ON A THEOREM OF H. F. BLICHFELDT

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In 1903 H. F. Blichfeldt[1] proved the following brilliant theorem: Let G be a matrix group of order g and of degree n. Let p be a prime divisor of g such that $p>(n-1)(2n+1)$. Then G contains the abelian normal p-Sylow subgroup. In 1941 applying his modular theory of the group representation, R. Brauer[2] improved this theorem in the case in which p divides g to the first power only. Further in 1943 H. F. Tuan[3] improved this result of R. Brauer one step more.

Now in the present paper we prove the following

THEOREM. *Let G be a Soluble matrix group of order g and of degree n. Let p be a prime divisor of g such that $p>n$. Then either* (i) *G contains the normal abelian p-Sylow subgroup or* (ii) *p is a Fermat prime $p=n+1$, $n=2^m$, g is even and G contains a subgroup of peculiar type* (*See the proof below*). (*The coefficient field is the field of all complex numbers.*)

Proof. Let P be a p-Sylow subgroup of G. Let P be not abelian. Then the degree of a faithful representation of P is greater than $p(>n)$. Hence P must be abelian. Therefore we have only to prove that P is normal in G, provided that the case (ii) does not occur.

Now we prove this by induction arguments with respect to the degree and the order of the group. In particular we assume that the p-Sylow subgroup is normal in all the proper subgroups of the group G.

Let G be reducible. Then G is decomposable and we may assume that $G=\begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix}$. Let $P=\begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}$ be a p-Sylow subgroup of G. Since G_i contains the normal p-Sylow subgroup by induction hypothesis, P_i is the normal p-Sylow subgroup of G_i $(i=1, 2)$. Then P is normal in G. Hence we may assume that G is irreducible.

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1) On the order of linear homogeneous groups, Transactions Am. Math. Soc., vol. **4** (1903), 387-397.

2) On groups whose order contains a prime number to the first power II, American Journal of Mathematics, vol. **54** (1942), 421-440.

3) On groups whose orders contain a prime number to the first power, Annals of Mathematics, vol. **45** (1944), 110-140.

Since the assertion is trivially true if G is a p-group, we may assume that G is not a p-group. Let g have at least three distinct prime divisors. By a theorem of P. Hall[4] G contains a (p, q)-Sylow subgroup $S(p, q)$ for any prime q which is distinct from p. Since $G \neq S(p, q)$, $S(p, q)$ contains the normal p-Sylow subgroup by induction hypothesis. Since this holds good for any q which is distinct from p, G contains clearly the normal p-Sylow subgroup. Hence we may assume that $g = p^a q^b$ for some prime $q \neq p$.

Let G contain a normal subgroup H of index q. Then H contains the normal p-Sylow subgroup which is also the normal p-Sylow subgroup of G by induction hypothesis. Hence G contains no normal subgroup of index q. Let N be a normal subgroup of G of index p. N contains the normal p-Sylow subgroup $P(N)$ by induction hypothesis. Let us consider the centralizer $Z(P(N))$ of $P(N)$ in G. If $Z(P(N)) \neq G$, $Z(P(N))$ contains the normal p-Sylow subgroup which is also the normal p-Sylow subgroup G by induction hypothesis. Hence $Z(P(N)) = G$ and $P(N)$ is cyclic and scalar matrices. Therefore G contains the normal q-Sylow subgroup Q, and every p-Sylow subgroup P of G is cyclic.

Let Q^* be a maximal normal subgroups of G which is properly contained in Q. Let us consider $P \cdot Q^*$. Since $P \cdot Q^* \neq G$, $P \cdot Q^*$ contains the normal p-Sylow subgroup P by induction hypothesis and $P \cdot Q^* = P \times Q^*$. Let $Z(Q^*)$ be the centralizer of Q^* in G. If $Z(Q^*) \neq G$, $Z(Q^*)$ contains the normal p-Sylow subgroup of G by induction hypothesis. Hence $Z(Q^*) = G$ and Q^* is cyclic and scalar matrices. Now if Q is abelian, n must be a power of p, which is a contradiction.[5] Hence Q is not abelian. Let $Z(Q)$ be the centre of Q. Then, since $Z(Q) \subsetneqq Q$, $Z(Q) = Q^*$ is the largest normal subgroup of G, which is properly contained in Q. In particular, since $Z(Q)$ contains the commutator subgroup of Q, the class of Q is 2. Further, $Q/Z(Q)$ is abelian of type $(q, \ldots, q)$.

Let X and Y be any two elements of Q. Then $Y^{-1}XY = XZ$, where Z belongs to $Z(Q)$. Since Y^q belongs to $Z(Q)$, $Z^q = E$. Let $D(Q)$ be the commutator subgroup of Q. Then the order of $D(Q)$ is q. Further, since the class of Q is 2, it holds the formula: $(XY)^s = Y^s X^s [X, Y]^{\frac{1}{2}s(s-1)}$ for every integer s. First let us consider the case where q is odd. Let $W_1(Q)$ be the totality of elements of Q with order at most q. Then $W_1(Q)$ constitutes a subgroup which is seen from the above formula. If $W_1(Q) \neq Q$, then $W_1(Q)$ is contained in $Z(Q)$ and is of order q. This implies that Q is cyclic, which is a contradiction. Hence $W_1(Q) = Q$. Then $Z(Q) = D(Q)$ is of order q. Next let us consider the case where q is even. Let $W_2(Q)$ be the totality of elements of Q with order at most 4. Then $W_2(Q)$ constitutes a subgroup which is seen

[4] A note on soluble groups, Journal London Math. Soc., 3 (1928), 98-105.

[5] N. Itô, On the degrees of irreducible representations of a finite group, These Journal, vol. 3 (1951), 5-6.

from the above formula. If $W_2(Q) \neq Q$, then $W_2(Q)$ is contained in $Z(Q)$ and is cyclic of order 4. This implies that Q is cyclic, which is a contradiction. Hence $W_2(Q) = Q$. If $Z(Q)$ is of order 4, let us consider $Q/D(Q)$. Now P induces an automorphism $\hat{P}$ of order p in $Q/D(Q)$. In other words we can consider $Q/D(Q)$ as a representation module of $\hat{P}$. Since $p \neq 2$, this representation module is completely reducible. Now $Q/D(Q)$ contains a submodule $Z(Q)/D(Q)$. Therefore Q contains a proper normal subgroup of G different from $Z(Q)$ which is a contradiction. Hence $Z(Q) = D(Q)$ is of order 2.

Since the order of Q is q^b, Q possesses q^{b-1} linear characters and $q-1$ algebraically conjugate faithful characters of degree $q^{\frac{b-1}{2}}$, which is seen from the well known fact that the group order is the sum of the squares of the degrees of the irreducible representations. In particular, b is odd.

Again let us consider $Q/D(Q)$. Then, since the order of $Q/D(Q)$ is q^{b-1}, it is clear that $q^{b-1} - 1 \equiv 0 \pmod{p}$. Now, since $Q/D(Q)$ is irreducible for $\hat{P}$ and b is odd, we have $q^{\frac{b-1}{2}} + 1 \equiv 0 \pmod{p}$.

Let N be a normal subgroup of G of index p. Then N either is irreducible or decomposed into p distinct irreducible parts by a theorem of Z. Suetuna.[6] Since $p > n$, the last case does not occur, that is, N is irreducible. Then, since the p-Sylow subgroup of N is scalar matrices, Q is irreducible. Therefore n must be equal to $q^{\frac{b-1}{2}}$.

Thus we have two condition: $q^{\frac{b-1}{2}} + 1 \equiv 0 \pmod{p}$ and $p > q^{\frac{b-1}{2}}$. These hold only when $q^{\frac{b-1}{2}} + 1 = p$. This implies clearly that $q = 2$. In the last case, on the contrary, such a group actually exists.[7] Q.E.D.

Remark. The assumption of solubility in our theorem may be weakend formally to p-solubility. In fact, by a theorem of S. A. Čunihin[8] there exists a (p, q)-Sylow subgroup in such a group.

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6) Zerlegung der Charaktere einer Gruppe in die ihres Normalteilers, Jap. J. of Math., **12** (1935), 95-98.
cf. I. Schur, Arithmetische Untersuchungen über endliche Gruppen linearer Substitutionen, Sitzb. Berlin, (1906), 164-184.

7) N. Itô, On the characters of soluble groups, These Journal, vol. **3** (1951), 31-48.

8) S. Čunihin, O Π-svoĭstvah konečnyh grupp, Math. sb., **25** (67) (1949), 321-346.