ON A PROBLEM OF CHEVALLEY

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In the present note we wish to deal with the same problem as the preceding paper [1] for the case of modular fields.

Let k be a field of characteristic $p \neq 0$ and $K = k(x_1, \ldots, x_p)$ a purely transcendental extension of k. Let S be the automorphism of K which is induced by the cyclic permutation $(x_1, \ldots, x_p)$ and L the fixed subfield of S. Then *L is a purely transcendental extension field over k.*

Proof. We put

$$\begin{aligned} u_1 &= x_1 + x_2 + x_3 + \ldots + x_p, \\ u_2 &= (1x_2 + 2x_3 + \ldots + (p-1)x_p)/u_1, \\ u_3 &= (1^2x_2 + 2^2x_3 + \ldots + (p-1)^2x_p)/u_1, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ u_p &= (1^{p-1}x_2 + 2^{p-1}x_3 + \ldots + (p-1)^{p-1}x_p)/u_1. \end{aligned}$$

Since $u_1, u_2u_1, u_3u_1, \ldots, u_pu_1$ are linear forms in $x_1, \ldots, x_p$ and their determinant is $\prod_{p > i > j \geqq 0} (i-j) \neq 0$,

$$K = k(x_1, \ldots, x_p) = k(u_1, u_2u_1, \ldots, u_pu_1) = k(u_1, u_2, \ldots, u_p).$$

To see the effect of S on u_i, we compute $S^{-1}u_i - u_i (= \Delta u_i)$ intstead of Su_i.

$$\begin{aligned} \Delta u_1 &= 0, \\ \Delta u_2 &= 1, \\ \Delta u_3 &= 2u_2 + 1, \\ &\ldots\ldots\ldots \\ \Delta u_{i+1} &= \binom{i}{1} u_i + \binom{i}{2} u_i + \ldots + \binom{i}{i-1} u_2 + 1, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

From these u_i we now construct new elements $v_2(= u_2), v_3, \ldots, v_p \in K$ such that

Received June 2, 1954.

$$\Delta v_i = 1,$$
$$v_i = u_i + f_i(v_2, \dots, v_{i-1}),$$

where f_i is a linear form of v_j^e, $j = 2, \dots, i-1$, $e = 0, \dots, i-1$, with coefficients in the prime field. We take, at first, $v_2 = u_2$, $v_3 = u_3 - u_2^2 + u_2$ and construct them by induction. If we get first $i-2$ terms $v_2, \dots, v_{i-1}$, then v_i is obtained as follows:

$$\Delta u_i = \binom{i-1}{1} u_{i-1} + \binom{i-1}{2} u_{i-2} + \dots + 1$$

(1)
$$= \binom{i-1}{1}(v_{i-1} - f_{i-1}(v_2, \dots, v_{i-2}))$$
$$+ \binom{i-1}{2}(v_{i-2} - f_{i-2}(v_2, \dots, v_{i-3})) + \dots + 1.$$

The right side of this relation is a linear form of v_j^e, $j = 2, \dots, i-2$, $e = 0, \dots, i-2$. We compute Δv_j^e, using the inductive assumption $\Delta v_j = 1$,

$$\Delta v_j = 1,$$
$$\Delta v_j^2 = 1 + 2v_j,$$
$$\Delta v_j^3 = 1 + 3v_j + 3v_j^2,$$
$$\dots\dots\dots\dots$$
$$\Delta v_j^e = 1 + \binom{e}{1} v_j + \binom{e}{2} v_j^2 + \dots + \binom{e}{e-1} v_j^{e-1}.$$

From these relations we solve v_j^e in a linear form of $\Delta v_j^{e'}$.

(2)
$$v_j^e = h_j(\Delta v_j, \Delta v_j^2, \dots, \Delta v_j^{e+1}) = \Delta h_j(v_j, v_j^2, \dots, v_j^{e+1}),$$
$$1 \leqq e \leqq i - 2 < p,$$

where h_j is a linear form in its arguments. We put (2) into (1), then

$$\Delta u_i = \Delta g_i(v_2, \dots, v_{i-1}),$$

where g_i is a linear form of v_j^e, $j = 2, \dots, i-1$, $e = 0, \dots, i-1$. Since

$$\Delta [u_i - g_i(v_2, \dots, v_{i-1})] = 0,$$

the element

$$u_i - g_i(v_2, \dots, v_{i-1}) + v_2$$

satisfies the inductive assumption and we may take it as v_i.

Now, we construct algebraically independent generators of L over k. We put

$$w_1 = u_1,$$
$$w_2 = v_2^p - v_2,$$
$$w_i = v_i - u_2, \qquad i = 3, \ldots, p.$$

Then $$\Delta w_i = 0, \qquad i = 1, \ldots, p,$$

hence $$k(w_1, \ldots, w_p) < L.$$

On the other hand

$$[k(w_1, \ldots, w_p, v_2) : k(w_1, \ldots, w_p)] \leqq p,$$
$$k(w_1, \ldots, w_p, v_2) \supset k(u_i, \ldots, u_p) = K,$$

and $[K : L] = p$. Therefore

$$L = k(w_1, \ldots, w_p).$$

Since L is an extension field of dimension (degree of transcendency) p over k, we see that $w_1, \ldots, w_p$ are algebraically independent over k.

REFERENCE

[1] K. Masuda: On a problem of Chevalley, this journal.

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