ON THE DIMENSION OF MODULES AND ALGEBRAS, I

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In [5], Ikeda-Nagao-Nakayama gave a characterization of algebras of cohomological dimension $\leqq n$. In a subsequent paper [4] Eilenberg gave an alternative treatment of the same question. The present paper is devoted to the discussion of a number of questions suggested by the results of [4] and [5]. Among others it is shown that the conditions employed in stating the main results in [4] and [5] are equivalent, so that the main results of these two papers are in accord. Further, the cohomological dimension of a residue-algebra is studied in terms of that of the original algebra and the (module-) dimension of the associated ideal. The terminology and notation employed here are that of [3].

§ 1. Modules and quasi-modules

Throughout this paper, Λ will denote an algebra over a commutative ring K. It is always assumed that Λ has a unit, and this unit acts as the identity on all Λ-modules.

In addition to Λ-modules we shall also consider quasi-modules in which it is no longer assumed that the unit element 1 of Λ operates as the identity; however the unit element ε of K still operates as the identity. Explicitly a (left) Λ-quasi-module is a K-module A together with a homomorphism

$$\Lambda \otimes_K A \to A$$

satisfying

$$\gamma(\lambda a) = (\gamma\lambda)a \qquad (\gamma, \lambda \in \Lambda; a \in A)$$

where λa is the image of $\lambda \otimes a$.

Clearly each Λ-module is a Λ-quasi-module. Further each K-module A may

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also be regarded as a Λ-quasi-module with $\lambda a = 0$ for all $\lambda \in \Lambda$, $a \in A$. In a sense, these two classes exhaust the picture. Indeed, for each Λ-quasi-module A we have the direct sum decomposition (due to Peirce)

$$A = 1A + A^{\bullet}$$

where $A^{\bullet}$ consists of all elements $a \in A$ with $1a = 0$. Clearly $1A$ is a Λ-module, while $A^{\bullet}$ is just a K-module converted into a Λ-quasi-module as above.

A Λ-module A is ***projective*** if for every epimorphism (i.e. onto-homomorphism)

$$\varphi : B \to A$$

of Λ-modules, there exists a Λ-homomorphism $\psi : A \to B$ such that $\varphi\psi = \text{identity}$. A is said to be ***injective*** if for each monomorphism (i.e. (into-)isomorphism)

$$\varphi : A \to C$$

of Λ-modules, there exists a Λ-homomorphism $\psi : C \to A$ with $\psi\varphi = \text{identity}$.

Replacing in the above definitions all modules by quasi-modules we obtain the notions of a projective quasi-module and of an injective quasi-module.

PROPOSITION 1. A Λ-quasi-module A is projective [injective] if and only if $1A$ is a projective [injective] Λ-module and $A^{\bullet}$ is a projective [injective] K-module.

Proof. Let $\varphi : B \to A$ be an epimorphism of Λ-quasi-modules. Then φ decomposes into two components

$$\varphi_1 : 1B \to 1A, \qquad \varphi_2 : B^{\bullet} \to A^{\bullet}$$

A map $\psi : A \to B$ with $\varphi\psi = \text{identity}$ exists if and only if such maps exist for φ_1 and φ_2. This yields the desired conclusion.

This proposition implies that a Λ-module A is projective [injective] if and only if it is projective [injective] as a quasi-module.

§ 2. The Hochschild quasi-operators

It will be convenient to denote by Λ^n the n-fold tensor product $\Lambda \otimes \ldots \otimes \Lambda$ where $\otimes = \otimes_K$. We may regard Λ^n as a two-sided Λ-module by setting

$$\lambda(\lambda_1 \otimes \ldots \otimes \lambda_n) = \lambda\lambda_1 \otimes \ldots \otimes \lambda_n,$$
$$(\lambda_1 \otimes \ldots \otimes \lambda_n)\lambda = \lambda_1 \otimes \ldots \otimes \lambda_n\lambda.$$

We consider the complex $S(\Lambda)$ with

$$S_n(\Lambda)=\Lambda^{n+2} \qquad n=0,1,\ldots,$$

$$d(\lambda_0\otimes\ldots\otimes\lambda_{n+1})=\sum_{i=0}^{n}(-1)^i\lambda_0\otimes\ldots\otimes\lambda_i\lambda_{i+1}\otimes\ldots\otimes\lambda_{n+1}$$

and with the augmentation

$$\varepsilon: S_0(\Lambda)=\Lambda\otimes\Lambda\to\Lambda$$

given by $\varepsilon(\lambda_0\otimes\lambda_1)=\lambda_0\lambda_1$. This complex is acyclic as can be easily seen using the homotopy operator $\zeta: S_n(\Lambda)\to S_{n+1}(\Lambda)$ given by $\zeta x=1\otimes x$, $x\in S_n(\Lambda)$.

If Λ is assumed to be K-projective, then each Λ^n $(n>1)$ is easily seen to be a $\Lambda\otimes\Lambda^*$-projective module, where Λ^* is the inverse ring of Λ. Thus in this case $S(\Lambda)$ is $\Lambda\otimes\Lambda^*$-projective resolution of Λ. This is the *standard complex* of Λ as defined in [3] (Ch. IX, § 2).

Now let A be a left Λ-module which is K-projective. We consider the complex (of left Λ-modules)

$$S(A)=S(\Lambda)\otimes_\Lambda A.$$

It is easy to see that $S(A)$ is a projective resolution of A. We have

$$\begin{aligned}S_n(A)&=S_n(\Lambda)\otimes_\Lambda A=\Lambda^{n+2}\otimes_\Lambda A\\&=\Lambda^{n+1}\otimes\Lambda\otimes_\Lambda A=\Lambda^{n+1}\otimes A.\end{aligned}$$

In this notation we have

$$\begin{aligned}d(\lambda_0\otimes\ldots\otimes\lambda_n\otimes a)=&\sum_{i=0}^{n-1}(-1)^i\lambda_0\otimes\ldots\otimes\lambda_i\lambda_{i+1}\otimes\ldots\otimes\lambda_n\otimes a\\&+(-1)^n\lambda_0\otimes\ldots\otimes\lambda_{n-1}\otimes\lambda_n a.\end{aligned}$$

Since the complex $S(A)$ is acyclic, we have

$$B_n(S(A))=Z_n(S(A)).$$

Consequently we have the exact sequence

$$0\to B_n(S(A))\to S_n(A)\to\ldots\to S_0(A)\to A\to 0.$$

Since $S_i(A)$ are Λ-projective, it follows that $B_n(S(A))$ is Λ-projective if and only if $\text{l.dim}_\Lambda A\leqq n+1$.

In addition to the already present Λ-operators on $S_n(A)=\Lambda^{n+1}\otimes A$ we introduce Λ-quasi-operators as follows

$$(*) \qquad \lambda * x = d(\lambda \otimes x) = \lambda x - \lambda \otimes dx.$$

We calculate

$$\gamma * (\lambda * x) = \gamma * d(\lambda \otimes x) = \gamma d(\lambda \otimes x) = d(\gamma\lambda \otimes x) = (\gamma\lambda) * x$$

so that indeed we have quasi-operators.

PROPOSITION 2. If Λ and the left Λ-module A are both K-projective then for each $n > 0$ the following properties are equivalent:

(i) $\text{l.dim}_\Lambda A \leqq n$,

(ii) the left Λ-module $B_{n-1}(S(A))$ is projective,

(iii) the left Λ-module $1 * (\Lambda^n \otimes A)$ is projective,

(iv) the left Λ-quasi-module $\Lambda^n \otimes A$ is projective.

Proof. The equivalence of (i) and (ii) has already been asserted above. We prove the equivalence of (ii) and (iii) by showing that $B_{n-1}(S(A))$ and $1 * (\Lambda^n \otimes A)$ coincide as Λ-modules. We have

$$1 * x = d(1 \otimes x) \in B_{n-1}(S(A)),$$
$$d(\lambda \otimes x) = \lambda * x = 1 * (\lambda * x) \in 1 * (\Lambda^n \otimes A)$$

which shows that $B_{n-1}(S(A))$ and $1 * (\Lambda^n \otimes A)$ coincide as groups. Further if $x \in B_{n-1}(S(A))$ then $dx = 0$ and thus $(*)$ yields $\lambda * x = \lambda x$ so that the Λ-operators also coincide.

To prove the equivalence of (iii) and (iv) consider the direct sum decomposition

$$\Lambda^n \otimes A = 1 * (\Lambda^n \otimes A) + (\Lambda^n \otimes A)^\bullet.$$

Since $\Lambda^n \otimes A$ is K-projective it follows that $(\Lambda^n \otimes A)^\bullet$ is K-projective. The conclusion thus follows from Prop. 1.

Remark. If $n = 0$ then $B_{-1}(S(A))$ should be interpreted as the image of the augmentation $\Lambda \otimes A \to A$; thus $B_{-1}(S(A)) = A$. Further if we interpret $\Lambda^0 = K$ then $\Lambda^0 \otimes A = A$. The quasi-operators are $\lambda * a = d(\lambda \otimes a) = \lambda a$ and coincide with the operators. With these interpretations Prop. 2 remains valid also for $n = 0$.

§ 3. Discussion of dim Λ.

Using the results of § 2 it is now possible to close the gap between [4] and [5]. First we give a glossary translating the terminology used here into that of [5] and [6]:

module——module M satisfying $M = 1\,M$,
quasi-module——module,
projective quasi-module——(M_0)-module,
injective quasi-module——(M_u)-module.

Let Λ be a K-algebra. The (cohomological) dimension of Λ may be defined as follows: $\dim \Lambda \leqq n$ if and only if the cohomology groups $H^q(\Lambda, A)$ vanish for all $q > n$ and all two-sided Λ-modules A.

Assume that K is a field and that $(\Lambda : K) < \infty$. Let N denote the radical of Λ. The main result of [5] may now be stated as follows:

For $n > 0$, the condition

(a) $$\dim \Lambda \leqq n$$

is equivalent with the set of two conditions

(b) $$\Lambda/N \text{ is separable,}$$

(c) $$1 * (\Lambda^{n-1} \otimes N) \text{ is projective.}$$

In view of Prop. 2 (c) is equivalent with

(c′) $$\text{l. dim}_\Lambda N \leqq n$$

which is in turn equivalent with

(c″) $$\text{l. dim}_\Lambda (\Lambda/N) \leqq n.$$

This is the form of the result as established in [4]. Actually if (c″) is used, the main result remains valid also for $n = 0$.

Remark. In [5] it is proved also that (a) implies

(c_0) $$1 * (\Lambda^{n-1} \otimes \mathfrak{l}) \text{ is projective for any left ideal } \mathfrak{l} \text{ of } \Lambda.$$

This is equivalent to

(c_0') $$\text{l. dim}_\Lambda \mathfrak{l} < n$$

or

(c_0'') $$\text{l. dim}_\Lambda (\Lambda/\mathfrak{l}) \leqq n.$$

This last inequality is a consequence of the general inequality l. gl. dim $\Lambda \leqq \dim \Lambda$ (see [4], Corollary 5).

§ 4. An inequality

Let Λ and Λ' be rings and

$$\varphi : \Lambda \to \Lambda'$$

a ring homomorphism. By means of this homomorphism, each left Λ'-module may also be regarded as a left Λ-module.

PROPOSITION 3. For each left Λ'-module A we have

$$\text{l.dim}_\Lambda A \leqq \text{l.dim}_{\Lambda'} A + \text{l.dim}_\Lambda \Lambda'.$$

Proof. This proposition could be derived directly from a spectral sequence established in [3] (Ch. XVI, § 5), however we shall give an elementary inductive proof here.

Let $p = \text{l.dim}_{\Lambda'} A$ and $q = \text{l.dim}_\Lambda \Lambda'$. Clearly we may assume that p and q are finite. For each free Λ'-module F we have $\text{l.dim}_\Lambda F = q$, and therefore for each direct summand P of F we have $\text{l.dim}_\Lambda P \leqq q$. This proves the proposition if A is Λ'-projective i.e. if $p = 0$.

From here we proceed by induction with respect to p. We assume $p > 0$ and assume that the proposition holds for Λ'-modules A of left dimension (over Λ') smaller than p. Let

$$0 \to B \to X \to A \to 0$$

be an exact sequence of Λ'-modules with X Λ'-projective. Then

$$\text{l.dim}_{\Lambda'} X = 0, \qquad \text{l.dim}_{\Lambda'} B = p - 1$$

and therefore by the inductive assumption

$$\text{l.dim}_\Lambda X \leqq p < p + q, \qquad \text{l.dim}_\Lambda B < p + q.$$

For each left Λ-module C we have the exact sequence

$$\text{Ext}_\Lambda^{p+q}(B, C) \to \text{Ext}_\Lambda^{p+q+1}(A, C) \to \text{Ext}_\Lambda^{p+q+1}(X, C)$$

and since the extreme terms are zero, so is $\text{Ext}_\Lambda^{p+q+1}(A, C)$. Thus $\text{l.dim}_\Lambda A \leqq p + q$, as required.

COROLLARY 4. If Λ' is semi-simple, then

$$\text{l.dim}_\Lambda A \leqq \text{l.dim}_\Lambda \Lambda'$$

for each left Λ'-module A.

THEOREM 5. *Let Λ be a K-algebra over a field K with $(\Lambda : K) < \infty$, and let $\mathfrak{l}$ be a two-sided ideal contained in the radical N of Λ. Denoting $\Lambda' = \Lambda/\mathfrak{l}$, we have*

$$\dim \Lambda \leqq \dim \Lambda' + \mathrm{l.\,dim}_\Lambda \Lambda'.$$

Proof. Let $N' = N/\mathfrak{l}$. Then N' is the radical of Λ' and $\Lambda/N \cong \Lambda'/N'$. Clearly we may assume that $\dim \Lambda' < \infty$. This implies that Λ'/N' is separable (see preceeding section). Since both Λ/N and Λ'/N' are separable it follows from the preceeding section that

$$\dim \Lambda = \mathrm{l.\,dim}_\Lambda (\Lambda/N) = \mathrm{l.\,dim}_\Lambda (\Lambda'/N')$$
$$\dim \Lambda' = \mathrm{l.\,dim}_{\Lambda'} (\Lambda'/N').$$

Thus the desired inequality follows from Prop. 3 with $A = \Lambda'/N'$.

Remark. If instead of $\mathfrak{l} \subset N$ we have $N \subset \mathfrak{l}$ then Cor. 4 is applicable.

§ 5. Cartan Matrix

In proving that if $\dim \Lambda < \infty$ then Λ/N is separable an important role is played by the Cartan matrix $M(\Lambda)$. In fact, denoting by Λ_L the algebra obtained from Λ by passing to the algebraic closure L of K, it was proved in [4] and [5] that if $\dim \Lambda < \infty$ then $\det M(\Lambda_L) = \pm 1$. An algebra Λ is called *primary* if Λ/N is simple. A direct product (sum) of a finite number of primary algebras is called *primarily decomposable.* An algebra Λ is called *absolutely primarily decomposable* if for each extension K' of K, the algebra $\Lambda_{K'}$ is primarily indecomposable. It suffices that this be the case for the algebraic closure L of K. For a structural characterization of absolutely primarily decomposable algebras see [1], § 1.

PROPOSITION 6. If the algebra Λ is absolutely primarily decomposable then $\dim \Lambda = 0, \infty$.

Proof. Since $\dim \Lambda$ remains unchanged under extensions of the ground field we may assume that K is algebraically closed. If is semi-simple (i.e. separable) then $\dim \Lambda = 0$. We may thus assume that Λ is not semi-simple. Let Λ_1 be one of the primary components of Λ with a non-zero radical N_1. Now all the primitive idempotents in Λ_1 are isomorphic and if e_1 is one of them then $e_1 N_1 e_1 \neq 0$. Thus

$$\det M(\Lambda_1) = (e_1\Lambda_1 e_1 : K) = (e_1 N_1 e_1 : K) + (e_1(\Lambda_1/N_1)e_1 : K) > 1.$$

Since $\det M(\Lambda)$ is the product of $\det M(\Lambda_i)$ where Λ_i runs through all the primary components of Λ it follows that $\det M(\Lambda) > 1$. Therefore by the result quoted above we have $\dim \Lambda = \infty$.

There are other situations in which it can be proved that $\dim \Lambda = \infty$ by showing that the matrix $M(\Lambda_L)$ is not inversible. The converse however is not true as will be shown by an example. Indeed, we shall construct an algebra Λ over any field K such that $\dim \Lambda = \infty$ but $\det M(\Lambda_L) = -1$.

Let K be an arbitrary field. Given $\alpha = (\alpha_1, \ldots, \alpha_{12})$, $\alpha_i \in K$, we consider the matrices

$$m_1(\alpha) = \begin{vmatrix} \alpha_1 & 0 & 0 & 0 & 0 \\ \alpha_3 & \alpha_2 & 0 & 0 & 0 \\ \alpha_4 & 0 & \alpha_2 & 0 & 0 \\ \alpha_5 & 0 & 0 & \alpha_2 & 0 \\ \alpha_{10} & \alpha_8 & \alpha_7 & \alpha_6 & \alpha_1 \end{vmatrix}, \qquad m_2(\alpha) = \begin{vmatrix} \alpha_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_{11} & \alpha_2 & 0 & 0 & 0 & 0 & 0 \\ \alpha_8 & 0 & \alpha_1 & 0 & 0 & 0 & 0 \\ \alpha_7 & 0 & 0 & \alpha_1 & 0 & 0 & 0 \\ \alpha_6 & 0 & 0 & 0 & \alpha_1 & 0 & 0 \\ \alpha_{12} & 0 & 0 & 0 & 0 & \alpha_2 & 0 \\ \alpha_9 & \alpha_{11} & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_{12} & \alpha_2 \end{vmatrix},$$

$$m(\alpha) = \begin{vmatrix} m_1(\alpha) & 0 \\ 0 & m_2(\alpha) \end{vmatrix}.$$

The matrices $m(\alpha)$ form an algebra Λ with $(\Lambda : K) = 12$. Basis elements $x_i \in \Lambda$ $(i = 1, \ldots, 12)$ are obtained by taking $x_i = m(\alpha)$ where $\alpha_j = \delta_{ij}$.

The elements x_1 and x_2 are primitive idempotents with $x_1 + x_2 = 1$. Further computation shows that

$$\begin{aligned} x_1\Lambda x_1 &= x_1K + x_{10}K, \\ x_1\Lambda x_2 &= x_6K + x_7K + x_8K, \\ x_2\Lambda x_1 &= x_3K + x_4K + x_5K, \\ x_2\Lambda x_2 &= x_2K + x_9K + x_{11}K + x_{12}K. \end{aligned}$$

This implies that the idempotents x_1 and x_2 are not isomorphic and thus form a maximal set of non-isomorphic idempotents in Λ. Thus the Cartan matrix of Λ is

$$M(\Lambda) = \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix}$$

with determinant -1. The ground field K played no role in the argument and the result remains valid for any extension of K.

Next consider the K-homomorphism $\varphi : \Lambda \to K$ given by

$$\varphi(m(\alpha)) = \alpha_9 + \alpha_{10}.$$

We have

$$\varphi(m(\alpha)m(\beta)) = \alpha_{11}\beta_{11} + \alpha_{12}\beta_{12} + \sum_{i=1}^{10}\alpha_i\beta_{10-i}.$$

This shows that

$$\varphi(m(\alpha)m(\beta)) = \varphi(m(\beta)m(\alpha))$$

and that if $\varphi(m(\alpha)m(\beta)) = 0$ for all $m(\alpha)$ then $m(\beta) = 0$. Thus the hyperplane $\varphi = 0$ contains no left ideals (except zero) and contains all commutators. Thus Λ is a symmetric algebra and therefore also a Frobenius algebra (see [2]). For such algebras it has been proved in [5] that $\dim \Lambda = 0, \infty$. However Λ is not semi-simple since $x_3, \ldots, x_{12}$ are nilpotent. Thus $\dim \Lambda = \infty$.

Remark. The argument that $\dim \Lambda = \infty$ remains valid if K is an arbitrary commutative ring (with a unit element). This follows from the generalized treatment of symmetric and Frobenius algebras that will appear in the next paper in this series.

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