

# SOME REMARKS ON THE EXCEPTIONAL SIMPLE LIE GROUP $\mathfrak{F}_4$

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1. Let  $\mathbb{C}$  be the Cayley algebra of dimension 8 over the field  $R$  of real numbers and let  $\mathfrak{J}$  be the set of all  $3 \times 3$  Hermitian matrices

$$(1) \quad X = \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix}$$

with coefficients in  $\mathbb{C}$ . We define the multiplication in  $\mathfrak{J}$  by

$$X \circ Y = \frac{1}{2}(XY + YX).$$

Then  $\mathfrak{J}$  becomes a distributive algebra over  $R$ . A non-singular linear transformation  $\alpha$  of  $\mathfrak{J}$  is said to be an automorphism of  $\mathfrak{J}$ , if

$$\alpha(X \circ Y) = \alpha X \circ \alpha Y$$

for all  $X, Y \in \mathfrak{J}$ . The group  $\mathfrak{A}$  of all the automorphisms of  $\mathfrak{J}$  is compact and the connected component containing the identity of  $\mathfrak{A}$  is the exceptional simple compact group  $\mathfrak{F}_4$ .<sup>1)</sup> Denote by  $E_i$  the matrix (1) with  $\xi_i = 1$ , all remaining terms zero. Let  $\mathfrak{A}_i$  be the subgroup of  $\mathfrak{F}_4$  consisting of all automorphisms  $\alpha$  such that  $\alpha E_i = E_i$  for  $i = 1, 2, 3$  and let  $\mathfrak{H}_i$  ( $i = 1, 2, 3$ ) be the subgroups of  $\mathfrak{F}_4$  consisting of all  $\alpha \in \mathfrak{F}_4$  such that  $\alpha E_i = E_i$ . Then the left coset spaces  $\mathfrak{F}_4/\mathfrak{H}_i$  are homomorphic to the set  $\Pi$  of all irreducible idempotents of  $\mathfrak{J}$  and  $\Pi$  is geometrically the “*plan projectif des octaves*.”<sup>2)</sup>

In this note we prove the following two theorems.

**THEOREM 1.**  *$\mathfrak{A}$  is connected and isomorphic to the universal covering group  $\widetilde{SO}(8)$  of the proper orthogonal group  $SO(8)$  of 8 dimensional euclidean space.*

**THEOREM 2.**  *$\mathfrak{H}_i$  are connected and isomorphic to the universal covering group  $\widetilde{SO}(9)$  of the proper orthogonal group  $SO(9)$  of 9 dimensional euclidean space.*

Theorem 2 gives a proof of a result announced by A. Borel.<sup>3)</sup>

Received October 22, 1951.

<sup>1)</sup> See, Chevalley-Schafer [2] and Freudenthal [3].

<sup>2)</sup> See, Freudenthal [3] §7 and Hirsch [4].

<sup>3)</sup> See, Borel [1], Théorème 1.

2. *Proof of Theorem 1.* Let  $F_i^a$  be the matrix (1) with  $x_i = a$  and all numbers except  $x_i$  zero. Then  $E_i \circ F_i^a = 0$ ,  $E_j \circ F_i^a = \frac{1}{2} F_i^a$  if  $i \neq j$ . Let  $\alpha \in \mathfrak{H}$ . Then  $E_i \circ \alpha F_i^a = 0$  and  $E_j \circ \alpha F_i^a = \frac{1}{2} \alpha F_i^a$ . It follows that

$$\alpha F_i^a = F_i^{\alpha_i a}, \quad (i = 1, 2, 3),$$

where  $\alpha_i$  are the linear transformations of  $\mathfrak{G}$ .

Now  $F_i^a \circ F_i^b = (a, b)(E_j + E_k)$ ,<sup>4)</sup> where  $\{i, j, k\}$  is a permutation of  $\{1, 2, 3\}$ , implies

$$(\alpha_i a, \alpha_i b) = (a, b).$$

Denote by  $O(8)$  the group of all linear transformations of  $\mathfrak{G}$  which leave the positive definite bilinear form  $(a, b)$  invariant. (i.e. orthogonal transformations of  $\mathfrak{G}$ .) Further  $F_1^{2x} \circ F_2^{2y} = F_3^{2(xy)}$ ,  $F_2^{2x} \circ F_3^{2y} = F_1^{2(xy)}$  and  $F_3^{2x} \circ F_1^{2y} = F_2^{2(xy)}$  imply

$$(2) \quad \begin{cases} \alpha_1(x)\alpha_2(y) = \kappa\alpha_3(xy), \\ \alpha_2(x)\alpha_3(y) = \kappa\alpha_1(xy), \\ \alpha_3(x)\alpha_1(y) = \kappa\alpha_2(xy), \end{cases}$$

where  $\kappa\alpha_i(x) = \overline{\alpha_i(\bar{x})}$ . Let  $\gamma$  be the orthogonal transformation of  $\mathfrak{G}$  defined by  $\gamma x = \bar{x}$  for all  $x \in \mathfrak{G}$ . Then  $\kappa\alpha_i = \gamma\alpha_i\gamma$  and  $\alpha_i \rightarrow \kappa\alpha_i$  is an automorphism of  $O(8)$ . We shall show that  $\alpha_i \in SO(8)$  i.e.  $\det. \alpha_i = 1$ .

LEMMA 1. (*Principle of Triality.*)<sup>5)</sup> For every  $\theta \in SO(8)$ , there exist  $\theta_1$  and  $\theta_2$  in  $SO(8)$  such that

$$\theta(x)\theta_1(y) = \theta_2(xy)$$

for all  $x, y \in \mathfrak{G}$ . If there exist the other  $\theta'_1$  and  $\theta'_2$  in  $SO(8)$  such that  $\theta(x)\theta'_1(y) = \theta'_2(xy)$ , then  $\theta'_1 = \pm\theta_1$  and  $\theta'_2 = \pm\theta_2$ . The same holds also, if we start from  $\theta_1$  or  $\theta_2$  instead of  $\theta$ .

LEMMA 2. Let  $\theta_i$  be in  $O(8)$  and let

$$(3) \quad \theta_1(x)\theta_2(y) = \kappa\theta_3(xy)$$

for all  $x, y \in \mathfrak{G}$ . Then  $\theta_2(x)\theta_3(y) = \kappa\theta_1(xy)$  and  $\theta_3(x)\theta_1(y) = \kappa\theta_2(xy)$  for all  $x, y \in \mathfrak{G}$ .

*Proof.* Multiplying the both sides of (3) by  $\overline{\theta_1(x)}/|x|^2$ ,<sup>6)</sup> we have

$$\theta_2(y) = \frac{1}{|x|^2} \overline{\theta_1(x)} \overline{\theta_3(xy)}.$$

<sup>4)</sup> The positive definite bilinear form  $(a, b)$  on  $\mathfrak{G}$  is defined by  $(a, b) = \text{Re}(ab)$ , where  $\text{Re}x = \frac{1}{2}(x + \bar{x})$ .

<sup>5)</sup> See, Freudenthal [3] p. 16.

<sup>6)</sup>  $|x|^2 = (x, x) = x \cdot \bar{x} = \bar{x} \cdot x$ . In the following proof, we use the formulae  $|\bar{x}| = |x|$ ,  $|xy| = |\kappa||y|$ ,  $\bar{x}(xa) = (\bar{x}x)a$  and  $(a\bar{x})x = a(\bar{x}x)$ . See, Freudenthal [3] p. 7.

Analogously we have

$$\frac{1}{|y|^2} \theta_2(y) \cdot \theta_3(\bar{y}\bar{x}) = \overline{\theta_1(x)}.$$

Let  $\bar{x} = yz$ . Then

$$\frac{1}{|\bar{y}|^2} \theta_2(y) \theta_3(\bar{y}(yz)) = \overline{\theta_1(\bar{y}\bar{z})}.$$

Hence  $\theta_2(y)\theta_3(z) = \kappa\theta_3(yz)$ .

LEMMA 3. Let  $\theta_i \in O(8)$  ( $i = 1, 2, 3$ ) and  $\theta_1(x)\theta_2(y) = \kappa\theta_3(xy)$  for all  $x, y \in \mathbb{C}$ . Then  $\theta_i \in SO(8)$  ( $i = 1, 2, 3$ ).

*Proof.* Suppose that  $\theta_1$  is not in  $SO(8)$ . For every  $\eta_1 \in SO(8)$ , there exist  $\eta_2$  and  $\eta_3$  in  $SO(8)$  such that

$$\eta_1\theta_1(x)\eta_2\theta_2(y) = \kappa\eta_3 \cdot \kappa\theta_3(xy) = \kappa(\eta_3 \cdot \theta_3)(xy).$$

Let us choose  $\eta_1$  such that  $\eta_1\theta_1 = \gamma$ , where  $\gamma x = \bar{x}$  for all  $x \in \mathbb{C}$ . Then

$$(4) \quad \bar{x}\zeta_2(y) = \kappa\zeta_3(xy)$$

for all  $x, y \in \mathbb{C}$ , where  $\zeta_2 = \eta_2\theta_2$  and  $\zeta_3 = \eta_3\theta_3$ . Putting  $x = 1$  in (4), we have  $\zeta_2(y) = \kappa\zeta_3(y)$ . Hence  $\zeta_2 = \kappa\zeta_3$  and

$$(5) \quad \bar{x}\zeta_2(y) = \zeta_2(xy).$$

Putting  $y = 1$  in (5), we have

$$(6) \quad \zeta_2(x) = \bar{x}\zeta_2(1).$$

Let  $\zeta_2(1) = a$ . Then  $a \neq 0$ . It follows from (5) and (6) that  $\bar{x}(\bar{y}a) = (\bar{y}\bar{x})a$ . Hence  $x(ya) = (yx)a$  for all  $x, y \in \mathbb{C}$ . It follows that  $a = 0$  and this is a contradiction. Hence  $\theta_1 \in SO(8)$ . We may prove analogously that  $\theta_2$  and  $\theta_3$  are also in  $SO(8)$ .

Thus  $\alpha_i$  ( $i = 1, 2, 3$ ) in (2) are in  $SO(8)$ . Thus if  $\alpha \in \mathfrak{N}$ , then

$$(7) \quad \alpha X = \begin{pmatrix} \hat{\xi}_1 & \alpha_2(x_3) & \kappa\alpha_2(\bar{x}_2) \\ \kappa\alpha_3(\bar{x}_3) & \hat{\xi}_2 & \alpha_1(x_1) \\ \alpha_2(x_2) & \kappa\alpha_1(\bar{x}_1) & \hat{\xi}_3 \end{pmatrix},$$

where  $X$  is the matrix (1) and  $\alpha_i$ 's satisfy the relations (2).

Conversely let  $\alpha_1$  be an arbitrary element in  $SO(8)$  and let  $\alpha_2$  and  $\alpha_3$  be the elements in  $SO(8)$  such that  $\alpha_1(x)\alpha_2(y) = \kappa\alpha_3(xy)$  for all  $x, y \in \mathbb{C}$  (cf. Lemma 1). Then the relations (2) hold for these  $\alpha_i$ 's by Lemma 2. Now we define the linear transformation  $\alpha(\alpha_1, \alpha_2, \alpha_3)$  of  $\mathfrak{J}$  by (7). For every  $\alpha_1 \in SO(8)$  we have thus two linear transformations  $\alpha(\alpha_1, \alpha_2, \alpha_3)$  and  $\alpha(\alpha_1, -\alpha_2, -\alpha_3)$  (cf. Lemma 1). We may easily verify that these linear transformations are the automorphisms of  $\mathfrak{J}$  and form a closed subgroup  $\mathfrak{M}$  of the group  $\mathfrak{A}$  of all automorphisms of  $\mathfrak{J}$ . It is clear that every automorphism in  $\mathfrak{M}$  leaves fixed the

elements  $E_i$  ( $i=1, 2, 3$ ) and  $\mathfrak{M} \cong \mathfrak{N}$ . The mapping  $f_1(\alpha(\alpha_1, \alpha_2, \alpha_3)) = \alpha_1$  is a homomorphism of  $\mathfrak{M}$  onto  $SO(8)$  and the kernel of  $f_1$  consists of  $\alpha(1, 1, 1)$  and  $\alpha(1, -1, -1)$ .<sup>7)</sup> Let  $\mathfrak{M}_0$  be the connected component of  $\mathfrak{M}$  containing the identity. Then  $f_1(\mathfrak{M}_0) = SO(8)$ . Since  $f_1^{-1}(\alpha_1) = \{\alpha(\alpha_1, \alpha_2, \alpha_3), \alpha(\alpha_1, -\alpha_2, -\alpha_3)\}$ , at least one of  $\alpha(\alpha_1, \alpha_2, \alpha_3)$  and  $\alpha(\alpha_1, -\alpha_2, -\alpha_3)$  is in  $\mathfrak{M}_0$ . We shall prove that  $\mathfrak{M} = \mathfrak{M}_0$ . Suppose, on the contrary, that  $\mathfrak{M} \neq \mathfrak{M}_0$ . Since  $\mathfrak{M}_0 \cup \alpha(1, -1, -1)\mathfrak{M}_0 = \mathfrak{M}$ ,  $\mathfrak{M}$  consists of two connected components and  $\alpha(1, -1, -1) \in \mathfrak{M}_0$ . Now  $\alpha(-1, 1, -1)$  and  $\alpha(-1, -1, 1)$  belong to the distinct components of  $\mathfrak{M}$ , for otherwise  $\alpha(-1, 1, -1)\alpha(-1, -1, 1) = \alpha(1, -1, -1)$  is in  $\mathfrak{M}_0$ . Let, for example,  $\alpha(-1, -1, 1) \in \mathfrak{M}_0$ . Let  $f_3(\alpha(\alpha_1, \alpha_2, \alpha_3)) = \alpha_3$ . Then  $f_3$  is also a homomorphism of  $\mathfrak{M}$  onto  $SO(8)$  and the kernel of  $f_3$  is  $\{\alpha(1, 1, 1), \alpha(-1, -1, 1)\}$ . Hence  $f_3$  is a local isomorphism and  $f_3(\mathfrak{M}_0) = SO(8)$ . By assumption the kernel of  $f_3$  is contained in  $\mathfrak{M}_0$  and hence  $\mathfrak{M} = \mathfrak{M}_0$  and this is a contradiction. Hence  $\mathfrak{M} = \mathfrak{M}_0$ . Moreover we have shown that  $\mathfrak{M}$  is a two sheeted covering group of  $SO(8)$ . Hence  $\mathfrak{M}$  is isomorphic to the universal covering group  $\widetilde{SO(8)}$  of  $SO(8)$ . Since  $\mathfrak{M}$  is connected,  $\mathfrak{M}$  is contained in  $\mathfrak{F}_4$  and each automorphism in  $\mathfrak{M}$  leaves fixed the elements  $E_i$ . Hence  $\mathfrak{M} \subseteq \mathfrak{N}$ . Since we have already shown that  $\mathfrak{M} \cong \mathfrak{N}$ , we have  $\mathfrak{M} = \mathfrak{N}$  and this completes the proof of Theorem 1.

**3. Proof of Theorem 2.** Since the subgroups  $\mathfrak{F}_i$  of  $\mathfrak{F}_4$  are conjugate to each other in  $\mathfrak{F}_4$ ,<sup>8)</sup> it is sufficient to consider the group  $\mathfrak{F}_1$ . The derivation  $\delta$  of  $\mathfrak{F}$  such that  $\delta E_1 = 0$  may be represented uniquely as the sum of two derivations

$$\delta = \tilde{A} + A,$$

where  $AE_i = 0$  ( $i=1, 2, 3$ ) and

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & -\bar{a} & 0 \end{pmatrix}, \quad a \in \mathbb{C},$$

and  $\tilde{A}X = [A, X] = AX - XA$ . Conversely for each such a matrix  $A$ ,  $\tilde{A}$  is a derivation of  $\mathfrak{F}$  such that  $\tilde{A}E_1 = 0$ .<sup>9)</sup> Since  $A$ 's form the Lie algebra of the group  $\mathfrak{N}$ ,  $\dim.\{A\} = 28$  and  $\dim.\{\tilde{A}\} = 8$ , where  $\{A\}$  and  $\{\tilde{A}\}$  denote the linear spaces consisting of  $A$ 's and  $\tilde{A}$ 's respectively. Hence the derivations which maps  $E_1$  to 0 form a Lie algebra of dimensions 36 and this is the Lie algebra of  $\mathfrak{F}_1$ . Hence  $\dim.\mathfrak{F} = 36$ . Now let  $\Pi$  be the set of all irreducible idempotents of  $\mathfrak{F}$ .<sup>10)</sup> Further let  $\Pi_1$  be the set of all  $X \in \Pi$  such that  $E_1 \circ X = 0$ . Then an element  $X \in \mathfrak{F}$  is in  $\Pi_1$  if and only if

<sup>7)</sup> We denote by 1 and  $-1$  the identity transformation and the transformation defined by  $x \rightarrow -x$  respectively.

<sup>8)</sup> For, there exist  $\alpha$  and  $\beta$  in  $\mathfrak{F}_4$  such that  $\alpha E_1 = E_2$  and  $\beta E_1 = E_3$ . See, Freudenthal [3] p. 27. This fact is also proved in the following.

<sup>9)</sup> Chevalley-Schafer [2] and Freudenthal [3] p. 20.

<sup>10)</sup> See, Freudenthal [3] §5. Note that the set  $\Pi$  is invariant under the transformations of  $\mathfrak{F}_4$ .

$$(8) \quad X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \bar{x}_1 & \xi_3 \end{pmatrix},$$

where  $\xi_2 = \xi_3^2 + x_1 \bar{x}_1$ ,  $\xi_2 + \xi_3 = 1$ . Then  $\xi_3 = \xi_3^2 + x_1 \bar{x}_1$ . Hence  $1 = \xi_2^2 + \xi_3^2 + 2x_1 \bar{x}_1$ . Now the bilinear form  $(X, Y) = Sp(X \circ Y)$  defined on  $\mathfrak{F}$  is positive definite and invariant under the transformations of  $\mathfrak{F}_4$ .<sup>11)</sup> Let  $\|X\|^2 = (X, X)$ . If  $X$  is the matrix (1), then  $\|X\|^2 = \sum_{i=1}^3 \xi_i + 2 \sum_{i=1}^3 x_i \bar{x}_i$ . Hence if  $X \in \Pi_1$ , then  $\|X\| = 1$ . Now let  $\mathfrak{F}_1$  be the 10 dimensional linear subspace of  $\mathfrak{F}$  consisting of the matrices of the form (8), and let  $S^9$  be the set of all  $X \in \mathfrak{F}_1$  such that  $\|X\| = 1$ . Then  $S^9$  is a 9 dimensional sphere and  $\Pi_1$  is the intersection of  $S^9$  and the hyper-plane  $\xi_2 + \xi_3 = 1$  in  $\mathfrak{F}_1$ . Hence  $\Pi_1$  is an 8 dimensional sphere. Let  $\alpha \in \mathfrak{H}_1$ . Then  $\alpha(E_1 \circ X) = E_1 \circ \alpha X$ , hence  $\alpha(\Pi_1) = \Pi_1$ . Thus  $\alpha$  induces a transformation  $R_\alpha$  of the sphere  $\Pi_1$ . Since  $\alpha$  is an orthogonal transformation of  $\mathfrak{F}$ ,  $R_\alpha$  is an isometric transformation of  $\Pi_1$  and hence a (proper or improper) rotation. Thus  $g(\alpha) = R_\alpha$  is a homomorphism of  $\mathfrak{H}_1$  into the group  $O(9)$ . Let  $\mathfrak{D}$  be the kernel of  $g$ . Since each  $\alpha \in \mathfrak{D}$  leaves fixed the elements  $E_i$ ,  $\alpha$  is contained in  $\mathfrak{N}$ . Hence  $\alpha(\in \mathfrak{D})$  is of the form  $\alpha = \alpha(\alpha_1, \alpha_2, \alpha_3)$  (see §1) and

$$\alpha X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi_2 & \alpha_1(x_1) \\ 0 & \kappa \alpha_1(\bar{x}_1) & \xi_3 \end{pmatrix} = X$$

for all  $X \in \Pi_1$ . We see easily that  $\alpha_1 = 1$  and hence  $\mathfrak{D}$  is the finite group of order 2. Since  $\dim \mathfrak{H}_1 = \dim O(9) = 36$ , the component  $\mathfrak{H}_1^0$  containing the identity is mapped by  $g$  onto  $SO(9)$ . As  $\mathfrak{H}_1^0 \supset \mathfrak{N} \supset \mathfrak{D}$  by Theorem 1,  $\mathfrak{H}_1^0$  is a two-sheeted covering group of  $SO(9)$  and hence it is isomorphic to the universal covering group  $\widetilde{SO}(9)$  of  $SO(9)$ . We may easily see that if  $\mathfrak{H}_1 \neq \mathfrak{H}_1^0$ , then the order of the group  $\mathfrak{H}_1/\mathfrak{H}_1^0$  is 2 and  $g(\mathfrak{H}_1) = O(9)$ . Now the mapping

$$X \rightarrow RX = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi_3 & x_1 \\ 0 & \bar{x}_1 & \xi_2 \end{pmatrix}$$

is an improper rotation of the sphere  $\Pi_1$ . If  $\mathfrak{H}_1 \neq \mathfrak{H}_1^0$ , there exists  $\alpha \in \mathfrak{H}_1$  such that  $\alpha X = RX$  for all  $X \in \Pi_1$ . Then  $\alpha E_1 = E_1$ ,  $\alpha E_2 = E_3$  and  $\alpha E_3 = E_2$ . Since  $g(\mathfrak{H}_1^0) = SO(9)$  and  $SO(9)$  is transitive on  $\Pi_1$ , there exists  $\beta \in \mathfrak{H}_1^0$  such that  $\beta E_2 = E_3$ .  $\beta(E_1 \circ E_3) = E_1 \circ \beta E_3 = 0$ ,  $\beta(E_2 \circ E_3) = E_3 \circ \beta E_3 = 0$  and  $\beta E_3 \circ \beta E_3 = \beta E_3$  imply  $\beta E_3 = E_2$ . Then  $\beta^{-1} \alpha E_i = E_i$  for  $i = 1, 2, 3$ . Thus  $\beta^{-1} \alpha \in \mathfrak{N} \cap \mathfrak{H}_1^0$ . Hence  $\alpha \in \mathfrak{H}_1^0$  and this is a contradiction. Thus  $\mathfrak{H}_1$  is connected and isomorphic to  $\widetilde{SO}(9)$ .

*Remark.* The group of all automorphisms of  $\mathfrak{F}$  is not connected. For example,

<sup>11)</sup> See, Freudenthal [3], §4.

$$X = \begin{pmatrix} \hat{\xi}_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \hat{\xi}_2 & x_1 \\ x_2 & \bar{x}_1 & \hat{\xi}_3 \end{pmatrix} \rightarrow \alpha X = \begin{pmatrix} \hat{\xi}_1 & x_2 & \bar{x}_3 \\ \bar{x}_2 & \hat{\xi}_3 & x_1 \\ x_3 & \bar{x}_1 & \hat{\xi}_2 \end{pmatrix}$$

is an automorphism of  $\mathfrak{J}$ .  $\alpha$  is an improper orthogonal transformation of  $\mathfrak{J}$  and hence  $\alpha \in \bar{\mathfrak{O}}_4$ .

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