ON THE BEHAVIOR OF AN ANALYTIC FUNCTION ABOUT AN ISOLATED BOUNDARY POINT

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Introduction. Let D be an open set in the z-plane, C its boundary, z_0 a point on C, and $f(z)$ a one-valued meromorphic function in D. Given a set $E \subset D+C$, we denote the intersection of E with $G_r = \{0<|z-z_0|<r\}$ by E_r, and the set of values $\{f(z)\,;\, z \in D_r\}$ by $f(D_r)$. The *cluster set* $S_{z_0}^{(D)}$ of $f(z)$ at z_0 in D is defined by $\bigcap_r [f(D_r)]^a$, where $[\ \ ]^a$ denotes the closure of the set in $[\ \ ]$, and the *range of values* $R_{z_0}^{(D)}$ is defined by $\bigcap_r f(D_r)$. Further the cluster set $S_{z_0}^{(E)}$ on E is defined by $\bigcap_r [\bigcup_{z \in F_r} S_z^{(D)}]^a$, where $S_z^{(D)}$ at an inner point z is put equal to $f(z)$. In the *theory of cluster sets* relations between $S_{z_0}^{(D)}$, $S_{z_0}^{(C)}$, $R_{z_0}^{(D)}$ are pursued chiefly.[1] Here we refer to the following two principal theorems under the assumption that z_0 is non-isolated:

(I) (Brelot[2]). $(S_{z_0}^{(D)})^b \subset S_{z_0}^{(C)}$, where $(\ \ )^b$ denotes the boundary of the set in $(\ \ )$.

(II) (Kunugui [5]). Each component of $S_{z_0}^{(D)} - S_{z_0}^{(C)}$, with two possible exceptions, is contained in $R_{z_0}^{(D)}$, provided that D is a domain.[3]

It is always assumed that z_0 is *non-isolated* in these theorems, and the case when z_0 is isolated is left to the well-known Picard's theorem.

Above the cluster sets are defined for a function which takes values in a plane. However, the definitions can be generalized to a function, which is defined in a plane domain and takes values on an *abstract Riemann surface*, and

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1) For various results and literatures, cf. [7].

2) See [2], Theorem in §6. The form of Brelot's theorem is different from (I), but the equivalency is proved as usual. Cf. [6], for instance.

3) This theorem can be proved also in the case where D is any open set as follows: Suppose that there exists a component Ω of $S_{z_0}^{(D)} - S_{z_0}^{(C)}$, at least three points of which do not belong to $R_{z_0}^{(D)}$. Let w_0 be such an exceptional value. Since $w_0 \in S_{z_0}^{(D)}$, we can choose $\{z_n\}$, $z_n \to z_0$, such that $f(z_n) \to w_0$. Among the inverse images in D of the segments $\overline{f(z_n)w_0}$ in Ω, we can find an inverse image l in D terminating at z_0. $f(z)$ has a limit $w_1 \in \Omega$ as $z \to z_0$ along l. Let D_1 be the component of D which contains l, and C_1 its boundary. Then $S_{z_0}^{(D_1)}$ contains w_1, and $S_{z_0}^{(D)} \supset S_{z_0}^{(D_1)}$, $S_{z_0}^{(C)} \supset S_{z_0}^{(C_1)}$, $R_{z_0}^{(D)} \supset R_{z_0}^{(D_1)}$. The component Ω_1, which contains w_1, of $S_{z_0}^{(D_1)} - S_{z_0}^{(C_1)}$ includes Ω by (I). Hence $R_{z_0}^{(D_1)}$ does not contain at least three values in Ω_1. This is contrary to (II).

some results are obtained (cf. [8], Chap. V, § 1). In this note we shall *investigate the behavior of such an analytic function about an isolated boundary point by making use of the methods in the theory of cluster sets.*

1. Let D be a domain in the z-plane, z_0 its isolated boundary point, $\mathfrak{R}$ an abstract Riemann surface in the sense of Weyl-Radó, and $f(z)$ an analytic function mapping D into $\mathfrak{R}$. Setting $\{0<|z-z_0|<r\}=G_r$ and $D\cap G_r=D_r$, we denote the set of values $\{f(z);\, z\in D_r\}$ by $\mathfrak{D}_r$. The cluster set $S_{z_0}^{(D)}$ of $f(z)$ in D at z_0 is defined by $\bigcap_r \mathfrak{D}_r^a$, where $\mathfrak{D}_r^a$ is the closure taken relatively to $\mathfrak{R}$ of $\mathfrak{D}_r$, and the range of values $R_{z_0}^{(D)}$ is defined by $\bigcap_r \mathfrak{D}_r$.[4]

We begin with the following lemma:

LEMMA. *Suppose that the cluster set* $S_{z_0}^{(D)}$ *is not empty. Then* $S_{z_0}^{(D)}$ *consists of either a point on* $\mathfrak{R}$ *or* $\mathfrak{R}$ *itself.*

Proof. Suppose that the assertion is not true. Then there is a neighborhood N on $\mathfrak{R}$ of a boundary point P_0 of $S_{z_0}^{(D)}$ such that $S_{z_0}^{(D)} \not\subset N^a$. Let $\Delta: |t|<1$ be a local parameter circle, corresponding to N and with $t=0$ as the image of P_0. Consider the inverse image D_1 in D of N, and denote the composed function $t(f(z))$ in D_1 by $t(z)$. Since $P_0\in S_{z_0}^{(D)}$, we can find a sequence $\{z_n\}$ tending to z_0 such that $f(z_n)\to P_0$. Hence z_0 is a boundary point of D_1. Further z_0 is not isolated, because there is a sequence $\{z_n'\}$, $z_n'\to z_0$, outside D_1 such that $f(z_n')$ tends to a certain point of $S_{z_0}^{(D)}$ outside N. Thus D_1 is an open subset of D, with z_0 as its non-isolated boundary point. The cluster set of $t(z)$ on the boundary of D_1 at z_0 consists of points on $|t|=1$ but does not contain $t=0$, whereas this point belongs to the boundary of the cluster set of $t(z)$ in D_1 at z_0. This contradicts (I) in the introduction.

2. Let us suppose first that $\mathfrak{R}$ is of genus finite. $\mathfrak{R}$ is then conformally equivalent to a subsurface of a certain closed Riemann surface $\underline{\mathfrak{R}}$. The transformed function, which takes values on $\underline{\mathfrak{R}}$, of $f(z)$ will be denoted by $F(z)$. We shall use notations $\underline{S}_{z_0}^{(D)}$ and $\underline{R}_{z_0}^{(D)}$ to represent the cluster set and the range of values of $F(z)$ respectively. Since $\underline{S}_{z_0}^{(D)}$ is non-empty, it consists of a point on $\underline{\mathfrak{R}}$ or of $\underline{\mathfrak{R}}$ itself by the above lemma.

In case $\underline{S}_{z_0}^{(D)}$ consists of one point on $\underline{\mathfrak{R}}$, the image $\mathfrak{D}_r$ on $\mathfrak{R}$ of D_r converges to an inner point of $\mathfrak{R}$ or to a parabolic ideal boundary component of $\mathfrak{R}$ as $r\to 0$.[5]

The case in which $\underline{S}_{z_0}^{(D)}=\underline{\mathfrak{R}}$ will be investigated in details in the sequel. We shall denote the genus of $\underline{\mathfrak{R}}$ by $\underline{p}$.

Case: $\underline{p}=0$. We suppose that $\underline{\mathfrak{R}}-\underline{R}_{z_0}^{(D)}$ contains at least three points,

[4] Notice that $f(z)$, $S_{z_0}^{(D)}$ and $R_{z_0}^{(D)}$ take values on a Riemann surface here, though the same notations as in the introduction are used.

[5] As for the definition of a parabolic ideal boundary component, see [8], Chap. III, § 5.

say, $\underline{P}_1$, $\underline{P}_2$, $\underline{P}_3$. Since $\underline{P}_1 \in \underline{S}_{z_0}^{(D)}$, there is a sequence $\{z_n\}$ tending to z_0 such that $F(z_n) \to \underline{P}_1$. Connect every $F(z_n)$ with $\underline{P}_1$ by a curve L_n such that L_n approaches $\underline{P}_1$ as $n \to \infty$. For a sufficiently large number n_0 the inverse image l_{n_0} with z_{n_0} as its starting point must lie near z_0 and hence terminate at z_0, because $\{F(z)\,;\, z \in l_n\} \subset L_n \to \underline{P}_1$ as $n \to \infty$. A part D_0 of D, near z_0 and cut by l_{n_0}, can be regarded as an angular domain with the opening 2π. $F(z)$ tends to a value $\underline{P}_0 \in L_{n_0}$ as $z \to z_0$ on l_{n_0}. Since $F(z) \neq \underline{P}_1$, $\underline{P}_2$, $\underline{P}_3$, near z_0, $F(z)$ tends to $\underline{P}_0$ uniformly as $z \to z_0$ in D_0 by Lindelöf-Iversen's theorem [3]. Thus $\underline{S}_{z_0}^{(D)} = \{\underline{P}_0\}$, and a contradiction is lead. Therefore when $\mathfrak{R}$ is of genus zero and $\underline{S}_{z_0}^{(D)} = \mathfrak{R}$, then $\underline{R}_{z_0}^{(D)}$ contains all points of $\mathfrak{R}$ with two possible exceptions. This fact is none other than Picard's theorem.

Case: $\underline{p} = 1$. Suppose that $\underline{R}_{z_0}^{(D)} \neq \underline{S}_{z_0}^{(D)} = \mathfrak{R}$, and take a point $\underline{P} \in \mathfrak{R} - \underline{R}_{z_0}^{(D)}$. In the mapping of the universal covering surface $\mathfrak{R}^\infty$ of $\mathfrak{R}$ onto the finite whole w-plane, $\underline{P}_1$ corresponds to an enumerably infinite number of points in the plane. Similarly as in the preceding case we get a curve l terminating at z_0 such that $F(z)$ tends to a value $\underline{P}_0$ on $\mathfrak{R}$ as $z \to z_0$ along l. In the angular domain D_0 cut by l, any branch $w(z)$ of the composed function $w(F(z))$ becomes one-valued regular by monodromy theorem. It tends to respective definite limits along both sides of l and does not take near z_0 the w-values corresponding to $\underline{P}_1$. Hence $w(z)$ tends to a certain value uniformly in D_0 by Lindelöf-Iversen's theorem. This shows $\underline{S}_{z_0}^{(D)} = \{\underline{P}_0\}$, contrary to the assumption that $\underline{S}_{z_0}^{(D)} = \mathfrak{R}$. Thus, when $\mathfrak{R}$ is of genus one and $\underline{S}_{z_0}^{(D)} = \mathfrak{R}$, then $\underline{R}_{z_0}^{(D)} = \mathfrak{R}$.

Case: $\underline{p} \geqq 2$. On mapping $\mathfrak{R}^\infty$ onto $|w| < 1$ it is shown from $\underline{S}_{z_0}^{(D)} = \mathfrak{R}$ as above that $\underline{R}_{z_0}^{(D)} = \mathfrak{R}$. $\mathfrak{R}$ is made of planar character by $\underline{p}$ disjoint simple closed curves $\{C_i\}$ $(i = 1, 2, \ldots, \underline{p})$. By connecting infinitely many samples along the opposite shores of $\{C_i\}$, we obtain a Schottky covering surface $\overline{\mathfrak{R}}$, of planar character and having no relative boundary, over $\mathfrak{R}$. $\overline{\mathfrak{R}}$ is mapped conformally onto a domain outside a perfect set F in the w-plane and any image of C_i is a closed curve. For any $\underline{P}_1 \in C_1$ there exists a sequence $\{z_n\}$ tending to z_0 such that $F(z_n) = \underline{P}_1$. We may suppose that on C_1 there is no image of a double point of $F(z)$. We denote by C_1' a conjugate curve, which intersects C_1 merely at $\underline{P}_1$ and on which no image of a double point lies. Let l_n be the inverse image through z_n of C_1. If no l_n terminates at z_0, there exists a number n_0 such that every l_n for $n \geqq n_0$ is a simple closed curve around z_0, because disjoint inverse images of C_1 can not cluster in D and no image is a closed curve surrounding a compact domain in D. Consider the inverse image l_{n_0}' of C_1', which starts from z_{n_0} and runs inside l_{n_0}. A domain near and inside l_{n_0} corresponds to one side of C_1 on $\mathfrak{R}$. Therefore l_{n_0}' can not intersect l_{n_0} again and hence must terminate at z_0. Thus the inverse image through z_n of C_1 or C_1' terminates at

z_0 for any large n. Without loss of generality we may suppose that an image l of C_1 terminates at z_0. In the angular domain D_0 cut by l, any branch $w(z)$ of the composed function $w(F(z))$ becomes one-valued regular. Its cluster sets S_1 and S_2 on the both sides of l at z_0 lie either on one and the same image Γ of C_1 or on two images Γ_1 and Γ_2 of C_1 respectively. In the former case $S_1 \cap S_2$ is not empty and the cluster set S of $w(z)$ at z_0 in D_0 coincides with $S_1 \cup S_2$ on account of (I), (II), because $w(z)$ does not take values of the perfect set F whose points lie both outside and inside Γ. Hence $\underline{S}_{z_0}^{(D)} \subset C_1$, but this contradicts the assumption: $\underline{S}_{z_0}^{(D)} = \underline{\mathfrak{R}}$. The latter case is impossible too by (I), (II), because S is a continuum but every component of the complement of $\Gamma_1 \cup \Gamma_2$ contains points of F. Hence it does not arise that $\underline{S}_{z_0}^{(D)} = \underline{\mathfrak{R}}$ for $\underline{\mathfrak{R}}$ of genus $\underline{p} \geqq 2$.

We have considered so far the case when the genus of the original $\mathfrak{R}$ is finite. Finally we suppose that $\mathfrak{R}$ is of genus infinite. If there is $r>0$ such that $\mathfrak{D}_r$ is of genus finite, the foregoing discussions apply. Consequently we suppose that every $\mathfrak{D}_r$ is of genus infinite. We can then take a mutually non-homotopic disjoint infinite sequence of loop cuts $\{C_n\}$, $C_n \subset \mathfrak{D}_{1/n}$, such that C_n does not divide $\mathfrak{R}$ and approaches the ideal boundary of $\mathfrak{R}$ as $n \to \infty$. As in the preceding case we find an inverse image, which terminates at z_0, of a certain C_n or its conjugate loop cut C_n'. The cluster set of $f(z)$ along it is contained in C_n or C_n' and hence non-empty. Accordingly by Lemma in § 1 $S_{z_0}^{(D)} = \mathfrak{R}$. By considering the Schottky covering surface of $\mathfrak{R}$ a contradiction will be lead as before.

We now summarize the results in the following:

THEOREM 1. *Let $f(z)$ be a function, which is defined in a plane domain D with an isolated boundary point z_0 and takes values on an abstract Riemann surface $\mathfrak{R}$. Then either the image of the ring domain $G_r: 0<|z-z_0|<r$ contained in D converges to an inner point of $\mathfrak{R}$ or to a parabolic ideal boundary component of $\mathfrak{R}$ as $r \to 0$, or the range of values of $f(z)$ in D at z_0 is conformally equivalent to a sphere with two possible exceptions or to a torus.*

It is easy to find functions which realize these cases.

3. When $\mathfrak{R}$ is of genus finite, Theorem 1 can be proved also by Ahlfors' theory of covering surfaces [1]. We shall give an outline of the proof.

Since there exists a one-valued non-constant meromorphic function on $\underline{\mathfrak{R}}$ of § 2, $\mathfrak{R}$ is conformally equivalent to a subsurface of a closed surface $\mathfrak{R}_\sigma$, which covers the Riemann sphere σ touching the w-plane at $w=0$ and with diameter of length 1. Denoting the composed function $w(f(z))$ by $w(z)$, we consider the Riemann surface $\overline{\mathfrak{R}}_w$ of the inverse function of $w(z)$. If $z=0$ is removable for $w(z)$, the image on $\mathfrak{R}_\sigma$ of G_r converges to a point on $\mathfrak{R}_\sigma$. The image on $\mathfrak{R}$ of G_r converges then to a point or to a parabolic ideal boundary component of $\mathfrak{R}$.

Hence suppose that $z=0$ is an essential singularity of $w(z)$. Similarly as for Riemann surfaces of parabolic type, it is seen that $\overline{\mathfrak{R}}_w$ is regularly exhaustible. Regard now $\overline{\mathfrak{R}}_w$ as a covering surface over $\mathfrak{R}_\sigma$ and denote it by $\overline{\mathfrak{R}}_\sigma$. Then $\overline{\mathfrak{R}}_\sigma$ is still a regularly exhaustible covering surface over $\mathfrak{R}_\sigma$, because the closed surface $\mathfrak{R}_\sigma$ covers σ only in finite times.

On the other hand, if the genus of $\mathfrak{R}_\sigma$ is $q \geqq 2$, Ahlfors' fundamental inequality gives

$$0 = \overset{+}{\rho} \geqq (2q-2)S(r) - hL(r)\,,$$

where the usual notations are used; especially, $S(r)$ is the average covering number over $\mathfrak{R}_\sigma$ of the part of $\overline{\mathfrak{R}}_\sigma$ corresponding to $D-G_r^a$. Hence

$$\frac{L(r)}{S(r)} \geqq \frac{2q-2}{h} > 0\,,$$

which contradicts the fact that $\overline{\mathfrak{R}}_\sigma$ is regularly exhaustible.

Next suppose that $\mathfrak{R}_\sigma$ is of genus one. If there is a number $r_0 > 0$ such that the part $\overline{\mathfrak{R}}'_\sigma$ of $\overline{\mathfrak{R}}_\sigma$ corresponding to G_{r_0} does not cover a point P_0 of $\mathfrak{R}_\sigma$, regard $\overline{\mathfrak{R}}'_\sigma$ as a covering surface over $\mathfrak{R}'_\sigma = \mathfrak{R}_\sigma - \{P_0\}$. Applying Ahlfors' inequality to them, there follows $L(r)/S(r) \geqq 1/h > 0$, which contradicts the regular exhaustibility of $\overline{\mathfrak{R}}'_\sigma$. As is known, Picard's theorem is proved by the same method.

It is not comprehensible to me, however, how such a method can be utilized in the case when $\mathfrak{R}$ is of genus infinite.

4. In [8], Chap. III, § 6, the following theorem was proved:

THEOREM 2. *Let $\mathfrak{R}$ be an abstract Riemann surface with universal covering surface $\mathfrak{R}^\infty$ of hyperbolic type. In the mapping of $\mathfrak{R}^\infty$ onto U: $|z|<1$, the parabolic ideal boundary components of $\mathfrak{R}$ and the classes of parabolic fixed points, equivalent with respect to a Fuchsian group, on Γ: $|z|=1$ correspond to each other in a one-to-one manner.*

The proof in [8] was different from the usual one given for a plane domain (e.g., [4], pp. 31-34). But once Theorem 1 is established, Theorem 2 can be proved in the usual way.

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