ON THE COMPACITY OF THE ORTHOGONAL GROUPS

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It is a well known fact on Lorenz groups that a quadratic form f is definite if and only if the corresponding orthogonal group $O_n(R_\infty, f)$, where R_∞ is the real number field, is compact. In this note, we shall show that the analogue of this holds for the case of the p-adic orthogonal group $O_n(R_p, f)$, where R_p is the rational p-adic number field, as a special result of the more general statement on the completely valued fields.

Let K be a field with non-trivial valuation $|\ |$, and of characteristic $\neq 2$. Let V be an n-dimensional vector space over K and let u_i $(i=1, \ldots, n)$ be some fixed basis of V over K. If we define norm of $x = \sum_{i=1}^{n} x_i u_i \in V$ by $\|x\| = \max_{i=1,\ldots,n} |x_i|$, then the space V is topologized as usual.[1] Now, let E be the algebra of endomorphisms of V over K. Using the above basis, we also define norm of transformation $X = (x_{ij})$ by $\|X\| = \max_{i,j=1,\ldots,n} |x_{ij}|$. It is easy to see that $\|X \cdot Y\| \leqq n\|X\| \cdot \|Y\|$. Thus, E becomes a normed algebra over K. A subset S of a normed space is called bounded if for some number $b > 0$ we have $\|x\| < b$ for all $x \in S$. For our normed space V, boundedness is independent of the choice of basis u_i. The same is true for the normed space E. If K is locally compact, then a bounded and closed subset of a normed space over K is the same thing as a compact subset. Now, let f be a non-degenerate symmetric bilinear form on V. The orthogonal group $O_n(K, f)$ is obviously a closed subset of E. If f and g are congruent, it is easy to see that their groups are homeomorphically isomorphic and if one of them is bounded in E so is the other. We say that a from f is of index ν if ν is the maximum dimension of $U \subset V$ such that U is a totally isotropic subspace of V.[2] $\nu = 0$ means that $f(x, x) = 0$ implies $x = 0$.

We prove the following

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1) See [1] p. 18.

2) See [2] p. 17.

THEOREM 1. *Let K be a completely (non-trivially) valued field with characteristic $\neq 2$ and let f be a non-degenerate symmetric bilinear form over K. Then the index ν of f is zero if and only if the orthogonal group $O_n(K,f)$ is bounded in E.*

Proof. If $n=1$, since then $\nu=0$ always and the group is of order 2, the statement is trivial. So we assume that $n \geqq 2$. Suppose that $\nu \geqq 1$. Then f is congruent to the form g whose matrix is of type

$$G=\begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & * \end{pmatrix}^{3)}$$

Since ${}^t\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ for all $x(\neq 0) \in K$, it follows that

$$X=\begin{pmatrix} x & 0 & & & & \\ 0 & x^{-1} & & & & \\ & & 1 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & 1 \end{pmatrix}$$

belongs to $O_n(K,g)$ for each $x(\neq 0) \in K$. Thus, $O_n(K,g)$ is not bounded in E. Hence, $O_n(K,f)$ is also not bounded. This proves the sufficiency. It is to be noted that we do not use the completeness of K.

Next, we shall prove the necessity.[4] Here the completeness of K is used essentially. Assume that $O_n(K,f)$ is not bounded. Without loss of generality, we may suppose that the matrix of f is of type

$$F=\begin{pmatrix} a_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & a_n \end{pmatrix} \quad \text{where } |a_i| \leqq 1,\ i=1,\ldots,n$$

By our assumption, for any $N>0$ there exists an $X \in O_n(K,f)$ such that $\|X\| > N$. Suppose that $\|X\|=|x_{pq}|$. Comparing the (q,q)-components of both sides in ${}^tXFX=F$, we get $\sum_{i=1}^{n} a_i x_{iq}^2 = a_q$. Multiplying x_{pq}^{-2} on both sides, we see that

[3] See [3] Satz 5.

[4] The following proof is inspired by Theorem 2, Dieudonné [4].

the inequality $\left|\sum_{i=1}^{n} a_i x_i^2\right| < |a_q| N^{-2}$ has a solution x_i such that $|x_i| \leqq 1$, $|x_p| = 1$. Now, if K is locally compact then the unit cube, i.e. the set of x with $\|x\| \leqq 1$ in V is compact. Thus, for increasing N we may select a sequence of vectors x_N in the unit cube satisfying an inequality as above one of whose component, say p_N-th, is of value 1. Taking a subsequence, if necessary, we may assume that p_N are all equal. It is obvious that $x = \lim_{N\to\infty} x_N$ gives a non-trivial solution of $f(x, x) = 0$. Thus, the necessity is proved for our special case, i.e. the case when K is archimedean (that is, when K is real or complex field) or K is a finite extension of the Hensel p-adic number field R_p with some prime p or a field of power series of one variable over a finite field of characteristic $\neq 2$. Therefore, there remains to be considered a case of a non-archimedean field K. We shall construct a non-trivial solution of $f(x, x) = 0$ by successive approximation. We fix an element $c \in K$ such that $|c| < 1$, and put $d = 2a_1 \ldots a_n \cdot c$. Then, from the above argument, the inequality $\sum_{i=1}^{n} a_i x_i^2 < |d|^3$ has a solution x_i such that $|x_i| \leqq 1$, $|x_p| = 1$. Then, we shall show by induction on μ that the inequality $\left|\sum_{i=1}^{n} a_i x_{i,\mu}^2\right| < |d|^{\mu+2}$ has a solution $x_{i,\mu}$ such that $|x_{i,\mu}| \leqq 1$, $|x_{p,\mu}| = 1$. For $\mu = 1$, it suffices to take $x_{i,1} = x_i$. Next, we assume that we have a solution for some μ. Put $\sum_{i=1}^{n} a_i x_{i,\mu}^2 = d^\mu e$, $e = d^2 f$. We have $|f| < 1$. And set $y = -e(2a_p x_{p,\mu})^{-1}$. Then, we get $|y| = |a_1 \ldots \overset{p}{\vee} \ldots a_n| \, |c| \, |d| \, |f| \, |x_{p,\mu}|^{-1} < |d| < 1$. Using this y, we put $x_{i,\mu+1} = x_{i,\mu}$ $(i \neq p)$, $x_{p,\mu+1} = x_{p,\mu} + d^\mu y$. Since the valuation is non-archimedean, we have $|x_{i,\mu+1}| = |x_{i,\mu}|$ $i = 1, \ldots, n$. From the definition of y, we have $\sum_{i=1}^{n} a_i x_{i,\mu+1}^2 = \sum_{i=1}^{n} a_i x_{i,\mu}^2 + 2a_p x_{p,\mu} d^\mu y + a_p d^{2\mu} y^2 = d^\mu (e + 2a_p x_{p,\mu} y) + a_p d^{2\mu} y^2 = a_p d^{2\mu} y^2$. Therefore, it follows that $\left|\sum_{i=1}^{n} a_i x_{i,\mu+1}^2\right| \leqq |d|^{2\mu} |y|^2 < |d|^{2\mu+2} \leqq |d|^{\mu+3}$. Thus, we get n Cauchy sequences $\{x_{i,\mu}\}$ in K. Since K is complete, there exist $x_i = \lim_{\mu\to\infty} x_{i,\mu}$. It is obvious that $x = \sum_{i=1}^{n} x_i u_i$ is a non-trivial solution of the equation $f(x, x) = 0$. This proves the necessity assertion.

As an immediate consequence of Theorem 1 we get the following

THEOREM 2. *Let K be a locally compactly valued field with characteristic $\neq 2$. Then, the index ν of f is zero if and only if the group $O_n(K, f)$ is compact.*[5)]

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5) Mr. A. Hattori has communicated to the writer an elegant alternative proof. Here we sketch his proof. Let P be the projective space corresponding to V. If we define the open set in P as the totality of lines in V each of which intersects with some given open set in V, then P becomes a compact space. If $\nu = 0$, then there is a homeomorphism between P and the set S of all symmetries with respect to the hyperplanes in V. Here, the topology in S is the one induced from E. Thus S is a compact set. Therefore, $O_n(K, f) = S^n$ (Cartan-Dieudonné) is also compact.

Now we shall apply the above results to the orthogonal group over a field K of algebraic numbers or algebraic functions of one variable over a finite field of characteristic $\neq 2$. Let $K_{\mathfrak{p}}$ be a $\mathfrak{p}$-adic completion of K with respect to a place $\mathfrak{p}$ in K. Suppose that a form f is given in K. Naturally f may be considered as a form over $K_{\mathfrak{p}}$ and $O_n(K,f)$ is contained in $O_n(K_{\mathfrak{p}}, f)$.[6] Let ν and $\nu_{\mathfrak{p}}$ be the global and local indices of f respectively. According to *Hasse's principle*, we have the relation $\nu = \min_{\mathfrak{p}} \nu_{\mathfrak{p}}$ between these indices.[7] If $\nu \geqq 1$, since we do not use the completeness of valuation in the proof of sufficiency in Theorem 1, if $\mathfrak{p}$ is any place of K, then $O_n(K,f)$ is unbounded with respect to the $\mathfrak{p}$-adic topology. Conversely, if $\nu = 0$, then by the above principle we get $\nu_{\mathfrak{p}} = 0$ for some $\mathfrak{p}$. Therefore $O_n(K_{\mathfrak{p}}, f)$ is compact for such $\mathfrak{p}$ (Theorem 2) and we see that $O_n(K,f)$ is bounded in the $\mathfrak{p}$-adic topology.

Thus we get

THEOREM 3. *Let K be a field of algebraic numbers or algebraic functions of one variable over a finite field of characteristic $\neq 2$. Then a form f is a zero-form[8] if and only if the orthogonal group $O_n(K,f)$ is unbounded for all $\mathfrak{p}$-adic topologies in K.*

REFERENCES

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[2] J. Dieudonné, Sur les groupes classiques, Act. Sci. et Ind., **1040** (1948).

[3] E. Witt, Theorie der Quadratischen Formen in beliebigen Körpern, J. Reine Angew. Math. **176** (1937).

[4] J. Dieudonné, Sur les groupes orthogonaux rationnels à trois et quatre variables. Compt. rend., t. **233** (1951).

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[6] By Cayley's parametrization we can see that $O_n(K,f)$ is dense in $O_n(K_{\mathfrak{p}}, f)$. But this fact is unnecessary to prove our Theorem 3.

[7] See [3] Satz 19. Though only the number field case is treated in [3], we know that the principle is also valid for the function field case.

[8] This means that f represents zero non-trivially.