ON THE DIMENSION OF MODULES AND ALGEBRAS, V.

DIMENSION OF RESIDUE RINGS

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We shall consider a semi-primary ring Λ with radical N (i.e. N is nilpotent and Λ/N is semi-simple (with minimum condition)). All modules considered are left Λ-modules. We refer to [1] for all notions relevant to homological algebra.

The objective of this paper is to establish the following two theorems:

THEOREM I. *Let a be a two-sided ideal in Λ such that*

$$a \subset N^2, \qquad \text{gl.dim}(\Lambda/a) \leqq 1.$$

Then $a = 0$.

THEOREM II. *Let r be a right ideal in Λ such that*

$$rN \subset Nr \subset N^2, \qquad \text{gl.dim}(\Lambda/Nr) \leqq n, \qquad n > 1.$$

Then $Nr^{n-1}N = 0$.

Taking $r = N^{k-1}$, $k > 1$ we obtain

COROLLARY II′. *If* $\text{gl.dim}(\Lambda/N^k) \leqq n$, $k > 1$, $n > 1$, *then* $N^{(n-1)(k-1)+2} = 0$.

COROLLARY II″. *If* $\text{gl.dim}(\Lambda/N^2) \leqq n$, $n \geqq 0$, *then* $N^{n+1} = 0$.

In this last corollary we admitted also the cases $n = 0$ (since $N/N^2 = 0$ implies $N = 0$) and $n = 1$ (by Theorem I). The result stated in Corollary II″ is the best possible. Indeed, in [3, Proposition 12 and Corollary 11], for each $n \geqq 0$, a semi-primary ring Λ was constructed such that

$$\text{gl.dim}\,\Lambda \leqq 1, \qquad \text{gl.dim}(\Lambda/N^2) = n, \qquad N^{n+1} = 0, \qquad N^n \neq 0.$$

Let $\varphi: P \to A$ be an epimorphism of Λ-modules. We say that φ is *minimal* if P is projective and $\text{Ker}\,\varphi \subset NP$. We see without much difficulty

(i) For each Λ-module A there is a minimal epimorphism $\varphi: P \to A$;

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(ii) If φ': $P' \to A$ is another minimal epimorphism then there exists an isomorphism π: $P \to P'$ such that $\varphi'\pi = \varphi$;

for the detailed account under a more general setting, see [2].

Let a be any subset of Λ. We define the orthogonality relation $a \perp A$ by the condition $aP = 0$, where P is the projective module occuring in the minimal epimorphism for A. Clearly $a \perp A$ implies $aA = 0$.

LEMMA 1. *If $B \subset NA$ then the relations $a \perp A$ and $a \perp A/B$ are equivalent.*

LEMMA 2. *If $B \subset NA$ and A/B is projective then $B = 0$.*

Proof. Consider the composition

$$P \xrightarrow{\varphi} A \xrightarrow{\psi} A/B$$

where φ is the minimal epimorphism for A and ψ is the natural factorization epimorphism. Since $\operatorname{Ker}\varphi \subset NP$ we have $\varphi^{-1}(B) \subset \varphi^{-1}(NA) = NP$. Thus $\operatorname{Ker}(\psi\varphi) \subset NP$ and $\psi\varphi$ is a minimal epimorphism for A/B. Consequently each of the conditions $a \perp A$, $a \perp A/B$ is equivalent with $aP = 0$. If A/B is projective, then, by (ii) $\psi\varphi$ is an isomorphism. Thus ψ is an isomorphism and $B = 0$.

LEMMA 3. *Let a be a two-sided ideal in Λ, A a Λ-module and B a submodule such that*

$$aA \subset B \subset NA, \qquad A/B \text{ is } (\Lambda/a)\text{-projective.}$$

Then $aA = B$.

Proof. Consider the ring $\Lambda' = \Lambda/a$ with radical $N' = (N + a)/a$. The Λ'-modules $A' = A/aA$, $B' = B/aA$ then satisfy

$$B' \subset N'A', \qquad A'/B' \text{ is } \Lambda'\text{-projective.}$$

Thus, by Lemma 2, $B' = 0$ i.e. $aA = B$.

Proof of Theorem I. Since $\text{gl.dim}\,(\Lambda/a) \leqq 1$ we have $\text{l.dim}_{\Lambda/a}(\Lambda/N) \leqq 1$. From the exact sequence $0 \to N/a \to \Lambda/a \to \Lambda/N \to 0$ it follows that N/a is (Λ/a)-projective. Since $aN \subset a \subset NN$ we may apply Lemma 3 with (A, B) replaced by (N, a). Thus $aN = a$ and $a = 0$.

PROPOSITION 4. *Let r be a right ideal in Λ and A a left Λ-module. If*

$$rN \subset Nr, \qquad rA = 0, \qquad \text{l.dim}_{\Lambda/Nr} A \leqq n, \qquad n > 0$$

then $Nr^n \perp A$.

Proof. Let $\varphi: P \to A$ be a minimal epimorphism. Since $rA = 0$ it follows that $rP \subset \operatorname{Ker} \varphi$. If we write $C = \operatorname{Ker} \varphi$, there results an exact sequence

$$0 \to C \to P \overset{\varphi}{\to} A \to 0$$

such that

$$rP \subset C \subset NP.$$

Since $NrP \subset \operatorname{Ker} \varphi$ we derive an exact sequence

$$0 \to c/NrP \to P/NrP \to A \to 0$$

of (Λ/Nr)-modules. Since P/NrP is (Λ/Nr)-projective (see [3, Prop. 1]), we have

$$(*) \qquad \text{l.dim}_{\Lambda/Nr}(C/NrP) \leqq n-1.$$

Now consider the case $n = 1$. Since

$$NrC \subset NrP \subset NC$$

we may apply Lemma 3 with (A, B, a) replaced by (C, NrP, Nr). We obtain $NrC = NrP$. Since $C \subset NP$ and $rN \subset Nr$ we have

$$NrP = NrC \subset NrNP \subset N^2rP.$$

Thus $NrP = 0$ i.e. $Nr \perp A$.

For $n > 1$ we proceed by induction and assume the proposition valid for $n-1$. Since $rC \subset rNP \subset NrP$ we have $r(C/NrP) = 0$ and thus $(*)$ yields

$$Nr^{n-1} \perp C/NrP.$$

However $NrP \subset NC$, so that, by Lemma 1, $Nr^{n-1} \perp C$. Consequently $Nr^nP \subset Nr^{n-1}C = 0$ and $Nr^n \perp A$.

Proof of Theorem II. Since $\text{gl.dim}(\Lambda/Nr) \leqq n$ we have $\text{l.dim}_{\Lambda/Nr}(\Lambda/N) \leqq n$. From the exact sequence $0 \to N/Nr \to \Lambda/Nr \to \Lambda/N \to 0$ it follows

$$\text{l.dim}_{\Lambda/Nr}(N/Nr) \leqq n-1.$$

Since $rN \subset Nr$ we have $r(N/Nr) = 0$ and thus, by Proposition 4,

$$Nr^{n-1} \perp N/Nr.$$

Since $Nr \subset NN$, it follows from Lemma 1 that

$$Nr^{n-1} \perp N.$$

Thus $Nr^{n-1}N = 0$, as required.

References

[1] H. Cartan and S. Eilenberg, Homological Algebra, Princeton Univ. Press, 1956.
[2] S. Eilenberg, Homological dimension and syzygies, Ann. Math., 64 (1956), 328-336.
[3] S. Eilenberg, H. Nagao and T. Nakayama, On the dimension of modules and algebras, IV, Nagoya Math. J., 10 (1956), 87-95.

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