(4,\,5,\,=,\,\mathrm{I})\end{array}}
\end{array}$$

$\iota*18$ $\quad (\iota\circ\sigma)\upharpoonright a=\sigma\upharpoonright a$

$$\begin{array}{lc}
1 & \iota*18\quad \text{Cut }\iota*16\\
\hline
2 & \iota\circ\sigma\neq\sigma\cap\mathrm{V}^2\quad \text{Cut }\upharpoonright*16\\
\hline
\multicolumn{2}{c}{(\sigma\cap\mathrm{V}^2)\upharpoonright a\neq\sigma\upharpoonright a}\\
\multicolumn{2}{c}{\scriptstyle(1,\,2,\,=,\,\mathrm{I})}
\end{array}$$

$\iota*19$ $\quad (\sigma\circ\iota)\upharpoonright a=\sigma\upharpoonright a$

(Added in proof) (i) The principle of arranging the formulas in this Part (III) is as follows. The main symbols used are classified into seven groups: (i) $=$, $\subseteq$; (ii) $\{a, b\}$, $\langle ab\rangle$; (iii) Un, Un_2, D_σ, W_σ, a^σ, A^σ; (iv) σ^{-1}; (v) $\sigma\circ\tau$; (vi) $\upharpoonright$; and (vii) ι. To these groups correspond respectively the seven Sections: $=$, El, Im, -1, $\circ$, $\upharpoonright$, and ι. If, among the symbols listed above and contained in a formula A to be proved, x is the symbol which belongs to the rightmost group, say y, then A is proved in the Section corresponding to the group y. Owing to this principle it happens that in the proof of -1*12 are used some cuts of which the cut formulas are proved after -1*12.

(ii) Most proofs are analytic (see the last page of Part (IV)). For instance, the proofs for -1*12, ι*18, -1*14, $\upharpoonright$*10, etc. are not analytic: note that in the proofs of $\upharpoonright$*10 is found the dependent variable $\langle rr^\sigma\rangle$ and in that of -1*14 the variables $a^{\sigma^{-1}(\sigma^{-1})^{-1}}$ and $\mathrm{W}_{\sigma^{-1}}$, which do not belong to the closure of the variables in the formulas to be proved. Some formulas with non-analytic proofs may be proved analytically.

(iii) All the proofs are given in the reduced form (see §20, Part (II)), or in the normal form except some practical change. The weakly irreducible proofs are those for *El*6, *El*7; and for *$\upharpoonright$*1, *$\upharpoonright$*4, *$\upharpoonright$*5, *$\upharpoonright$*6 and *ι*6. Thus, the proofs are given without roundabout way and without superfluity as far as we can.

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