ON ONE-PARAMETER SUBGROUPS IN FINITE DIMENSIONAL LOCALLY COMPACT GROUP WITH NO SMALL SUBGROUPS

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Let G be a locally compact topological group and let U be a neighborhood of the identity in G. A curve $g(\lambda)$ $(|\lambda| \leqq 1)$ in G, which satisfies the conditions,

$$g(s)g(t) = g(s+t) \quad (|s|,\ |t|,\ |s+t| \leqq 1),$$

is called a one-parameter subgroup of G. If there exists a neighborhood U_1 of the identity in G such that for every element x of U_1 there exists a unique one-parameter subgroup $g(\lambda)$ which is contained in U and $g(1) = x$, we shall call, for the sake of simplicity, that U has the property (S)*. It is well known that the neighborhoods of the identity in a Lie group have the property (S)*. More generally it is proved that if G is finite dimensional, locally connected, and is without small subgroups,[1] G has the same property.[2] In this note, these theorems will be generalized to the case when G is finite dimensional and without small subgroups.

The writer's proof is based on the theorems recently developed by D. Montgomery and A. Gleason.[3] Their theorems, which will be used in this note, are summarized in §1. In §2 it will be proved that the group G, which is finite dimensional and without small subgroups, is locally connected and our theorem is reduced to the known case.

§1. THEOREM 1 (*Montgomery*).[4] *Let G be a locally compact locally connected n-dimensional group $(n < \infty)$. Then there exists a neighborhood V of the identity in G possessing the following properties:*

Let A and B, $(B \subset A)$, be compact subsets of V. Then the sufficient con-

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1) G is called to be without small subgroups, if there exists a compact neighborhood of the identity in G which does not contain non-trivial subgroups of G.

2) Cf. Chevalley, C. [1]. C. Chevalley proved the case when G is locally euclidean and without small subgroups. K. Iwasawa communicated to the present author that D. Montgomery pointed out that the Chevalley's method may be applicable even when G is locally connected and without small subgroups. It is also informed that H. Yamabe obtained the same result.

3) D. Montgomery [7], [8], [9], [10], A. Gleason [2].

4) This Theorem and its Corollaries are valid when G is a locally connected finite dimensional homogeneous space, or more generally, G is a locally homogeneous space. See D. Montgomery [8].

ditions for $A-B$ *to be an open subset of* G *are*

1) B *carries an* $(n-1)$*-cycle* z[5] *which is not homologous to zero in* B, *and*

2) A *is minimal with respect to the properties*

   a) $B\subset A$

   b) z *is homologous to zero in* A.

COROLLARY 1 to THEOREM 1. (*Invariance theorem of domain*). *Let* G_1 *and* G_2 *be locally compact locally connceted groups. Suppose that* $\dim G_1=\dim G_2=n<\infty$. *Let* M *be an open subset of* G_1 *and* f *be a topological mapping of* M *into* G_2. *Then the image* $f(M)$ *of* M *under the mapping* f *is an open subset of* G_2.

Proof. Let V_i be the neighborhood of the identity in G_i pointed out in Theorom 1 $(i=1, 2)$. Let p_2 be a point of $f(M)$ and let p_1 be the ponit M such that $f(p_1)=p_2$. We can take a neighborhood V_1' of the identity in G_1 such that $\overline{V}_1'\subseteqq V_1$, $\overline{V}_1'p_1\subseteqq M$, $f(\overline{V}_1'p_1)\subseteqq V_2p_2$. Since the dimension of $\overline{V}_1'$ is n, there exist compact subsets A_1 and B_1 of $\overline{V}_1'$ satisfying the conditions *1*) and *2*) of the Theorem 1.[6] Moreover, we can assume that the identity in G_1 is contained in A_1-B_1. Then $f(A_1p_1)$ and $f(B_1p_1)$ are subsets of V_2p_2 and satisfy the conditions *1*) and *2*) of the Theorem 1. Hence by Theorem 1 $f(A_1p_1)-f(B_1p_1)$ is an open subset of G_2. Since $p_2\in f(A_1p_1)-f(B_1p_1)\subseteqq f(M)$, $f(M)$ is an open subset of G_2.

COROLLARY 2 to THEOREM 1. *Under the same notations and assumptions as in the Corollary 1, let* N *be an open subset of* M *such that* $\overline{N}\subseteqq M$, *and let* x *be an arbitrary point of* $f(N)$. *Then*

$$C_x(G_2-f(bdry\,N))^{7)}\subseteqq f(N).$$

Proof. From the Corollary 1, it is easy to prove that

$$bdry\,f(N)=f(bdry\,N).$$

Hence, $G_2-f(bdry\,N)=G_2-bdry\,f(N)=f(N)\cup(G_2-\overline{f(N)})$, $f(N)\cap(G_2-\overline{f(N)})=\phi$,[8] and both $f(N)$ and $(G_2-\overline{f(N)})$ are open subsets of G_2. Since $x\in f(N)$, it follows that

$$C_x(G_2-f(bdry\,N)\subseteqq f(N)).$$

THEOREM 2[9] (*Montgomery*). *Let* G *be a locally compact* n*-dimensional*

5) Cycles are in the sense of Cech.

6) Cf. Hurewicz and Wallman [2], p. 151.

7) If x is a point of topological space A, $C_x(A)$ is the connected component of A which contains x.

8) ϕ denotes the empty set.

9) This is a part of Theorem 7 of D. Montgomery [9].

group ($n<\infty$). *Then there exists a locally compact locally connected group* G^* *of dimension* n *and a continuous one-to-one mapping* α *of* G^* *into* G *satisfying the following conditions.*

Let C^* *be a neighborhood of the identity in* G^*, *then* $\alpha(C^*)=C$ *is an invariant local snbgroup of* G *and the factor local group of* G *by* C *is zero-dimensional.*

§2. A neighborhood U of the identity in a topological group G is called to have the property (S), if for every element x of U there exists an integer n such that $x^{2^n}\notin U$.

LEMMA 1 (*Yamabe*).[10] *Let* G *be a locally compact group, and suppose that* G *is without small subgroups. Let* U *be a neighborhood of the identity* e *in* G *such that* U *contains no non-trivial subgroups. For every neighborhood* V *of* e *there exists a neighborhood* V^* *of* e *satisfying the following conditions.*

If x *and* x^k *are contained in* V^* *and if* x^i $(1\leqq i\leqq k)$ *are elements of* U, *then* x^i *is contained in* V *for* $i=1,2,\ldots,k$.

COROLLARY to LEMMA 1 (*Yamabe and Gotô*).[11] *If a locally compact group* G *is without small subgroups,* G *has the property* (S).

LEMMA 2.[12] *Let* G *be a locally compact group which is without small subgroups. Then there exists a neighborhood* U *of the identity in* G, *in which the square root is unique. More strictly, if* x *and* y *are elements of* U, *and if* $x^2=y^2$, *it follows that* $x=y$.

In this case the mapping $\varphi(x)=x^2$ of U into G is one-to-one.

LEMMA 3.[12] *Let* G *be a locally compact group which is without small subgroups. Then on a sufficiently small neighborhood* U *of the identity in* G *we can define a real valued continuous function* $f(x)$ *satisfying the following conditions.*

$$f(x^2)\geqq 2f(x) \quad \text{for} \quad x,\ x^2\in U, \tag{3}$$

$$f(x)=0 \text{ if and only if } x \text{ is the identity.} \tag{4}$$

Now let U be a local group and let C be an invariant local subgroup of U. If we take a sufficiently small neighborhood W of the identity in U the factor local group W/C is defined as follows.[13]

(i) The element X of W/C is the coset $W\cap Cx$ for $x\in W$.

(ii) We shall consider that the product XY of a pair of elements X, Y of W/C is defined if and only if there exist elements $x\in X$ and $y\in Y$ such that

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10) For the proof, see H. Yamabe [12].

11) H. Yamabe and M. Gotô [4].

12) See Kuranishi [5] and [6].

13) Pontrjagin [11], p. 83.

xy is contained in W. The product XY is equal to $W \cap Cxy$, which is independent of the choices of x and y.

(iii) The natural mapping $W \to W/C$ is continuous and open.

Let G be a locally compact finite dimensional group. Suppose that G is without small subgroups. Let G^* and α be the locally compact locally connected group and the continuous one-to-one mapping of G^* into G stated in Theorem 2. Let U be the sufficiently small neighborhood of the identity in G on which the function $f(x)$ of Lemma 2 is defined. U is naturally a local group. Take a sufficiently small open neighborhood C^* of the identity in G^* and let $C = \alpha(C^*)$. By Theorem 2 C is an invariant local subgroup of U. Take a sufficiently small neighborhood W_1 of the identity in U so that the factor local group W_1/C is defined. By Theorem 2 W_1/C is a zero-dimensional locally compact local group. Let β be the natural mapping $W_1 \to W_1/C$ and let φ be a mapping $\varphi(x) = x^2$. Take an open neighborhood W of the identity in U such that $\overline{W} \subseteqq W_1$. Let V_1 be the neighborhood of the identiy in U such that

(5) $$\varphi(bdry\ W) \cap V_1^2 = \phi,$$

(6) $$V_1^2 \subseteqq W,$$

(7) $$V_1 \cap C \text{ is connected.}$$

Let V be a neighborhood of the identity in U such that $V^4 \subseteqq V_1$, $V = V^{-1}$.

LEMMA 4. *Let X be an element of $\beta(V)$ such that X^2 is contained in $\beta(V)$. Then for every element y of $X^2 \cap \overline{V}$, there exists an element x of X such that $y = x^2$.*

Proof. Let
$$X = W_1 \cap Cx_0, \quad x_0 \in V,$$
and
$$M^* = \alpha^{-1}((W_1 \cap Cx_0)x_0^{-1}).$$

We define the topological mapping $\psi(a)$ of M^* into G^* by
$$\psi(a) = \alpha^{-1}((\varphi((\alpha(a))x_0))x_0^{-2}).^{14)}$$

Since $N^* = \alpha^{-1}((W \cap Cx_0)x_0^{-1})$ is an open set containing the identity e^* in G^* and $\overline{N}^* \subseteqq M^*$, by Corollary 2 to Theorem 1,

(8) $$C_{e^*}(G^* - \psi(bdry\ N^*)) \subseteqq \psi(N^*).$$

Since
$$\begin{aligned} &\alpha(\alpha^{-1}(V_1 \cap C) \cap \psi(bdry\ N^*)) \\ &\subseteqq (V_1 \cap C) \cap (\varphi(bdry\ (W \cap Cx_0)))x_0^{-2} \\ &\subseteqq [V_1 x_0^2 \cap \varphi(bdry\ W)]x_0^{-2} \\ &\subseteqq [V_1^2 \cap \varphi(bdry\ W)]x_0^{-2} = \phi \qquad \text{(by condition (5))} \end{aligned}$$

14) α is the injection of G^* into G.

and since $V_1 \cap C$ is connected, it follows that

$$\alpha^{-1}(V_1 \cap C) \subseteqq C_{e^*}(G^* - \psi(bdry\, N^*)). \tag{9}$$

If $cx_0^2 \in X^2 \cap \overline{V} = (W_1 \cap Cx_0^2) \cap \overline{V} = \overline{V} \cap Cx_0^2$, it follows that

$$c \in \overline{V} x_0^{-2} \cap C \subseteqq \overline{V} V^{-2} \cap C \subseteqq V_1 \cup C,$$

that is,

$$x^2 \cap \overline{V} \subseteqq (V_1 \cap C) x_0^2$$

$$\alpha^{-1}[(X^2 \cap \overline{V}) x_0^{-2}] \subseteqq \alpha^{-1}(V_1 \cap C). \tag{10}$$

From (8), (9) and (10), it follows that

$$\alpha^{-1}[(X^2 \cap \overline{V}) x_0^{-2}] \subseteqq \psi(N^*) = \alpha^{-1}[\varphi(W \cap Cx_0) x_0^{-2}],$$

that is,

$$X^2 \cap V \subseteqq \varphi(W \cap Cx_0).$$

Hence the lemma is proved.

We now define the function $F(X)$ on $\beta(\overline{W})$ by

$$F(X) = \inf_{x \in X \cap \overline{V}} f(x).^{15)} \tag{11}$$

LEMMA 5. *Let V be the neighborhood of the identity in G stated in Lemma 4. We can assume without loss of generality that $V = \{x | f(x) \leqq \delta\}$, where $f(x)$ is the function of Lemma 3. Then*

$$F(X^2) \geqq 2F(X) \quad \textit{if } X,\ X^2 \in \beta(V), \tag{12}$$

$$F(X) = 0 \quad \textit{if and only if } X \textit{ is the identity}, \tag{13}$$

$$F(X) \quad \textit{is continuous.} \tag{14}$$

Proof. Continuity of $F(X)$: Let $X_n \in \beta(V)$, and $X_n \to X \in \beta(V)$. There exists a sequence x_n $(n = 1, 2, \dots)$ of V such that $F(X_n) = f(x_n)$. We can assume without loss of generality that $x_n \to x \in V \cap X$. Then

$$F(X) \leqq f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} F(X_n). \tag{15}$$

Let x be the element of X such that $F(X) = f(x)$. For arbitrary positive number ε, there exists a neighborhood V_2 of the identity in G such that

$$f(y) \leqq f(x) + \varepsilon \quad \text{for} \quad y \in V_2 x.$$

Since β is an open mapping, there exists an integer N' such that

$$X_n \in \beta(V_2 x) \quad \text{for} \quad n > N'.$$

[15] $f(x)$ is the function of Lemma 3.

Let x_n be a point of $X_n \cap V_2 x$, $(n = N'+1, N'+2, \ldots)$. Then

(16) $$F(X_n) \leqq f(x_n) \leqq f(x) + \varepsilon = F(X) + \varepsilon \quad \text{for} \quad n \leqq N',$$

from (15) and (16) it follows that $F(X)$ is a continuous function on $\beta(\overline{W})$.

(13) is obvious. We shall prove (12). Suppose that X and X^2 are elements of $\beta(V)$. There exists an element y of $X^2 \cap V$ such that

$$F(X^2) = f(y).$$

From Lemma 4 and the fact that $V = \{x | f(x) \leqq \delta\}$, there exists an element x of $X \cap V$ such that $x^2 = y$.
Hence

$$F(X^2) = f(y) = f(x^2) \geqq 2f(x) \geqq 2F(X).$$

LEMMA 6. *Let G be a locally compact finite dimensional group. Suppose that G is witout small subgroups. Then G is locally connected.*

Proof. Let V be a sufficiently small neighborhood of the identity in G. Since W/C is a zero-dimensional local group, $\beta(V)$ contains an open and compact subgroup H of W/C. We can take H so that H is the group in the large, i.e., the product is defined for every pair of elements of H and is contained in H.[17] By Lemma 5 there is defined the function $F(X)$ on the compact group H and satisfies the conditions

(12)′ $$F(X^2) \geqq 2F(X) \quad \text{for every element } X \text{ of } H.$$

(13). and (14). Hence H must be the group consisting of the identity element only. Since H is an open subset of W/C, W/C must be a discrete space. Thus W is locally connected.

THEOREM 3. *Let G be a finite dimensional locally compact group. Suppose that G is without small subgroups. Then for every neighborhood U of the identity in G there exists a neighborhood U_1 satisfying the following conditions.*

"For every element x of U_1, there exists a unique one-parameter subgroup $g(\lambda)$ $(0 \leqq \lambda \leqq 1)$ contained in U such that $g(1) = x$."

Proof. We can suppose without loss of generality that

(18) the function $f(x)$ of Lemma 3 is defined on U, and that

(19) the mapping $\varphi(x) = x^2$ of U into G is one-to-one. (Lemma 2.)

Take a neighborhood V of the identity in G such that $V^2 \subseteqq U$ and let V^* be an open neighborhood of the identity in G of the Lemma 1 with respect to V. By Lemma 6, G is locally connected. Hence from the condition (19) and

[17] This can be proved in the same way as in the case of the locally compact zero-dimensional groups.

the Corollary 1 to Theorem 1, $\varphi(V^*)$ is an open subset G and contains the identity. Choose a sufficiently small positive number δ such that

$$U_1 = \{x \mid f(x) < \delta\} \subseqq V^* \cap \varphi(V^*). \tag{20}$$

For every element x of U_1, there exists an element x_1 of V^* such that $x = x_1^2$. Since $f(x_1) \leqq \frac{1}{2} f(x_1^2) = \frac{1}{2} f(x) < \delta$, x_1 is contained in U_1. Thus there exists a suquence x_n $(n = 1, 2, \ldots)$ of elements of U_1 such that

$$x = x_n^{2^n}.$$

Since the square root is unique (Lemma 2),

$$x_n x_m = x_m x_n$$

and

$$x_n = x_m^{2^{m-n}} \quad \text{for} \quad m \geqq n.$$

Then there exists a unique one-parameter subgroup $g(\lambda)$ such that $g\left(\frac{1}{2^n}\right) = x_n$ for $n = 1, 2, \ldots$.[17] Suppose that

$$g\left(\frac{m}{2^n}\right) \subseteqq V \quad \text{for} \quad m = 1, 2, \ldots, 2^n.$$

Put $y = g\left(\frac{1}{2^{n+1}}\right) \in U_1$. For $m = 2m' + 1$,

$$y^m = g\left(\frac{m}{2^{n+1}}\right) = g\left(\frac{m'}{2^n}\right) g\left(\frac{1}{2^{n+1}}\right) \in V^2 \subseqq U.$$

Hence

$$y^m \in U \quad \text{for} \quad m = 1, 2, \ldots, 2^{n+1},$$

and

$$y,\ y^{2^{n+1}} \in U_1 \subseqq V^*.$$

By Lemma 1, $\quad y^m \in V \quad \text{for} \quad m = 1, 2, \ldots 2^{n+1}.$

Hence

$$g\left(\frac{m}{2^n}\right) \in V \subseqq U \quad \text{for} \quad m = 1, 2, \ldots, 2^n, \quad n = 1, 2, \ldots.$$

Thus

$$g(\lambda) \in V \subseqq U \quad \text{for} \quad 0 \leqq \lambda \leqq 1.$$

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