CYCLES AND ENDOMORPHISMS OF ABELIAN VARIETIES

HISASI MORIKAWA

In the present paper we shall remark that to each class of algebraically equivalent cycles on a Jacobian variety we can attach a symmetric element of the ring of endomorphisms of the variety and shall prove some formulae concerning attached symmetric elements.

§ 1. Endomorphism $\delta(\mathbf{X}, \mathbf{Y})$

We shall denote by $\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \ldots$ cycles on an abelian variety $\mathbf{A}$. Throughout in this section we shall fix this abelian variety $\mathbf{A}$ of dimension n. When $\mathbf{X}$ and $\mathbf{Y}$ have complemental dimensions and $\mathbf{X}\cdot\mathbf{Y}$ is defined, $\mathrm{S}[\mathbf{X}\cdot\mathbf{Y}]$ means $e_1x_1+\ldots+e_fx_f$, where $x_1, x_2, \ldots, x_f$ are points contained in $\mathbf{X}\cdot\mathbf{Y}$ with multiplicities $e_1, e_2, \ldots, e_f$ respectively. When $\mathbf{A}, \mathbf{X}, \mathbf{Y}$ are defined over a field k, $\mathrm{S}[\mathbf{X}\cdot(\mathbf{Y}_t-\mathbf{Y})]$ defines a function on $\mathbf{A}$ within values in $\mathbf{A}$ and defined over k. Since $\mathrm{S}[\mathbf{X}\cdot(\mathbf{Y}-\mathbf{Y})]=0$, this function is an endomorphism of $\mathbf{A}$. We denote it by $\delta(\mathbf{X}, \mathbf{Y})$. Even when $\mathbf{X}\cdot\mathbf{Y}_{t_0}$ is not defined, we define $\mathrm{S}[\mathbf{X}\cdot(\mathbf{Y}_{t_0}-\mathbf{Y})]$ by $\delta(\mathbf{X}, \mathbf{Y})t_0$.

We say that two cycles $\mathbf{X}, \mathbf{Y}$ are immediately algebraically equivalent if and only if there exists a complete curve Γ, a cycle $\mathbf{Z}$ on $\mathbf{A}\times\Gamma$ and two points $\mathbf{M}, \mathbf{M}'$ on Γ such that $\mathbf{Z}\cdot(\mathbf{A}\times\mathbf{P})$ is defined for any point $\mathbf{P}$ on Γ and $\mathbf{X}\times\mathbf{M}=\mathbf{Z}\cdot(\mathbf{A}\times\mathbf{M})$, $\mathbf{Y}\times\mathbf{M}'=\mathbf{Z}\cdot(\mathbf{A}\times\mathbf{M}')$.

We say that two cycles $\mathbf{X}, \mathbf{Y}$ are algebraically equivalent——$\mathbf{X}\equiv\mathbf{Y}$ in notation——if and only if there exist a finite number of cycles $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_f$ such that $\mathbf{X}_1=\mathbf{X}$, $\mathbf{X}_f=\mathbf{Y}$ and $\mathbf{X}_i$ is immediately algebraically equivalent to $\mathbf{X}_{i+1}$ for each $i=1, 2, \ldots, f-1$.

PROPOSITION 1. *If* $\mathbf{X}\equiv\mathbf{X}'$, *then* $\delta(\mathbf{X}, \mathbf{Y})=\delta(\mathbf{X}', \mathbf{Y})$.

Proof. It is sufficient to prove this for immediately algebraically equiva-

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lent $\mathbf{X}$, $\mathbf{X}'$. Let Γ be a complete curve and let $\mathbf{Z}$ be a cycle on $\mathbf{A}\times\Gamma$ such that $\mathbf{Z}\cdot(\mathbf{A}\times\mathbf{P})$ is defined for any point $\mathbf{P}$ on Γ and $\mathbf{X}\times\mathbf{M}=\mathbf{Z}\cdot(\mathbf{A}\times\mathbf{M})$, $\mathbf{X}\times\mathbf{M}'=\mathbf{Z}\cdot(\mathbf{A}\times\mathbf{M}')$ with points $\mathbf{M}$, $\mathbf{M}'$ on Γ. Let Γ' be a non-singular model of Γ and let J be the Jacobian variety of Γ' over a definition field of Γ'. We can replace Γ by Γ'. $\mathbf{S}[(\mathbf{X}(\mathbf{N})-\mathbf{X})\cdot(\mathbf{Y}_t-\mathbf{Y})]$ defines a function on $\Gamma'\times\mathbf{A}$, where $\mathbf{X}(\mathbf{N})\times\mathbf{N}=\mathbf{Z}\cdot(\mathbf{A}\times\mathbf{N})$. This function can be extended to a function on $\mathbf{J}\times\mathbf{A}$. Since $\mathbf{S}[(\mathbf{X}-\mathbf{X})\cdot(\mathbf{Y}_t-\mathbf{Y})]=0$, it is a homomorphism from $\mathbf{J}\times\mathbf{A}$ into $\mathbf{A}$. A homomorphism from a product of abelian varieties is a sum of homomorphisms from each component. Hence $\mathbf{S}[(\mathbf{X}(\mathbf{N})-\mathbf{X})\cdot(\mathbf{Y}_t-\mathbf{Y})]=\alpha_1(\varphi(\mathbf{N}))+\alpha_2 t$, where φ is the canonical function from Γ' into $\mathbf{J}$. Since $\mathbf{S}[(\mathbf{X}(\mathbf{N})-\mathbf{X})\cdot(\mathbf{Y}-\mathbf{Y})]=0$, $\alpha_1(\varphi(\mathbf{N}))=0$ for all $\mathbf{N}$ on Γ'. This shows that $\mathbf{S}[\mathbf{X}(\mathbf{N})\cdot(\mathbf{Y}_t-\mathbf{Y})]=\mathbf{S}[\mathbf{X}\cdot(\mathbf{Y}_t-\mathbf{Y})]$, namely $\delta(\mathbf{X}(\mathbf{N}),\mathbf{Y})=\delta(\mathbf{X},\mathbf{Y})$.

Similarly we get

PROPOSITION 2. *If* $\mathbf{X}\equiv\mathbf{X}'$, *then* $\delta(\mathbf{XY},\mathbf{Z})=\delta(\mathbf{X}'\mathbf{Y},\mathbf{Z})$.

By repeated application of this proposition we get

PROPOSITION 3. *If* $\underset{1}{\mathbf{X}}\equiv\underset{1}{\mathbf{X}'}$, $\underset{2}{\mathbf{X}}\equiv\underset{2}{\mathbf{X}'},\ldots,\underset{r}{\mathbf{X}}\equiv\underset{r}{\mathbf{X}'}$, *then* $\delta(\underset{1}{\mathbf{X}}\cdot\underset{2}{\mathbf{X}}\cdot\underset{3}{\mathbf{X}}\cdot\ldots\cdot\underset{r}{\mathbf{X}},\mathbf{Y})=\delta(\underset{1}{\mathbf{X}'}\cdot\underset{2}{\mathbf{X}'}\cdot\ldots\cdot\underset{r}{\mathbf{X}'},\mathbf{Y})$.

Since any two points t, s are connected by a finite number of curves on $\mathbf{A}$, $\mathbf{X}_t$ and $\mathbf{X}_s$ are algebraically equivalent.

PROPOSITION 4. $\delta(\mathbf{X},\mathbf{Y})+\delta(\mathbf{Y},\mathbf{X})=deg(\mathbf{X}\cdot\mathbf{Y})\,\delta$[1].

Proof.

$$\begin{aligned}\mathbf{S}[\mathbf{X}\cdot(\mathbf{Y}_t-\mathbf{Y})]&=\mathbf{S}[\mathbf{X}\cdot\mathbf{Y}_t]-\mathbf{S}[\mathbf{X}\cdot\mathbf{Y}]\\&=\mathbf{S}[(\mathbf{X}\cdot\mathbf{Y}_t)_{-t}]-(deg(\mathbf{X}\cdot\mathbf{Y}))(-t)-\mathbf{S}[\mathbf{X}\cdot\mathbf{Y}]\\&=\mathbf{S}[\mathbf{X}_{-t}\cdot\mathbf{Y}]-\mathbf{S}[\mathbf{X}\cdot\mathbf{Y}]+(deg(\mathbf{X}\cdot\mathbf{Y}))t\\&=(deg(\mathbf{X}\cdot\mathbf{Y}))t-\delta(\mathbf{Y}\cdot\mathbf{X})t.\end{aligned}$$

From Proposition 3 and Proposition 4 we get:

PROPOSITION 5. *If* $\underset{1}{\mathbf{X}}\equiv\underset{1}{\mathbf{X}'},\ldots,\underset{r}{\mathbf{X}}\equiv\underset{r}{\mathbf{X}'}$, $\underset{1}{\mathbf{Y}}\equiv\underset{1}{\mathbf{Y}'},\ldots,\underset{e}{\mathbf{Y}}\equiv\underset{e}{\mathbf{Y}'}$, *then* $\delta(\underset{1}{\mathbf{X}}\cdot\underset{2}{\mathbf{X}}\cdot\ldots\cdot\underset{r}{\mathbf{X}},\underset{1}{\mathbf{Y}}\cdot\underset{2}{\mathbf{Y}}\cdot\ldots\cdot\underset{e}{\mathbf{Y}})=\delta(\underset{1}{\mathbf{X}'}\cdot\underset{2}{\mathbf{X}'}\ldots\underset{r}{\mathbf{X}'},\underset{1}{\mathbf{Y}'}\cdot\underset{2}{\mathbf{Y}}\ldots\underset{e}{\mathbf{Y}'})$.

Thus $\delta(\mathbf{X},\mathbf{Y})$ depends only on the algebraic equivalence of $\mathbf{X}$ and $\mathbf{Y}$.

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[1] δ is the identity endomorphism of $\mathbf{A}$.

PROPOSITION 6. $\delta(X, \underset{1}{Y} \cdot \underset{2}{Y} \ldots \underset{r}{Y}) = \delta(X \cdot \underset{2}{Y} \ldots \underset{r}{Y}, \underset{1}{Y})$
$+ \ldots + \delta(X \cdot \underset{1}{Y} \ldots \underset{i-1}{Y} \cdot \underset{i+1}{Y} \ldots \underset{r}{Y}, \underset{i}{Y})$
$+ \ldots + \delta(X \cdot \underset{1}{Y} \ldots \underset{r-1}{Y}, \underset{r}{Y})$.

Proof.

$$\delta(X, \underset{1}{Y} \cdot \underset{2}{Y} \ldots \underset{r}{Y})t = S[X((\underset{1}{Y} \circ \underset{2}{Y} \ldots \underset{r}{Y})_t - (\underset{1}{Y} \cdot \underset{2}{Y} \ldots \underset{r}{Y}))]$$
$$= S[X \cdot (\underset{1}{Y_t} \cdot \underset{2}{Y_t} \ldots \underset{r}{Y_t} - \underset{1}{Y_t} \cdot \underset{2}{Y_t} \ldots \underset{r-1}{Y_t} \cdot \underset{r}{Y} + \underset{1}{Y_t} \cdot \underset{2}{Y_t} \ldots \underset{r-1}{Y_t} \cdot \underset{r}{Y}$$
$$- \underset{1}{Y_t} \cdot \underset{2}{Y_t} \ldots \underset{r-2}{Y_t} \cdot \underset{r-1}{Y} \cdot \underset{r}{Y} + \ldots + \underset{1}{Y_t} \cdot \underset{2}{Y} \ldots \underset{r}{Y} - \underset{1}{Y} \cdot \underset{2}{Y} \ldots \underset{r}{Y})]$$
$$= S[X \cdot \underset{1}{Y_t} \cdot \underset{2}{Y_t} \ldots \underset{r-1}{Y_t} \cdot (\underset{r}{Y_t} - \underset{r}{Y})]$$
$$+ \ldots + S[X \cdot \underset{1}{Y_t} \ldots \underset{i-1}{Y_t} \cdot \underset{i+1}{Y} \ldots \underset{r}{Y}(\underset{i}{Y_t} - \underset{i}{Y})]$$
$$+ \ldots + S[X \cdot \underset{2}{Y} \ldots \underset{r}{Y} \cdot (\underset{1}{Y_t} - \underset{1}{Y})]$$
$$= \delta(X \cdot \underset{1}{Y} \ldots \underset{r-1}{Y}, \underset{r}{Y})t + \ldots + \delta(X \cdot \underset{1}{Y} \ldots \underset{i-1}{Y} \cdot \underset{i+1}{Y} \ldots \underset{r}{Y}, \underset{i}{Y})t$$
$$+ \ldots + \delta(X \cdot \underset{2}{Y} \ldots \underset{r}{Y}, \underset{1}{Y})t.$$

PROPOSITION 7. $\delta(\underset{1}{X}, \underset{2}{X} \cdot \underset{3}{X} \ldots \underset{r}{X}) + \ldots + \delta(\underset{i}{X}, \underset{1}{X} \ldots \underset{i-1}{X} \cdot \underset{i+1}{X} \ldots \underset{r}{X})$
$+ \ldots + \delta(\underset{r}{X}, \underset{1}{X} \cdot \underset{2}{X} \ldots \underset{r-1}{X}) = (r-1)\, deg(\underset{1}{X} \ldots \underset{r}{X})\delta$.

Proof. We shall prove this by induction on r. We assume that this is true for $r-1$ and we put $Y = \underset{r-1}{X} \cdot \underset{r}{X}$. Then

$$\delta(\underset{1}{X}, \underset{2}{X} \ldots \underset{r-2}{X} \cdot Y) + \ldots + \delta(\underset{i}{X}, \underset{1}{X} \ldots \underset{i-1}{X} \cdot \underset{i+1}{X} \ldots \underset{r-2}{X} \cdot Y)$$
$$+ \ldots + \delta(\underset{r-2}{X}, \underset{1}{X} \ldots \underset{r-3}{X} \cdot Y) + \delta(Y, \underset{1}{X} \ldots \underset{r-2}{X})$$
$$= (r-2)\, deg(\underset{1}{X} \ldots \underset{r-2}{X} \cdot Y)\delta = (r-2)\, deg(\underset{1}{X} \ldots \underset{r}{X})\delta.$$

On the other hand, by virtue of Proposition 4 and Proposition 5,

$$\delta(Y, \underset{1}{X} \ldots \underset{r-2}{X}) = \delta(\underset{r-1}{X} \cdot \underset{r}{X}, \underset{1}{X} \ldots \underset{r-2}{X})$$
$$= deg(\underset{1}{X} \ldots \underset{r}{X})\delta - \delta(\underset{1}{X} \ldots \underset{r-2}{X}, \underset{r-1}{X} \cdot \underset{r}{X})$$
$$= deg(\underset{1}{X} \ldots \underset{r}{X})\delta - \delta(\underset{1}{X} \ldots \underset{r-2}{X} \cdot \underset{r-1}{X}, \underset{r}{X}) - \delta(\underset{1}{X} \ldots \underset{r-2}{X} \cdot \underset{r}{X}, \underset{r-1}{X})$$
$$= deg(\underset{1}{X} \ldots \underset{r}{X})\delta - (deg(\underset{1}{X} \ldots \underset{r}{X})\delta - \delta(\underset{r}{X}, \underset{1}{X} \ldots \underset{r-1}{X}))$$
$$- (deg(\underset{1}{X} \ldots \underset{r}{X})\delta - \delta(\underset{r-1}{X}, \underset{1}{X} \ldots \underset{r-2}{X} \cdot \underset{r}{X}))$$
$$= \delta(\underset{r-1}{X}, \underset{1}{X} \ldots \underset{r-2}{X} \cdot \underset{r}{X}) + \delta(\underset{r}{X}, \underset{1}{X} \ldots \underset{r-1}{X}) - deg(\underset{1}{X} \ldots \underset{r}{X})\delta.$$

Hence we have

$$\delta(\underset{1}{X}, \underset{2}{X} \ldots \underset{r}{X}) + \ldots + \delta(\underset{i}{X}, \underset{1}{X} \ldots \underset{i-1}{X} \cdot \underset{i+1}{X} \ldots \underset{r}{X})$$
$$+ \ldots + \delta(\underset{r}{X}, \underset{1}{X} \ldots \underset{r-1}{X}) = (r-1)\, deg(\underset{1}{X} \cdot \underset{2}{X} \ldots \underset{r}{X})\delta.$$

The case $r=1$ is Proposition 4 itself.

COROLLARY 1. *If* X *is a non-degenerate divisor, then*

$$\delta(\mathbf{X}, \overbrace{\mathbf{X}\ldots\mathbf{X}}^{n-1})^{2)} = (n-1)/n\, deg(\overbrace{\mathbf{X}\ldots\mathbf{X}}^{n})\delta.$$

COROLLARY 2. *If* X *is a non-degenerate divisor, then*

$$\delta(\overbrace{\mathbf{X}\ldots\mathbf{X}}^{r}, \overbrace{\mathbf{X}\ldots\mathbf{X}}^{n-r}) = (n-r)/n\, deg(\overbrace{\mathbf{X}\ldots\mathbf{X}}^{n})\,\delta.$$

§ 2. **Case of Jacobian variety**

Let Γ be a non-singular curve of genus n defined over k, let $\mathbf{J}$ be the Jacobian variety of Γ and let φ be the canonical function from Γ into $\mathbf{J}$. We assume that $\mathbf{J}$ and φ are also defined over k. We denote by $\mathbf{W}^{(1)}, \mathbf{W}^{(2)}, \ldots, \mathbf{W}^{(n-1)}$ the locuses of $\varphi(\mathbf{P}_1)$, $\varphi(\mathbf{P}_1)+\varphi(\mathbf{P}_2), \ldots, \varphi(\mathbf{P}_1)+\ldots+\varphi(\mathbf{P}_{n-1})$, respectively over k, where $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_{n-1}$ are independent generic points over k of Γ. We denote by $\mathbf{W}'^{(1)}, \ldots, \mathbf{W}'^{(n-1)}$ the locuses of $-\varphi(\mathbf{P}_1)$, $-\varphi(\mathbf{P}_1)-\varphi(\mathbf{P}_2), \ldots, -\varphi(P_1)-\ldots-\varphi(\mathbf{P}_{n-1})$, respectively, over k.

PROPOSITION 8. $\delta(\mathbf{W}^{(n-r)}, \mathbf{W}'^{(r)}) = \binom{n-1}{r}\delta.$

Proof. Let $x=\varphi(\mathbf{P}_1)+\ldots+\varphi(\mathbf{P}_n)$ where $\mathbf{P}_1, \ldots, \mathbf{P}_n$ are independent generic points of Γ over k. Then by virtue of Proposition 17, N° 40, § V, [1], $\mathbf{W}^{(n+r)}\mathbf{W}'^{(r)}_x = \sum_{(i)}(w_{i_1 i_2\ldots i_{n-r}})$, where $w_{i_1 i_2\ldots i_{n-1}} = \varphi(\mathbf{P}_{i_1})+\varphi(\mathbf{P}_{i_2})+\ldots+\varphi(\mathbf{P}_{i_{n-r}})$ and the summation which means the summation as cycles of dimension zero runs over all $\binom{n}{n-r}$ combinations of indices $1, 2, \ldots, n$. Hence

$$\begin{aligned} S[\mathbf{W}^{(n-r)}(\mathbf{W}'^{(r)}_x - \mathbf{W}^{(r)})] &= \sum_{i_1<i_2<\ldots<i_{n-r}}(\varphi(\mathbf{P}_{i_1})+\ldots+\varphi(\mathbf{P}_{i_{n-r}})) \\ &= \frac{n-r}{n}\binom{n}{n-r}[\varphi(\mathbf{P}_1)+\ldots+\varphi(\mathbf{P}_n)] = \binom{n-1}{r}x. \end{aligned}$$

LEMMA. $(n-r)\mathbf{W}^{(r)} = \mathbf{W}^{(r+1)}\cdot\theta_a{}^{3)}$ *with suitable* a.

This is proved in a similar manner as in the proof of Proposition 17, N° 40, § V, [1].

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2) $\delta(\mathbf{X}, \overbrace{\mathbf{X}\ldots\mathbf{X}}^{n-1})$ means $\delta(\mathbf{X}, \mathbf{X}_{a_1}\cdot\mathbf{X}_{a_2}\ldots\mathbf{X}_{a_{n-1}})$ with suitable $a_1, a_2, \ldots, a_{n-1}$ which is independent of the choice of $a_1, \ldots, a_{n-1}$.

3) θ means $\mathbf{W}^{(n-1)}$.

Consequently we get:

PROPOSITION 9. $(n-r)\mathbf{W}^{(r)} = \theta \cdot \theta_{a_1} \ldots \theta_{a_{n-r-1}}$ *with suitable* $a_1, \ldots, a_{n-r-1}$.

PROPOSITION 10. $\delta(\mathbf{W}^{(n-r)}, \mathbf{W}^{(r)}) = \delta(\mathbf{W}^{(n-r)}, \mathbf{W}'^{(r)}) = \binom{n-1}{r}\delta.$

Proof.
$$\delta(\mathbf{W}^{(n-r)}, \mathbf{W}^{(r)}) = \frac{1}{(n-r)!}\delta(\mathbf{W}^{(n-1)}, \overbrace{\theta \ldots \theta}^{n-r})$$
$$= \frac{1}{(n-r)!}\delta(\mathbf{W}^{(n-r)}, \overbrace{\theta_{-c} \ldots \theta_{-c}}^{n-r}) = \frac{1}{(n-r)!}\delta(\mathbf{W}^{(n-r)}, \overbrace{\theta' \ldots \theta'}^{n-r})$$
$$= \delta(\mathbf{W}^{(n-r)}, \mathbf{W}'^{(r)}) = \binom{n-1}{r}\delta,$$

where $c = \varphi(\mathfrak{K})$ with a canonical divisor $\mathfrak{K}$.

PROPOSITION 11.
$$\delta(\mathbf{W}^{(r)}\mathbf{W}^{(s)}, \mathbf{W}^{(2n-r-s)}) = \frac{(2n-r-s)!}{(n-r)!\,(n-s)!}\delta(\mathbf{W}^{(r+s-n)}, \mathbf{W}^{(2n-r-s)}).$$

Proof.
$$\delta(\mathbf{W}^{(r)}\mathbf{W}^{(s)}, \mathbf{W}^{(2n-r-s)}) = \frac{1}{(n-r)!\,(n-s)!}(\overbrace{\theta \ldots \theta}^{2n-s-r}, \mathbf{W}^{(2n-s-r)})$$
$$= \frac{(2n-r-s)!}{(n-r)!\,(n-s)!}\delta(\mathbf{W}^{(r+s-n)}, \mathbf{W}^{(2n-s-r)}).$$

Similarly we get

PROPOSITION 12.
$$\delta(\mathbf{W}^{(r)}\mathbf{W}^{(s)}\mathbf{X}, \mathbf{Y}) = \frac{(2n-r-s)!}{(n-r)!\,(n-s)!}\delta(\mathbf{W}^{(r+s-n)}\mathbf{X}, \mathbf{Y}).$$

PROPOSITION 13. *Let* $\mathbf{C}$ *be a cycle of dimension one. Then* $\delta(\mathbf{C}, \theta)$, $\delta(\theta, \mathbf{C})$ *are symmetric.*

Proof. It is sufficient to show this for simple irreducible curves. Let Γ_1 be a non-singular model of $\mathbf{C}$ and let f be the birational correspondence from Γ_1 to $\mathbf{C}$ regular at each simple point of $\mathbf{C}$. Let $\mathbf{J}_1$ be the Jacobian variety of Γ_1 and φ be the canonical function of Γ_1 into $\mathbf{J}_1$. We denote by λ the extension of f onto a homomorphism from $\mathbf{J}_1$ into $\mathbf{J}$. Then $\mathbf{S}[\mathbf{C}\cdot(\theta_t - \theta)] = \lambda\mathbf{S}[\varphi_1(\mathbf{P}_{r\Gamma_1} \wedge_f \Gamma_1 \times (\theta_t - \theta))] = \lambda\lambda'_\theta t$, where $\wedge_1$ is the graph of f in $\Gamma_1 \times \mathbf{J}$. By virtue of formulae in N° 77, § XI, [1],

$$\mathbf{M}_l((\lambda\lambda'_\theta)') = \mathbf{E}_l(\theta)^{-1}{}^t\mathbf{M}_l(\lambda\lambda'_\theta)\mathbf{E}_l(\theta) = \mathbf{E}_l(\theta)^{-1}{}^t\mathbf{M}_l(\lambda'_\theta){}^t\mathbf{M}_l(\lambda)\mathbf{E}_l(\theta)$$
$$= \mathbf{E}_l(\theta)^{-1}{}^t(\mathbf{E}_l(\theta)^{-1}{}^t\mathbf{M}_l(\lambda)\mathbf{E}_l(\theta)){}^t\mathbf{M}_l(\lambda)\mathbf{E}_l(\theta)$$
$$= \mathbf{E}_l(\theta)^{-1}{}^t\mathbf{E}_l(\theta)\mathbf{M}_l(\lambda){}^t\mathbf{E}_l(\theta)^{-1}{}^t\mathbf{M}_l(\lambda)\mathbf{E}_l(\theta)$$

$$= \mathbf{E}_l(\theta)^{-1}(-\mathbf{E}_l(\theta))\mathbf{M}_l(\lambda)(-\mathbf{E}_l(\theta_1))^{-1\,t}\mathbf{M}_l(\lambda)\mathbf{E}_l(\theta)$$
$$= \mathbf{M}_l(\lambda)\mathbf{E}_l(\theta_1)^{-1\,t}\mathbf{M}_l(\lambda)\mathbf{E}_l(\theta) = \mathbf{M}_l(\lambda\lambda'_\theta)$$

where θ_1 is the theta divisor of $\mathbf{J}_1$. This shows that $\delta(\mathbf{C}, \theta)$ is symmetric. Since $\delta(\theta, \mathbf{C}) = (deg\,\mathbf{C}\cdot\theta)\delta - \delta(\mathbf{C}, \theta)$, $\delta(\theta, \mathbf{C})$ is also symmetric.

PROPOSITION 14. *Let* $\mathbf{X}$ *be a cycle of dimension* r. *Then*

$$\delta(\mathbf{X}, \mathbf{W}^{(n-r)}) = \delta(\mathbf{X}\cdot\mathbf{W}^{(n-r+1)}, \theta).$$

Proof. $\delta(\mathbf{X}, \mathbf{W}^{(n-r)}) = \delta\left(\mathbf{X}, \frac{1}{r!}\overbrace{\theta\ldots\theta}^{r}\right) = \frac{1}{r!}\delta(\mathbf{X}, \overbrace{\theta\ldots\theta}^{r})$

$$= \frac{1}{r!}r\delta(\mathbf{X}, \overbrace{\theta\ldots\theta}^{r-1}) = \frac{1}{(r-1)!}\delta(\mathbf{X}\overbrace{\theta\theta\ldots\theta}^{r-1}, \theta) = \delta(\mathbf{X}\mathbf{W}^{(n-r+1)}, \theta).$$

From this proposition and Proposition 13 we get

THEOREM 1. $\delta(\mathbf{X}, \mathbf{W}^{(n-r)})$, $\delta(\mathbf{W}^{(n-r)}, \mathbf{X})$ *are symmetric.*

PROPOSITION 15. $\delta(\lambda^{-1}(\theta)\mathbf{W}^{(2)}, \theta) = (deg\,\lambda^{-1}(\theta)\mathbf{W}^{(2)})\delta - \lambda'\lambda$

$$= \frac{1}{2}\sigma(\lambda'\lambda)\delta - \lambda'\lambda.$$

Proof.

$$(\delta(\lambda^{-1}(\theta))\mathbf{W}^{(2)}, \theta) = (deg\,\lambda^{-1}(\theta)\mathbf{W}^{(2)}\theta) - \delta(\theta, \lambda^{-1}(\theta)\mathbf{W}^{(2)})$$
$$= (n-1)(deg\,\lambda^{-1}(\theta)\cdot\mathbf{W}^{(1)})\delta - \delta(\theta\cdot\mathbf{W}^{(2)}, \lambda^{-1}(\theta)) - \delta(\theta\cdot\lambda^{-1}(\theta), \mathbf{W}^{(2)})$$
$$= (n-1)(deg\,\lambda^{-1}(\theta)\cdot\mathbf{W}^{(1)})\delta - (n-1)\delta(\mathbf{W}^{(1)}, \lambda^{-1}(\theta)) - \delta(\theta\cdot\lambda^{-1}(\theta), \mathbf{W}^{(2)})$$
$$= (n-1)(deg\,\lambda^{-1}(\theta)\cdot\mathbf{W}^{(1)})\delta - (n-1)\delta(\mathbf{W}^{(1)}, \lambda^{-1}(\theta)) - (n-2)\delta(\lambda^{-1}(\theta)\mathbf{W}^{(2)}, \theta).$$

Hence

$$\delta(\lambda^{-1}(\theta)\mathbf{W}^{(2)}, \theta) = (deg\,\lambda^{-1}(\theta)\cdot\mathbf{W}^{(1)})\delta - \delta(\mathbf{W}^{(1)}, \lambda^{-1}(\theta))$$
$$= \frac{1}{2}\sigma(\lambda'\lambda)\delta - \lambda'\lambda^{4)}.$$

PROPOSITION 16. $\delta(\theta, \lambda^{-1}(\theta)\mathbf{W}^{(2)}) = \frac{(n-2)}{2}\sigma(\lambda'\lambda)\delta + \lambda'\lambda.$

Proof. $\delta(\theta, \lambda^{-1}(\theta)\mathbf{W}^{(2)}) = (deg\,\theta\lambda^{-1}(\theta)\mathbf{W}^{(2)}) - \delta(\lambda^{-1}(\theta)\mathbf{W}^{(2)}, \theta)$

$$= (n-1)(deg\,\lambda^{-1}(\theta)\cdot\mathbf{W}^{(1)})\delta - \left(\frac{1}{2}\sigma(\lambda'\lambda)\delta - \lambda'\lambda\right)$$
$$= \frac{(n-1)}{2}\sigma(\lambda'\lambda)\delta - \frac{1}{2}\sigma(\lambda'\lambda)\delta + \lambda'\lambda$$

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4) $\sigma(\alpha) = \mathrm{Spur}\, M_l(\alpha)$. See N° 49, § VI, [1].

$$= \lambda'\lambda + \frac{(n-2)}{2}\sigma(\lambda'\lambda)\delta.$$

PROPOSITION 17. $\delta(\mathbf{W}^{(r+1)}\lambda^{-1}(\theta), \mathbf{W}^{(n-r)})$

$$= \binom{n-2}{r-1}\left(\frac{1}{2}\sigma(\lambda'\lambda)\delta - \lambda'\lambda\right).$$

Proof. $\delta(\mathbf{W}^{(r+1)}\lambda^{-1}(\theta), \mathbf{W}^{(n-r)}) = (deg\,\mathbf{W}^{(r+1)}\mathbf{W}^{(n-r)}\lambda^{-1}(\theta))\delta$

$$- \delta(\mathbf{W}^{(n-r)}\lambda^{-1}(\theta), \mathbf{W}^{(r+1)}) - \delta(\mathbf{W}^{(n-r)}\mathbf{W}^{(r+1)}\lambda^{-1}(\theta))$$

$$= \frac{(n-1)!}{r!\,(n-r-1)!}(deg\,\mathbf{W}^{(1)}\lambda^{-1}(\theta))\delta - \delta(\mathbf{W}^{(n-r)}\lambda^{-1}(\theta)\mathbf{W}^{(r+2)}, \theta)$$

$$- \frac{(n-1)!}{r!\,(n-r-1)!}\delta(\mathbf{W}^{(1)}, \lambda^{-1}(\theta)) = \frac{(n-1)!}{2r!\,(n-r-1)!}\sigma(\lambda'\lambda)\delta$$

$$- \frac{1}{r}\delta(\mathbf{W}^{(n-r+1)}\theta\lambda^{-1}(\theta)\mathbf{W}^{(r+2)}, \theta) - \frac{(n-1)!}{r!\,(n-r-1)!}\lambda'\lambda$$

$$= \frac{(n-1)!}{r!\,(n-r-1)!}\left(\frac{1}{2}\sigma(\lambda'\lambda)\delta - \lambda'\lambda\right) - \frac{1}{r}\delta(\theta\lambda^{-1}(\theta)\mathbf{W}^{(r+2)}, \mathbf{W}^{(n-r)})$$

$$= \frac{(n-1)!}{r!\,(n-r-1)!}\left(\frac{1}{2}\sigma(\lambda'\lambda)\delta - \lambda'\lambda\right)$$

$$- \frac{(n-r-1)}{r}\delta(\lambda^{-1}(\theta)\mathbf{W}^{(r+1)}, \mathbf{W}^{(n-r)}).$$

Hence $\delta(\mathbf{W}^{(r+1)}\lambda^{-1}(\theta), \mathbf{W}^{(n-r)}) = \frac{(n-2)!}{r!\,(n-r-1)!}\left(\frac{1}{2}\sigma(\lambda'\lambda)\delta - \lambda'\lambda\right)$

$$= \binom{n-2}{r-1}\left(\frac{1}{2}\sigma(\lambda'\lambda)\delta - \lambda'\lambda\right).$$

PROPOSITION 18. $\delta(\mathbf{W}^{(n-r)}, \mathbf{W}^{(r+1)}\lambda^{-1}(\theta)) = \binom{n-2}{r-1}\lambda'\lambda + \binom{n-2}{r}\frac{\sigma(\lambda'\lambda)}{2}.$

Proof. $\delta(\mathbf{W}^{(n-r)}, \mathbf{W}^{(r+1)}\lambda^{-1}(\theta))$

$$= (deg\,\mathbf{W}^{(n-r)}\mathbf{W}^{(r+1)}\lambda^{-1}(\theta))\delta - \delta(\mathbf{W}^{(r+1)}\lambda^{-1}(\theta), \mathbf{W}^{(n-r)})$$

$$= \frac{(n-1)!}{2r!\,(n-r-1)!}\sigma(\lambda'\lambda)\delta - \binom{n-2}{r-1}\left(\frac{1}{2}\sigma(\lambda'\lambda)\delta - \lambda'\lambda\right)$$

$$= \binom{n-2}{r-1}\lambda'\lambda + \binom{n-2}{r}\frac{\sigma(\lambda'\lambda)}{2}.$$

PROPOSITION 19. $\delta(\lambda_1^{-1}(\theta)\ldots\lambda_r^{-1}(\theta), \mathbf{W}^{(r)})$

$$= a_0\delta - \sum_i a_i\lambda_i'\lambda_i + \sum_{i_1, i_2, i_1\neq i_2} a_{i_1 i_2}\lambda_{i_2}'\lambda_{i_1}'\lambda_{i_1}\lambda_{i_2}$$

$$- \ldots \pm \sum_{i_1, i_2, \ldots, i_r, i_j\neq i_k} a_{i_1 i_2, \ldots, i_r}\lambda_{i_r}'\ldots\lambda_{i_1}'\lambda_{i_1}\ldots\lambda_{i_r}$$

where $a_{i_1 i_2\ldots i_h} = deg\,\lambda_{j_1}^{-1}(\theta)\ldots\lambda_{j_{r-h}}^{-1}(\theta)\cdot\mathbf{W}^{(r-h)}\, i_1, i_2, \ldots, i_h \neq j_1, \ldots, j_h.$

Proof. $\delta(\lambda_1^{-1}(\theta)\ldots\lambda_r^{-1}(\theta), \mathbf{W}^{(r)})$

$$= a_0\delta - \delta(\mathbf{W}^{(r)}, \lambda_1^{-1}(\theta)\ldots\lambda_r^{-1}(\theta))$$

$$= a_0\delta - \sum_i \delta(\mathbf{W}^{(r)} \cdot \lambda_1^{-1}(\theta) \ldots \lambda_{i-1}^{-1}(\theta)\lambda_{i+1}^{-1}(\theta) \ldots \lambda_r^{-1}(\theta), \lambda_i^{-1}(\theta))$$
$$= a_0\delta - \sum_i \lambda_i' \delta(\mathbf{W}^{(r)} \lambda_1^{-1}(\theta) \ldots \lambda_{i-1}^{-1}(\theta)\lambda_{i+1}^{-1}(\theta) \ldots \lambda_r^{-1}(\theta), \theta)$$
$$= a_0\delta - \sum_i \lambda_i' \delta(\lambda_1^{-1}(\theta) \ldots \lambda_{i-1}^{-1}(\theta)\lambda_{i+1}^{-1}(\theta) \ldots \lambda_r^{-1}(\theta), \mathbf{W}^{(r-1)}).$$

If this proposition is true for $r-1$, then by virtue of the above calculations it is also true for r. It is clearly true for $r=1$. Hence our proof is finished.

Consequently we have

PROPOSITION 20. $\delta(\mathbf{W}^{(r)}, \lambda_1^{-1}(\theta) \ldots \lambda_r^{-1}(\theta))$

$$= \sum_i a_i \lambda_i' \lambda_i - \sum_{i_1, i_2, i_1 \neq i_2} a_{i_1 i_2} \lambda_{i_2}' \lambda_{i_1}' \lambda_{i_1} \lambda_{i_2}$$
$$- \ldots \pm \sum_{i_1, i_2, \ldots, i_r\, i_j \neq i_k} a_{i_1 i_2, \ldots, i_r} \lambda_{i_r}' \ldots \lambda_{i_1}' \lambda_{i_1} \ldots \lambda_{i_r},$$

where $a_{i_1 i_2 \ldots i_h} = deg\, \lambda_{j_1}^{-1}(\theta) \ldots \lambda_{j_{r-h}}^{-1}(\theta)\mathbf{W}^{(r-h)}\, i_1, i_2, \ldots, i_h \neq j_1, \ldots, j_h.$

From these propositions, we get

THEOREM 2. *Let r be an arbitrary integer with $1 \leqq r < n$. Then there exists an integer c such that for all symmetric element α there exists a cycle* $\mathbf{X}$ *of dimension r satisfying*

$$c\alpha = \delta(\mathbf{W}^{(n-r)}, \mathbf{X}).$$

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Mathematical Institute,
Nagoya University