NOTE ON A-GROUPS

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Let us consider soluble groups whose Sylow subgroups are all abelian. Such groups we call A-groups, following P. Hall. A-groups were investigated thoroughly by P. Hall and D. R. Taunt from the view point of the structure theory.[1] In this note, we want to give some remarks concerning representation theoretical properties of A-groups.

§ **1. Definition.** A group $\mathfrak{G}$ is called an M-group if all its irreducible representations are similar to those of monomial forms.

PROPOSITION 1. Every A-group is an M-group.

Proof. Let $\mathfrak{G}$ be an A-group and let $\mathfrak{Z}$ be an irreducible representation of $\mathfrak{G}$. Obviously the A-property is hereditary to subgroups and factor groups. Therefore, applying the induction argument with respect to the order of $\mathfrak{G}$, we see that we have only to consider faithful, primitive irreducible representations of $\mathfrak{G}$. Let $\mathfrak{Z}=\mathfrak{G}$ be such a one. Let $\mathfrak{N}$ be the radical, that is, the largest nilpotent normal subgroup of $\mathfrak{G}$. Since $\mathfrak{G}$ is an A-group, the radical $\mathfrak{N}$ is abelian. Therefore by a theorem of H. Blichfeld,[2] $\mathfrak{N}$ must coincide with the centre of $\mathfrak{G}$. If $\mathfrak{G}=\mathfrak{N}$, the assertion is trivial. If $\mathfrak{G}\neq\mathfrak{N}$, let $\mathfrak{N}_1$ be a normal subgroup of $\mathfrak{G}$, which is minimal over $\mathfrak{N}$. Then obviously $\mathfrak{N}_1$ is nilpotent and therefore $\mathfrak{N}_1=\mathfrak{N}$ which is a contradiction. Q.E.D.

Imposing some strong restriction on $\mathfrak{G}$, M. Tazawa proved the proposition 1.[3]

The M-property is not always hereditary to subgroups. First we remark the following well known fact:

(A) Let us consider a matrix group $\mathfrak{M}$ whose character is denoted by χ. Then $\mathfrak{M}$ is irreducible if and only if $\sum\chi\bar{\chi}=$ the order of $\mathfrak{M}$.

Example. Let $\mathfrak{G}$ be the hyperoctahedral group of degree 4 (and of order $2^4 . 4!$). Then $\mathfrak{G}$ is irreducible, which is easily verified applying (A). Let

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[1] P. Hall, The construction of soluble groups. J. Reine Angew. Math. **182**, 206-214 (1940). D. R. Taunt, On A-groups. Proc. Cambridge Philos. Soc. **45**, 24-42 (1949). The latter is not yet accessible to me.

[2] H. Blichfeld, Finite Collineation Groups. Chicago (1917).

[3] M. Tazawa, Über die monomial darstellbaren endlichen Substitutionsgruppen. Proc. Acad. Jap. **10**, 397-398 (1934).

$\mathfrak{Z}=\left\{\begin{pmatrix}-1&&&\\&-1&&\\&&-1&\\&&&-1\end{pmatrix}\right\}$ be the centre of $\mathfrak{G}$. Then $\mathfrak{G}/\mathfrak{Z}$ contains the abelian normal subgroup $\mathfrak{N}/\mathfrak{Z}$ of order 2^4 such that $\mathfrak{G}/\mathfrak{N}$ is a group of Jordan-Dedekind type, as is easily verified. Therefore $\mathfrak{G}/\mathfrak{Z}$ is an M-group by a theorem of K. Taketa.[4] Furthermore, all the faithful irreducible representations of $\mathfrak{G}$ are given by the Kronecker products of $\mathfrak{G}$ and the irreducible representations of $\mathfrak{G}/\mathfrak{N}$, as can also be easily verified by applying (A). Thus all the irreducible representations of $\mathfrak{G}$ are similar to those of monomial forms. Therefore $\mathfrak{G}$ is an M-group. On the other hand, let us consider the subgroup $\mathfrak{H}$ of $\mathfrak{G}$ generated by $\begin{pmatrix}&&-1&0\\&&0&-1\\1&0&&\\0&1&&\end{pmatrix}$ and $\begin{pmatrix}0&1&0&\\0&0&1&\\1&0&0&\\&&&1\end{pmatrix}$. Then it is easily seen that $\mathfrak{H}$ is isomorphic to the holomorph of the quaternion group by its automorphism of order 3. Therefore $\mathfrak{H}$ possesses a primitive irreducible representation of degree 2, and $\mathfrak{H}$ is not an M-group.

§ **2.** Let $\mathfrak{X}(\mathfrak{G})$ denote the set of all the elements of a group $\mathfrak{G}$ such that $\chi(G)\neq 0$ for any simple character χ of $\mathfrak{G}$.

PROPOSITION 2. Let $\mathfrak{G}$ be an A-group and let $\mathfrak{N}$ be the radical of $\mathfrak{G}$. Then $\{\mathfrak{X}(\mathfrak{G})\}=\mathfrak{N}$.

Proof. First we prove $\mathfrak{X}(\mathfrak{G})\subset\mathfrak{N}$. Let $\mathfrak{M}_p$ be the largest normal p-subgroup of $\mathfrak{G}$. Then $\mathfrak{G}/\mathfrak{M}_p$ contains no normal p-subgroup distinct from $\{e\}$. We proved in the previous paper[5] that in such a group there exists a character of defect 0 for p. Such a character vanishes for all the p-singular elements by a theorem of R. Brauer and C. Nesbitt.[6] Let G be an element of $\mathfrak{X}(\mathfrak{G})$. Then the p-part of G is contained in $\mathfrak{M}_p$. Therefore G belongs to $\mathfrak{N}$ and $\mathfrak{X}(\mathfrak{G})\subset\mathfrak{N}$.

Secondly we prove $\{\mathfrak{X}(\mathfrak{G})\}\subset\mathfrak{N}$. Let P be an element of $\mathfrak{M}_p$ and let χ be a simple character of $\mathfrak{G}$. Every p-block contains a character belonging to $\mathfrak{M}_p$.[7] Let $g(P)$ denote the number of conjugate elements of P. Then

$$g(P)\frac{\chi(P)}{\chi(e)}\equiv g(P)\pmod{\mathfrak{p}}$$

4) K. Taketa, Über die Gruppen, deren Darstellungen sich sämtlich auf monomiale Gestalt transformieren lassen. Proc. Acad. Jap. **6**, 31-33 (1930).

5) N. Itô, On the characters of soluble groups. These Journal **3**, 31-48 (1951).

6) R. Brauer and C. Nesbitt, On the modular characters of groups. Ann. Math. **42**, 556-590 (1941).

7) R. Brauer, On the arithmetic in a group ring. Proc. Nat. Acad. Sci. U.S.A. 109-114 (1944). N. Itô, Some studies on group characters. These Journal **2**, 17-28 (1951). (A remark to my paper: It was evident that $f/e_\kappa f_\kappa=1$ by a theorem of I. Schur, from which the description can be rather shortened.)

where $\mathfrak{p}$ is a prime ideal divisor of p in the character field of χ. Since $(g(P), p)=1$, we have $\chi(P)\neq 0$. Therefore P belongs to $\mathfrak{X}(\mathfrak{G})$ and $\{\mathfrak{X}(\mathfrak{G})\}\supset\mathfrak{N}$. Q.E.D.

Especially when the order of $\mathfrak{G}$ is divisible by only two distinct prime numbers, we have precisely $\mathfrak{X}(\mathfrak{G})=\mathfrak{N}$. To prove this, let P and Q be elements of $\mathfrak{M}_p$ and $\mathfrak{M}_q$ respectively. Considering P, Q and PQ in the group ring of $\mathfrak{G}$ and denoting by $\tilde{P}$, $\tilde{Q}$ and $\widetilde{PQ}$ the sum of conjugate elements of P, Q and PQ respectively, we have clearly $\widetilde{PQ}=\tilde{P}\tilde{Q}$. Then for any simple character χ of $\mathfrak{G}$, we have

$$g(PQ)\frac{\chi(PQ)}{\chi(e)}=g(P)\frac{\chi(P)}{\chi(e)}\cdot g(Q)\frac{\chi(Q)}{\chi(e)}\,.$$

Since $\chi(P)\neq 0$ and $\chi(Q)\neq 0$, we have $\chi(PQ)\neq 0$. Therefore PQ belongs $\mathfrak{X}(\mathfrak{G})$ and $\mathfrak{X}(\mathfrak{G})\supset\mathfrak{N}_x$ whence $\mathfrak{X}(\mathfrak{G})=\mathfrak{N}$.

This is not always valid for general A-groups.

Example. Let $\mathfrak{G}$ be a group generated by following two matrices of degree 7:

$$A=\begin{pmatrix}1&&&&&&\\&-1&&&&&\\&&1&&&&\\&&&-1&&&\\&&&&\rho^2&&\\&&&&&-\rho&\\&&&&&&1\end{pmatrix}\quad\text{and}\quad B=\begin{pmatrix}&1&&&&&\\&&1&&&&\\&&&1&&&\\&&&&1&&\\&&&&&1&\\&&&&&&1\\1&&&&&&\end{pmatrix},$$

where ρ is a primitive 6-th root of unity.

Put $A_2=B^{-1}A_1B$, $A_3=B^{-1}A_2B,\ldots,A_6=B^{-1}A_5B$. We can easily substantiate that $A_1, A_2,\ldots,A_6$ are independent one another. Therefore $\mathfrak{G}$ is an A-group of order $2^6\cdot 3^6\cdot 7$. Since the degree of any non-linear irreducible representation of $\mathfrak{G}$ is 7,[8] we have that $\mathfrak{G}$ is irreducible. Here the character of A is clearly equal to 0.

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[8] N. Itô, On the degrees of irreducible representations of a finite group. These Journal 3, 5-6 (1951).