ON THE TRANSITION PROBABILITY OF A RENEWAL PROCESS

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J. L. Doob, D. Blackwell, W. Feller and other authors have obtained several results concerning the renewal theorem. Especially Doob [1] has considered the renewal process and has showed that it becomes a stationary Markov process if we add a certain initial random variable to it. In the present note, we shall study this stationary Markov process and try to determine its transition probability by virtue of a pair of partial differential equations.

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§ 1. Preliminary notions

Most of the results in this section will be obtained by referring to the Doob's paper [1].

Let $X_0(\omega)$[1], $X_2(\omega)$, $X_3(\omega)$, . . . be mutually independent non-negative random variables with the common distribution function $G(x)$ such that

$$(1)\qquad P(X_i \leqq x) = G(x) = \begin{cases} \int_0^x g(t)dt & , \text{ if } x \geqq 0, \\ 0 & , \text{ if } x < 0, \end{cases}$$

$$i = 0, 2, 3, \ldots .$$

Furthermore we assume that $g(x) \geqq 0$ belongs to C^1-class and X_i has finite mean and variance.

In the renewal theory, these $\{X_i\}$ $(i = 0, 2, 3, \ldots)$ denote the lifetimes of individuals born successively and especially X_0 denotes the lifetime of the one which survives at $t = 0$.

Let x (for which $G(x) < 1$) be the age of it at $t = 0$. Then $X_1 = X_0 - x$ is

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1) ω is the probability parameter. We shall omit it unless we need it specially.

the left-time for survival. The (conditional) distribution function of X_1 is expressible as

$$(2) \qquad G_x(y) = \frac{G(x+y) - G(x)}{1 - G(x)}$$

Here, it is noted that the initial age x is a random variable. If the distribution of X_0 is determined by some $\Phi(x)$, the probability distribution $\Psi(y)$ of X_1 is obtained immediately:

$$(3) \quad \begin{cases} \Psi(y) = P(X_1 \leqq y) = \int_0^\infty P(x < X_0 \leqq x + y / X_0 \geqq x) d\Phi(x) \\ \qquad = \begin{cases} \int_0^\infty \frac{G(x+y) - G(x)}{1 - G(x)} d\Phi(x) & , \text{ if } \ y \geqq 0, \\ 0 & , \text{ if } \ y < 0. \end{cases} \end{cases}$$

Let $n(t, \omega)$ be the number of sums

$$X_1(\omega), \ X_1(\omega) + X_2(\omega), \ldots .$$

which are less than t. Then the *renewal process* is defined by

$$(4) \qquad x(t, \omega) = \begin{cases} t - \{X_1(\omega) + \ldots + X_{n(t)}(\omega)\} & , \text{ if } n(t) > 0, \\ t + X_0(\omega) & , \text{ if } n(t) = 0. \end{cases}$$

That is, it indicates the age of individual at time t. Doob [1] has proved that $x(t)$ process is a temporally homogeneous Markov process and its transition probability is

$$(5) \quad \begin{cases} P(x(h+s) \leqq y / x(s) = x) = P(x(h) \leqq y / x(0) = x) = P(h, x, y) \\ \quad = \begin{cases} 1 & , \text{ if } \ h + x \leqq y, \\ G_x(h) & , \text{ if } \ h \leqq y < h + x, \\ \int_0^y -(1 - G(u)) d_u U_x(h-u) & , \text{ if } \ 0 \leqq y < h, \\ 0 & , \text{ if } \ y < 0 \text{ or } G(x) = 1, \end{cases} \end{cases}$$

where $U_x(t) = E(n(t)/x(0) = x)$, and it can be written as

$$(6) \qquad U_x(t) = G_x(t) + G_x * H(t) \text{ and } H(t) = G(t) + G^{2*}(t) + G^{3*}(t) + \ldots$$[2].

Now, if we restrict the initial distribution to

2) $G^{2*} = G * G$ and $G^{n*} = G * G^{(n-1)*}$.

$$\Phi(x)=\frac{1}{m}\int_0^x(1-G(u))du, \quad \text{where} \quad m=\int_0^\infty x dG(x), \tag{7}$$

then it is proved that $E\{U_x(t)\}=\frac{t}{m}$. In this case the distribution function of $x(t)$ is

$$\begin{cases} P(x(t)\leqq y)=\int_0^\infty P(x(t)\leqq y/x(0)=x)d\Phi(x) \\ =\begin{cases}\int_0^{y-t}d\Phi(x)+\frac{1}{m}\int_{y-t}^\infty \frac{G(x+t)-G(x)}{1-G(x)}\cdot(1-G(x))dx=\frac{1}{m}\int_0^y(1-G(u))du \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad, \text{ if } t\leqq y, \\ \int_0^y(1-G(u))\frac{du}{m}\cdot\frac{1}{m}\int_0^\infty(1-G(x))dx=\frac{1}{m}\int_0^y(1-G(u))du \quad, \text{ if } t>y,\end{cases}\end{cases} \tag{8}$$

which is equal to $\Phi(y)$ and is independent of t. Therefore $x(t)$ becomes a stationary Markov process.

Hereafter we shall deal with such a stationary renewal process. It is noted that such a process is uniquely determined by the transition probability.

§ 2. Fundamental Differential Equations

We shall discuss only the case where $G(x)<1$.

THEOREM 1. *For every* $x(G(x)<1)$,

$$K(x)=\lim_{\tau\downarrow 0}\frac{1}{\tau}P(\tau, x, \tau) \tag{9}$$

exists aud satisfies the following three conditions:

$$\begin{cases} 1^\circ) \ K(x) \textit{ is differentiable and non-negative,} \\ 2^\circ) \ \varphi(a)=\int_0^a K(x)dx<\infty \textit{ for every } a \textit{ such that } G(a)<1, \\ 3^\circ) \ \lim_{a\to\infty}\varphi(a)=\infty. \end{cases} \tag{10}$$

Proof. We have

$$\lim_{\tau\downarrow 0}\frac{1}{\tau}P(\tau, x, \tau)=\lim_{\tau\downarrow 0}\frac{1}{\tau}\frac{G(\tau+x)-G(x)}{1-G(x)}=\frac{g(x)}{1-G(x)},$$

from which all the statements of (10) follow immediately. Q.E.D.

This theorem shows that $x(t)$ will have at least one jump point in the time interval $(t, t+\tau)$ with probability $K(x)\cdot\tau+o(\tau)$ when $x(t)=x$.

Now, from (5), we have for every $\varepsilon>0$

$$(11)\quad \begin{cases} \lim\limits_{\tau\downarrow 0} P(x-\varepsilon < x(t+\tau) \leqq x+\varepsilon/x(t)=x) \\ = \lim\limits_{\tau\downarrow 0} \{P(\tau, x, x+\varepsilon) - P(\tau, x, x-\varepsilon)\} \\ = \lim\limits_{\tau\downarrow 0} \left\{1 - \dfrac{G(x+\tau)-G(x)}{1-G(x)}\right\} = 1, \end{cases}$$

which proves the continuity (in probability) of $x(t)$ at any t and x.

The domains D_1, D_2 and D_3 in the 3-dimensional Euclidean (x, y, h) space are defined as follows:

$$(12)\quad \begin{cases} D_1: \ x,\ h \ \text{ and } \ y \geqq 0, & h+x \leqq y. \\ D_2: \ h \ \text{ and } \ x \geqq 0, & h \leqq y < h+x. \\ D_3: \ x \geqq 0, & 0 \leqq y < h. \end{cases}$$

THEOREM 2. $\dfrac{\partial P}{\partial h}$, $\dfrac{\partial P}{\partial x}$ *and* $\dfrac{\partial P}{\partial y}$ *exist in each domain defined in* (12).

Proof. First of all we consider $\dfrac{\partial P}{\partial h}$. In the domain D_1, our assertion is obvious. In the domain D_2, we have

$$\lim_{\tau\to 0} \frac{1}{\tau}\left\{\frac{G(x+h+\tau)-G(x)}{1-G(x)} - \frac{G(x+h)-G(x)}{1-G(x)}\right\} = \frac{g(h+x)}{1-G(x)},$$

although $\lim\limits_{\tau\to 0}$ must be taken as $\lim\limits_{\tau\downarrow 0}$ on the boundary where $h=0$. Thus P is differentiable with respect to h and

$$\frac{\partial P}{\partial h} = \frac{g(x+h)}{1-G(x)}.$$

By the assumption that $G(x) \in C^2$, we can easily see by simple calculations that $G_x(h-y)$ and $G_x * H(h-y)$ also belong to C^2-class with respect to h. Therefore, in the domain D_3

$$P(h, x, y) = \int_0^y -(1-G(u))d_u\{G_x(h-u) + G_x * H(h-u)\}$$

is differentiable under the integral sign, which proves the existence of $\dfrac{\partial P}{\partial h}$. The care on the boundary must be also taken of in this case.

Concerning $\dfrac{\partial P}{\partial x}$, the existence of it in every domain can be proved similarly. And it is easy to see that $\dfrac{\partial P}{\partial y}$ exists except for the derivatives at $y=h$ and $y=h+x$. Q.E.D.

From two theorems stated above, we can derive the fundamental differential equations. From the Chapman-Kolmogorov's equation and (5) we have

$$(13)\quad \begin{cases} P(h+\tau,\ x,\ y)=\displaystyle\int_0^\infty P(\tau,\ x,\ d\xi)P(h,\ \xi,\ y) \\ \qquad\qquad =\displaystyle\int_0^\tau P(\tau,\ x,\ d\xi)P(h,\ \xi,\ y)+\{1-P(\tau,\ x,\ \tau)\}\cdot P(h,\ \tau+x,\ y) \end{cases}$$

$$(\tau>0).$$

When $0\leqq y<h+x$, from (13), we have

$$\lim_{\tau\downarrow 0}\frac{1}{\tau}\{P(h+\tau,\ x,\ y)-P(h,\ x,\ y)\}=\lim_{\tau\downarrow 0}\frac{1}{\tau}\int_0^\tau P(\tau,\ x,\ d\xi)P(h,\ \xi,\ y)$$
$$+\lim_{\tau\downarrow 0}\frac{1}{\tau}\{P(h,\ \tau+x,\ y)-P(h,\ x,\ y)\}-\lim_{\tau\downarrow 0}P(h,\ \tau+x,\ y)\cdot\frac{1}{\tau}P(\tau,\ x,\ \tau).$$

Noting that the first term is

$$\lim_{\tau\downarrow 0}\frac{1}{\tau}\int_0^\tau P(\tau,\ x,\ d\xi)P(h,\ \xi,\ y)=\lim_{\tau\downarrow 0}\frac{1}{\tau}P(\tau,\ x,\ \tau)P(h,\ \theta\tau,\ y)\qquad(0<\theta<1)$$
$$=K(x)P(h,\ 0,\ y),$$

we have

$$\frac{\partial P(h,\ x,\ y)}{\partial h}=K(x)P(h,\ 0,\ y)+\frac{\partial P(h,\ x,\ y)}{\partial x}-K(x)P(h,\ x,\ y).$$

As an additional fact, we have a trivial equation in the domain D_1

$$\frac{\partial P(h,\ x,\ y)}{\partial h}=0.$$

Writing these equations in the following form, we shall call them *the first fundamental differential equations.*

$$(14)\quad \frac{\partial P(h,\ x,\ y)}{\partial h}-\frac{\partial P(h,\ x,\ y)}{\partial x}=K(x)\{P(h,\ 0,\ y)-P(h,\ x,\ y)\}\quad\text{in } D_2 \text{ and } D_3,$$

$$(15)\qquad \frac{\partial P(h,\ x,\ y)}{\partial h}=0\quad\text{in } D_1.$$

Next we consider the following equation that corresponds to (13).

$$(16)\quad \begin{cases} P(h,\ x,\ y)=\displaystyle\int_0^\infty P(\tau,\ \xi,\ y)P(h-\tau,\ x,\ d\xi) \qquad\qquad (\tau>0) \\ =\displaystyle\int_0^{h-\tau} P(\tau,\ \xi,\ y)P(h-\tau,\ x,\ d\xi)+\{1-P(h-\tau,\ x,\ h-\tau)\}P(\tau,\ x+h-\tau,\ y) \\ =\displaystyle\int_0^{h-\tau} P(0,\ \xi,\ y)P(h-\tau,\ x,\ d\xi)+\int_0^{h-\tau}\{P(\tau,\ \xi,\ y)-P(0,\ \xi,\ y)\}P(h-\tau,\ x,\ d\xi) \\ +\{1-P(h-\tau,\ x,\ h-\tau)\}P(\tau,\ x+h-\tau,\ y). \end{cases}$$

When $0\leqq y<h$, this implies

$$\frac{1}{\tau}\{P(h, x, y)-P(h-\tau, x, y)\}$$

$$=\frac{1}{\tau}\Big\{\int_0^{y-\tau}+\int_{y-\tau}^{y}+\int_y^{h-\tau}\Big\}\{P(\tau, \xi, y)-P(0, \xi, y)\{P(h-\tau, x, d\xi)$$

$$+\frac{1}{\tau}P(\tau, x+h-\tau, \tau)\{1-P(h-\tau, x, h-\tau)\}$$

$$=\frac{1}{\tau}\int_{y-\tau}^{y}P(\tau, \xi, \tau)P(h-\tau, x, d\xi)-\frac{1}{\tau}\{P(h-\tau, x, y)-P(h-\tau, x, y-\tau)\}$$

$$+\frac{1}{\tau}\int_y^{h-\tau}P(\tau, \xi, \tau)P(h-\tau, x, d\xi)+\frac{1}{\tau}P(\tau, x+h-\tau, \tau)\{1-P(h-\tau, x, h-\tau)\}.$$

Letting τ tend to 0, we have

$$\frac{\partial P(h, x, y)}{\partial h}=-\frac{\partial P(h, x, y)}{\partial y}+\int_y^h K(\xi)P(h, x, d\xi)+K(x+h)\{1-P(h, x, h)\}.$$

Thus we have

$$\frac{\partial p(h, x, y)}{\partial h}+\frac{\partial p(h, x, y)}{\partial y}=-K(y)p(h, x, y) \quad \text{in } D_3, \tag{17}$$

where $p(h, x, y)$ is the density function in y of $P(h, x, y)$. For the particular choice $x=0$ in (17), we have

$$\frac{\partial p(h, 0, y)}{\partial h}+\frac{\partial p(h, 0, y)}{\partial y}=-K(y)p(h, 0, y), \quad \text{if } 0\leqq y<h. \tag{17'}$$

And we have a trivial equation

$$\frac{\partial P(h, x, y)}{\partial y}=0 \quad \text{in } D_1 \text{ and } D_2. \tag{18}$$

(17′) (instead of (17)) and (18) will be called *the second fundamental differential equations.*

THEOREM 3. *The transition probability of the renewal process satisfies the first and the second fundamental differential equations.*

§ 3. Integrations of the fundamental differential equations

We intend to integrate the fundamental differential equations under the following conditions:

$$(19)\begin{cases} 1^\circ)\ P(h,x,y)=0, \text{ if } \int_0^x K(t)dt=\infty \text{ or } y\leqq 0, \\ 2^\circ)\ P(h,x,y) \text{ is an absolutely continuous distribution function in } y \\ \qquad \text{in each domain } D_1, D_2 \text{ and } D_3 \text{ respectively,} \\ 3^\circ)\ P(0,x,y)=\begin{cases}0 & , \text{ if } x<y, \\ 1 & , \text{ if } x\leqq y,\end{cases} \\ 4^\circ)\ \lim\limits_{\substack{x\downarrow 0 \\ h+x>y\geqq h}} P(h,x,y)=\lim\limits_{y\uparrow h} P(h,0,y). \end{cases}$$

For this purpose, we further assume the conditions:

$$(20)\begin{cases} 1^\circ)\ K(x) \text{ is non-negative and belongs to } C^1\text{-class,} \\ 2^\circ)\ \lim\limits_{x\to\infty}\int_0^x K(t)dt=\infty, \\ 3^\circ)\ 0<\int_0^\infty t^2 d\{1-\exp(-\int K(t)dt)\}<\infty. \end{cases}$$

Now the integration is performed in each of the domain D_1, D_2 and D_3.

Case I. In the domain D_1. In this case P depends neither y nor h by virtue of (15) and (18). Hence we may write

$$(21)\qquad P(h,x,y)=P(0,x,y)=1.$$

The last equality is implied by (19) 3°) since $x\leqq y$.

Case II. In the domain D_2. From (18) we see that P is independent of y. Being reduced to the Case I, it is proved that $P(h,0,y)=1$.

Therefore we may write (14) in the following form

$$(14')\qquad \frac{\partial P(h,x,y)}{\partial h}-\frac{\partial P(h,x,y)}{\partial x}=K(x)\{1-P(h,x,y)\}.$$

Solving the characteristic equation

$$\frac{dh}{1}=\frac{dx}{-1}=\frac{dP}{K(x)(1-P)},$$

we have

$$\begin{cases} x+h=\alpha \\ (1-P)\exp(-\int^x K(t)dt)=\beta, \end{cases}$$

where α and β are constants. Let $g(x)$ be an arbitrary function. Then the solution of (14′) may be written as

$$(1-P)\exp(-\int^{x}K(t)dt)=g(x+h),$$

that is,

$$P(h,\ x,\ y)=1-\exp(\int^{x}K(t)dt)\cdot g(x+h). \tag{22}$$

We can determine the explicit form of $g(x)$ by virtue of the boundary condition (19) 3°):

$$P(0,\ x,\ y)=1-\exp(\int^{x}K(t)dt)g(x)=0.$$

Hence we have

$$P(h,\ x,\ y)=1-\exp(-\int_{x}^{x+h}K(t)dt). \tag{23}$$

Cose III. In the domain D_3. First we must find the functional form of $P(h,\ 0,\ y)$ in this domain. Solving the characteristic equation of (17′)

$$\frac{dh}{1}=\frac{dy}{1}=\frac{dp}{-K(y)p},$$

we have

$$\begin{cases} h-y=\alpha \\ p\ \exp(\int^{y}K(t)dt)=\beta, \end{cases}$$

where α and β are constants. Let f be an arbitrary function. Then we have

$$p(h,\ 0,\ y)=\exp(-\int_{0}^{y}K(t)dt)f(h-y).$$

Because of the probabilistic interpretation, $f(h)$ must be non-negative for $h\geqq 0$ and $f(h)=0$ for $h<0$. Therefore we have

$$P(h,\ 0,\ y)=\int_{0}^{y}E(u)f(h-u)du, \tag{24}$$

where $$E(u)=\begin{cases}\exp(-\int_{0}^{u}K(t)dt) & ,\ \text{if}\ \ u\geqq 0, \\ 1 & ,\ \text{if}\ \ u<0.\end{cases}$$

Letting y increase to h, $P(h,\ 0,\ y)$ tends to

$$\int_0^h E(u)f(h-u)du \tag{25}$$

which is equal to $1-E(h)$, the limiting value of (23), by virtue of the continuity assumption (19) 4°). Let $F(h)$ be $\int_0^h f(u)du$. Then (25) becomes

$$F(h)-(1-E)*F(h).$$

As this is equal to $1-E(h)$, we have

$$F(h)=1-E(h)+(1-E)*F(h).$$

By induction, for any positive integer n, we have

$$F(h)=1-E(h)+(1-E)^{2*}(h)+\dots+(1-E)^{n*}*F(h).$$

This convolution can be defined since $1-E(h)$ is a distribution function and $F(h)$ is monotone non-decreasing function vanishing for $h<0$.

Now, $1-E(x)$ is a distribution function with finite variance by the assumption (20) 3°). Therefore we have

$$(1-E)^{n*}*F(h)\to 0^{3)}(n\to\infty) \quad \text{for every } h \geqq 0.$$

It is well known that the series $\sum_{n=1}^{\infty}(1-E)^{n*}(h)$ converges. Hence we have

$$F(h)=\sum_{n=1}^{\infty}(1-E)^{n*}(h). \tag{26}$$

Therefore we obtain

$$P(h,0,y)=\int_0^y -E(u)d_uF(h-u). \tag{27}$$

Using this, we can find the solution of (14). Let $Z=\exp\left(-\int_0^x K(t)dt\right)\cdot P$. Then, by a simple calculation, (14) becomes

$$\frac{\partial Z}{\partial h}-\frac{\partial Z}{\partial x}=-E'(x)P(h,0,y). \tag{14'}$$

The characteristic curve is given by

$$x=-s+a, \qquad h=s+b,$$

$$Z=\int_0^s -E'(-\sigma+a)\int_0^y -E(u)d_uF(\sigma+b-u)d\sigma+c,$$

where s is a parameter and a, b and c are constants. It is noted that y is con-

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[3] To prove it, it suffices to show that $(1-E)^{n*}(h)\to 0$ $(n\to\infty)$. If we apply the law of large numbers to the independent non-negative random variables which have the common distribution function $1-E(x)$, we can easily prove the convergence.

sidered as a constant there. Hence we may determine the initial curve on the boundary line $x=0$, $h \geqq y$ in the (x, h) plane as follows:

$$x=0,\ h=t+y, \qquad Z=P(h,\ 0,\ y)=\int_0^y -E(u)d_uF(y+t-u),$$

where t is a parameter. The integral surface of (14′) in question is given by

$$x=x(s,\ t)=-s, \qquad h=h(s,\ t)=s+t+y,$$
$$Z=Z(s,\ t)=\int_0^s\int_0^y E'(-\sigma)E(u)d_uF(\sigma+t+y-u)d\sigma+\int_0^y -E(u)d_uF(y+t-u).$$

Eliminating the parameters, we have

$$\begin{aligned} Z&=\int_0^y\int_0^{-x} E'(-\sigma)E(u)d_uF(\sigma+h+x-u)d\sigma+\int_0^y -E(u)d_uF(h+x-u)\\ &=-\int_0^y\int_0^x E'(\sigma)E(u)d_uF(h+x-\sigma-u)d\sigma+\int_0^y -E(u)d_uF(h+x-u). \end{aligned}$$

Thus we obtain

$$(28)\quad \left\{ \begin{aligned} P(h,\ x,\ y)=\frac{1}{E(x)}\Big\{\int_0^y\int_0^x -E(u)d_uF(h+x-\sigma-u)dE(\sigma)\\ +\int_0^y -E(u)d_uF(h+x-u)\Big\}. \end{aligned}\right.$$

There remains only to show that $P(h, x, y)$ given by (21), (23) and (28) is certainly a transition probability of the renewal process considered at the begining. For this purpose, it is sufficient to show that $P(h, x, y)$ given by (21), (23) and (28) is the same as the one given by (15). In the domain D_1 and D_2, our assertion is clear, if we let $1-E(x)$ and $F(x)$ correspond to $G(x)$ and $H(x)$ respectively. Let us compare two density functions with respect to y in the domain D_3. Noting that

$$\begin{aligned} (G_x*H)(h)&=\frac{1}{1-G(x)}\Big\{\int_0^h\Big\{G(x+h-s)-G(x)\Big\}dH(s)\Big\}\\ &=\frac{1}{1-G(x)}\Big\{\int_0^h G(x+h-s)dH(s)-G(x)H(h)\Big\}, \end{aligned}$$

we have the density function of (5)

$$(29)\qquad \frac{1-G(y)}{1-G(x)}\Big\{\int_0^{h-y} G'(x+h-y-s)H'(s)ds+G'(x+h-y)\Big\}.$$

On the other hand, the density function of (28) is

$$(30)\qquad \frac{E(y)}{E(x)}\left\{\int_0^x F'(x+h-y-\sigma)dE(\sigma)+F'(h+x-y)\right\}.$$

By replacing $E(x)$ and $F(x)$ by $1-G(x)$ and $H(x)$ respectively, (30) can be written in the form

$$\frac{1-G(y)}{1-G(x)}\left\{-\int_0^{x+h-y} G'(x+h-y-s)H'(s)ds-\int_{h-y}^0 G'(x+h-y-s)H'(s)ds\right.$$
$$\left.+H'(h+x-y)\right\}=\frac{1-G(y)}{1-G(x)}\left\{\int_0^{h-y} G'(x+h-y-s)H'(s)ds+G'(x+h-y)\right\},$$

which is identical with (29). Thus we can conclude that the assertion stated above holds.

Furthermore we can prove, putting $y=h$ in (28), the existence of $\lim_{h\downarrow 0}\frac{1}{h}$ $P(h, x, h)$. This is just the $K(x)$ obtained in Theorem 1. And it is obvious that the function $P(h, x, y)$ defined by (21), (23) and (28) satisfies the differential equation (14), (15), (17) and (18). Summing up we have

THEOREM 4. *The transition probability of the renewal process is completely determined by the first and the second fundamental differential equations.*

From this theorem, we may conclude that the stationary renewal process is uniquely determined by the first and the second fundamental differential equations since even the initial distribution can be determined uniquely by such a distribution function

$$\Phi(x)=\frac{1}{m}\int_0^x \exp(-\int_0^u K(t)dt)du,$$

where

$$m=\int_0^\infty xd\{1-\exp(-\int_0^x K(t)dt)\}.$$

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