ON THE DIMENSION OF MODULES AND ALGEBRAS, VI

COMPARISON OF GLOBAL AND ALGEBRA DIMENSION

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Throughout this paper all rings are assumed to have unit elements. A ring Λ is said to be semi-primary if its Jacobson radical N is nilpotent and $\Gamma = \Lambda/N$ satisfies the minimum condition. The main objective of this paper is

THEOREM I. *Let Λ be a semi-primary algebra over a field K. Let N be the radical of Λ and $\Gamma = \Lambda/N$. If*

$$\dim \Lambda < \infty \textit{ and } (\Gamma : K) < \infty,$$

Then

$$\dim \Lambda = \text{gl.}\dim \Lambda.$$

Here $\dim \Lambda$ denotes the dimension of Λ as a K-algebra, i.e. $\dim \Lambda = \text{l.}\dim_{\Lambda^e} \Lambda$ where $\Lambda^e = \Lambda \otimes_K \Lambda^*$.

We do not know whether the condition $(\Gamma : K) < \infty$ follows from the condition that Λ is a semi-primary ring such that $\text{gl.}\dim \Lambda = \dim \Lambda < \infty$. The theorem has been previously proven in [3] and [4] under the stronger assumption $(\Lambda : K) < \infty$. In this case it was further shown that Γ is separable (i.e. $\dim \Gamma = 0$). We do not know whether this is true without the assumption $(\Lambda : K) < \infty$.

1. Tensor product of semi-simple algebras

A semi-primary ring Λ with radical N is called *primary* if Λ/N is a simple ring.

PROPOSITION 1. *Let Λ and Σ be rings and $\varphi : \Lambda \longrightarrow \Sigma$ a ring epimorphism. If Λ is a semi-primary ring with radical N, then Σ is a semi-primary ring with radical $\varphi(N)$.*

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Proof: Since N is a nilpotent two-sided ideal in Λ, $\varphi(N)$ is a nilpotent two-sided ideal in Σ. The epimorphism $\varphi : \Lambda \longrightarrow \Sigma$ induces an epimorphism $\bar{\varphi} : \Lambda/N \longrightarrow \Sigma/\varphi(N)$. Since Λ/N is semi-simple, it follows that $\Sigma/\varphi(N)$ is semi-simple. Thus $\varphi(N)$ is the Jacobson radical of Σ, which shows that Σ is semi-primary.

The following proposition, which we state without proof, is due to Nakayama and Azumaya (see [5], theorem 9).

PROPOSITION 2. *Let Λ_1 and Λ_2 be simple K-algebras with centers C_1 and C_2. Then $C_1 \otimes_K C_2$ is the center of $\Lambda_1 \otimes_K \Lambda_2$ and the two-sided ideals in $\Lambda_1 \otimes_K \Lambda_2$ are in a one-to-one lattice preserving correspondence with the ideals in $C_1 \otimes_K C_2$. Under this correspondence a two-sided ideal I in $\Lambda_1 \otimes_K \Lambda_2$ corresponds with the ideal $I \cap (C_1 \otimes_K C_2)$ in $C_1 \otimes_K C_2$ and an ideal J in $C_1 \otimes_K C_2$ corresponds with the two-sided ideal $(\Lambda_1 \otimes_K \Lambda_2) J$ in $\Lambda_1 \otimes_K \Lambda_2$.*

PROPOSITION 3. *Let Λ_1 and Λ_2 be semi-simple algebras over a field K with centers C_1 and C_2. If $\Lambda_1 \otimes_K \Lambda_2$ is semi-primary, then each of the algebras $C_1 \otimes_K C_2$ and $\Lambda_1 \otimes_K \Lambda_2$ is a finite direct product of primary K-algebras.*

Proof: Since Λ_1 and Λ_2 are finite direct products of simple K-algebras we have that $\Lambda_1 \otimes_K \Lambda_2$ is the finite direct product of K-algebras of the form $\Sigma_1 \otimes_K \Sigma_2$, where Σ_1 and Σ_2 are simple algebras which are direct summands of Λ_1 and Λ_2. It follows from Proposition 1, that if $\Lambda_1 \otimes_K \Lambda_2$ is semi-primary, then so are the algebras $\Sigma_1 \otimes_K \Sigma_2$, which are homomorphic images of $\Lambda_1 \otimes_K \Lambda_2$. Thus it suffices to prove the proposition in the event that Λ_1 and Λ_2 are simple K-algebras.

Let N be the radical of $\Lambda_1 \otimes_K \Lambda_2$. Since $(\Lambda_1 \otimes_K \Lambda_2)/N$ is semi-simple, it satisfies the minimum condition. Hence we have by Proposition 2 that $(C_1 \otimes_K C_2)/N \cap (C_1 \otimes_K C_2)$ satisfies the minimum condition. Since N is the maximal nilpotent two-sided ideal in $\Lambda_1 \otimes_K \Lambda_2$, it follows from Proposition 2 that $N \cap (C_1 \otimes_K C_2)$ is the maximal nilpotent ideal in $C_1 \otimes_K C_2$. Therefore $(C_1 \otimes_K C_2)/N \cap (C_1 \otimes_K C_2)$ is semi-simple. Since $N \cap (C_1 \otimes_K C_2)$ is nilpotent, every set of orthogonal idempotents in $(C_1 \otimes_K C_2)/N \cap (C_1 \otimes_K C_2)$ can be "lifted" to an orthogonal set of idempotents in $C_1 \otimes_K C_2$. From this and the commutativity of $C_1 \otimes_K C_2$, it follows that $C_1 \otimes_K C_2$ is a finite direct product of primary K-algebras.

Let $C_1 \otimes_K C_2 = \Sigma_1 + \ . \ . \ . \ + \Sigma_n$ (direct product) where each Σ_i is a primary K-algebra with radical N_1 and let $\Gamma_i = \Sigma_i/N_i$. Since C_2 is a field we have for each i the exact sequence

$$0 \longrightarrow N_i \otimes_{C_2} \Lambda_2 \longrightarrow \Sigma_i \otimes_{C_2} \Lambda_2 \longrightarrow \Gamma_i \otimes_{C_2} \Lambda_2 \longrightarrow 0.$$

Since C_1 is a field, we deduce from the above exact sequence the exact sequence

$$(*) \qquad 0 \longrightarrow \Lambda_1 \otimes_{C_1} N_i \otimes_{C_2} \Lambda_2 \longrightarrow \Lambda_1 \otimes_{C_1} \Sigma_i \otimes_{C_2} \Lambda_2 \longrightarrow \Lambda_1 \otimes_{C_1} \Gamma_i \otimes_{C_2} \Lambda_2 \longrightarrow 0.$$

By Proposition 2, we have that the center of $\Lambda_1 \otimes_{C_1} \Gamma_i \otimes_{C_2} \Lambda_2$ is $C_1 \otimes_{C_1} \Gamma_i \otimes_{C_2} C_2 = \Gamma_i$ which is a field. Thus by Proposition 2, $\Lambda_1 \otimes_{C_1} \Gamma_i \otimes_{C_2} \Lambda_2$ has only the trivial two-sided ideals.

Now $\Lambda_1 \otimes_K \Lambda_2 = \Lambda_1 \otimes_{C_1} C_1 \otimes_K C_2 \otimes_{C_2} \Lambda_2 = \Lambda_1 \otimes_{C_1} (\Sigma_1 + \ldots + \Sigma_n) \otimes_{C_2} \Lambda_2 = \sum_{i=1}^{n} \Lambda_1 \otimes_{C_1} \Sigma_i \otimes_{C_2} \Lambda_3$. Since each $\Lambda_1 \otimes_{C_1} \Sigma_i \otimes_{C_2} \Lambda_2$ is a homomorphic image of $\Lambda_1 \otimes_K \Lambda_2$, we have by Proposition 1, that each $\Lambda_1 \otimes_{C_1} \Sigma_i \otimes_{C_2} \Lambda_2$ is semi-primary. It follows from the fact that each N_i is a nilpotent two-sided ideal that each $\Lambda_1 \otimes_{C_1} N_i \otimes_{C_2} \Lambda_2$ is a nilpotent two-sided ideal in $\Lambda_1 \otimes_{C_1} \Sigma_i \otimes_{C_2} \Lambda_2$. Hence we deduce from (*) and Proposition 1 that $\Lambda_1 \otimes_{C_1} \Gamma_i \otimes_{C_2} \Lambda_2$ satisfies the minimum condition and is thus simple. Therefore each $\Lambda_1 \otimes_{C_1} \Sigma_i \otimes_{C_2} \Lambda_2$ is a primary K-algebra, which establishes that $\Lambda_1 \otimes_K \Lambda_2$ is a direct product of primary K-algebras.

Remark. It should be noted that while the hypothesis of Proposition 3 is satisfied if $(\Lambda_1 : K) < \infty$, it can also be satisfied without any finiteness restrictions on the linear dimension of the algebras. For example, let Λ_1 be a pure transcendental field extension of K and Λ_2 an arbitrary algebraic extension of K. Then $\Lambda_1 \otimes_K \Lambda_2$ is a semi-primary K-algebra. On the other hand, it can be shown that if C is a commutative semi-simple K-algebra such that $C \otimes_K C$ is semi-primary, then $(C : K) < \infty$. Thus if Λ_1 and Λ_2 are semi-simple K-algebras with $C_1 = C_2$, we have by Proposition 3 that $\Lambda_1 \otimes_K \Lambda_2$ being semi-primary implies that $(C : K) < \infty$.

2. Tensor product of semi-primary algebras

LEMMA 4. *Let* $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ *be an exact sequence of left* Λ-*modules such that*

$$\text{l.dim}_\Lambda A < \sup(\text{l.dim}_\Lambda A', \text{l.dim}_\Lambda A'').$$

Then $\text{l.dim}_\Lambda A'' = 1 + \text{l.dim}_\Lambda A'$.

Proof: Let $n = \text{l.dim}_\Lambda A$, which is finite by hypothesis. Then $\text{Ext}_\Lambda^p(A, C) = 0$ for $p > n$ and all left Λ-modules C. Thus by the homology sequence for the functor Ext we have that $\operatorname{Ext}^p_\Lambda(A', C) \approx \operatorname{Ext}^{p+1}_\Lambda(A'', C)$ for $p > n$. Thus if $\text{l.dim}_\Lambda A' > n$ we are done. If $\text{l.dim}_\Lambda A' = n$, then $\text{l.dim}_\Lambda A'' \leqq n+1$. But then by hypothesis $\text{l.dim}_\Lambda A''$ would have to be greater than or equal to $n+1$. From the exactness of the sequence $\operatorname{Ext}^n_\Lambda(A', C) \longrightarrow \operatorname{Ext}^{n+1}_\Lambda(A'', C) \longrightarrow 0$ we see that if $\text{l.dim}_\Lambda A' < n$, then $\text{l.dim}_\Lambda A'' \leqq n$, which is impossible.

THEOREM 5. *Let Λ_1 and Λ_2 be semi-primary algebras over a field K. Let N_i be the radical of Λ_i and let $\Gamma_i = \Lambda_i/N_i$, $i=1, 2$. If $\Gamma_1 \otimes_K \Gamma_2$ is semi-primary, then $\Lambda_1 \otimes_K \Lambda_2$ is semi-primary. If further*

$$\text{gl.dim}\, \Lambda_1 \otimes_K \Lambda_2 < \infty$$

then

$$\text{gl.dim}\, \Lambda_1 \otimes_K \Lambda_2 = \text{gl.dim}\, \Lambda_1 + \text{gl.dim}\, \Lambda_2 = \text{l.dim}_{\Lambda_1 \otimes_K \Lambda_2} \Gamma_1 \otimes_K \Gamma_2.$$

Proof: Consider the exact sequence

$$0 \longrightarrow R \longrightarrow \Lambda_1 \otimes_K \Lambda_2 \longrightarrow \Gamma_1 \otimes_K \Gamma_2 \longrightarrow 0$$

where $R = N_1 \otimes_K \Lambda_2 + \Lambda_1 \otimes_K N_2$. Since R is nilpotent and $\Gamma_1 \otimes_K \Gamma_2$ is semi-primary, it follows that $\Lambda_1 \otimes_K \Lambda_2$ is semi-primary.

The inequality

$$\text{gl.dim}\, \Lambda_1 + \text{gl.dim}\, \Lambda_2 \leqq \text{gl.dim}\, (\Lambda_1 \otimes_K \Lambda_2)$$

follows from [1] Theorem 16. The inequality

$$\text{l.dim}_{\Lambda_1 \otimes_K \Lambda_2} \Gamma_1 \otimes_K \Gamma_2 \leqq \text{gl.dim}\, \Lambda_1 + \text{gl.dim}\, \Lambda_2$$

follows from the general inequality

$$\text{l.dim}_{\Lambda_1 \otimes_K \Lambda_2} A_1 \otimes_K A_2 \leqq \text{l.dim}_{\Lambda_1} A_1 + \text{l.dim}_{\Lambda_2} A_2$$

(See [2], Chapter XI, 3.2).

Assume $\text{l.dim}_{\Lambda_1 \otimes_K \Lambda_2} \Gamma_1 \otimes_K \Gamma_2 = m < n = \text{gl.dim}\, \Lambda_1 \otimes_K \Lambda_2$. There exists then by [1], Corollary 11, a simple $\Lambda_1 \otimes_K \Lambda_2$-module A such that $\text{l.dim}_{\Lambda_1 \otimes_K \Lambda_2} A = n$. Since R is nilpotent, $RA = 0$ and it follows that A is also a simple $\Gamma_1 \otimes_K \Gamma_2$-module. By Proposition 3 we know that $\Gamma_1 \otimes_K \Gamma_2$ is a direct product of primary rings. Thus A is isomorphic with a left ideal I in $\Gamma_1 \otimes_K \Gamma_2$ (See [1], Proposition 15). Then $\text{l.dim}_{\Lambda_1 \otimes_K \Lambda_2} I < \text{l.dim}_{\Lambda_1 \otimes_K \Lambda_2} \Gamma_1 \otimes_K \Gamma_2$. Thus by Lemma 4 we deduce from the exact sequence

$$0 \longrightarrow I \longrightarrow \Gamma_1 \otimes_K \Gamma_2 \longrightarrow (\Gamma_1 \otimes_K \Gamma_2)/I \longrightarrow 0$$

that $\mathrm{l.dim}\,(\Gamma_1 \otimes_K \Gamma_2)/I = 1 + \mathrm{l.dim}_{\Lambda_1 \otimes_K \Lambda_2} I = 1 + n$, a contradiction.

Remark. It should be noted that Theorem 5 is false without the assumption $\mathrm{gl.dim}\, \Lambda_1 \otimes_K \Lambda_2 < \infty$. Indeed, let Λ be a finite inseparable field extension of K. Then $\mathrm{gl.dim}\, \Lambda = 0$. By Proposition 3 $\Lambda \otimes_K \Lambda$ is a direct product of semi-primary K-algebras. Since $\Lambda \otimes_K \Lambda$ is not semi-simple, $\mathrm{gl.dim}\, \Lambda \otimes_K \Lambda = \infty$ (See [1], Proposition 15).

3. Proof of Theorem I.

By [3], Proposition 9, we have that

$$\dim(\Lambda) = \mathrm{gl.dim}\, \Lambda \otimes_K \Gamma^*.$$

Since $(\Gamma^* : K) = (\Gamma : K) < \infty$, it follows that $(\Gamma \otimes_K \Gamma^* : K) < \infty$. Thus we have that $\Gamma \otimes_K \Gamma^*$ is a semi-primary K-algebra. Since by hypothesis $\mathrm{gl.dim}\, \Lambda \otimes_K \Gamma^* = \dim \Lambda < \infty$, we have applying Theorem 5 that

$$\mathrm{gl.dim}\, \Lambda \otimes_K \Gamma^* = \mathrm{gl.dim}\, \Lambda + \mathrm{gl.dim}\, \Gamma^* = \mathrm{gl.dim}\, \Lambda.$$

Therefore $\dim \Lambda = \mathrm{gl.dim}\, \Lambda$.

Bibliography

[1] M. Auslander, On the dimension of modules and algebras (III), global dimension, Nagoya Math. J., **9** (1955), 67–77.

[2] H. Cartan and S. Eilenberg, Homological Algebra, Princeton University Press, 1956.

[3] S. Eilenberg, Algebras of cohomologically finite dimension, Comment. Math. Helv., 28 (1954), 310–319.

[4] M. Ikeda, H. Nagao and T. Nakayama, Algebras with vanishing n-cohomology groups, Nagoya Math. J. **7** (1954), 115–131.

[5] T. Nakayama and G. Azumaya, On irreducible rings, Ann. of Math. 48 (1947), 949–965.

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