

STRONG REGULARITY IN ARBITRARY RINGS

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An element a of a ring R is called regular, if there exists an element x of R such that $axa = a$, and a two-sided ideal α in R is said to be regular if each of its elements is regular. B. Brown and N. H. McCoy [1] has recently proved that every ring R has a unique maximal regular two-sided ideal $M(R)$, and that $M(R)$ has the following radical-like property: (i) $M(R/M(R)) = 0$; (ii) if α is a two-sided ideal of R , then $M(\alpha) = \alpha \cap M(R)$; (iii) $M(R_n) = (M(R))_n$, where R_n denotes a full matrix ring of order n over R . Arens and Kaplansky [2] has defined an element a of R to be strongly regular when there exists an element x of R such that $a^2x = a$. We shall prove in this note that replacing "regularity" by "strong regularity," we have also a unique maximal strongly regular ideal $N(R)$, and shall investigate some of its properties.

1. Existence and properties of $N(R)$.

The existence proof of $N(R)$ can be accomplished along the line of Brown and McCoy [1].

Definition 1. An element a of a ring R is called *strongly regular*, if and only if there exists an element x of R such that $a^2x = a$. A (two-sided) ideal α in R is called a *strongly regular ideal*, if each of its element is strongly regular. Finally, we call an element $a \in R$ *properly strongly regular*, if the principal ideal (a) generated by a is strongly regular.

LEMMA 1. *If $a^2y - a$ is strongly regular, so is a too.*

Proof. By virtue of strong regularity of $a^2y - a$, there exists an element z such that $(a^2y - a)^2z = a^2y - a$. Setting $x = y - z + ayz + yaz - ya^2yz$, we have readily $a^2x = a$.

LEMMA 2. *The set $N(R)$ of all properly strongly regular elements of R is a strongly regular ideal.*

Proof. That $z \in N(R)$ and $t \in R$ implies $(zt) \subset N(R)$ whence $zt \in N(R)$; similarly, $tz \in N(R)$. On the other hand, let z_1 and z_2 be any elements of $N(R)$ and let $a \in (z_1 - z_2)$. Then we have $a = u_1 - u_2$, where $u_i \in (z_i)$. By strong regularity of (z_1) , we have $u_1^2r = u_1$ for some $r \in R$. Then $a^2r - a = (u_1 - u_2)^2r - (u_1 - u_2) = u_2 + u_2^2r - u_1u_2r - u_2u_1r \in (u_2) \subset (z_2)$, and $a^2r - a$ is strongly regular. Then, Lemma 1 implies that a is strongly regular, and the proof is complete.

Received September 1, 1951.

From Lemma 2, we have immediately

THEOREM 1. *Every ring R has a unique maximal strongly regular ideal $N(R)$.*

THEOREM 2. *For every ring R we have $N(R/N(R)) = 0$.*

Proof. Let us denote by \bar{a} the residue class mod $N(R)$ containing a . Suppose that $\bar{b} \in N(R/N(R))$ and a is any element of (b) . Then \bar{a} is as an element of (\bar{b}) strongly regular in the ring $R/N(R)$: $\bar{a}^2\bar{x} = \bar{a}$, that is $a^2x - a \in N(R)$, and hence $a^2x - a$ is strongly regular. It follows therefore from Lemma 1 that a is strongly regular. Since we have proved that every element of the ideal (b) is strongly regular, we have $b \in N(R)$, i.e. $\bar{b} = 0$.

LEMMA 3. *Let a be a two-sided ideal of R . Then, an element a of a is properly strongly regular in the ring a if and only if it is strongly regular in the ring R .*

Proof. Let a be properly strongly regular in a , and let b be any element of the ideal (a) generated by a in R . Then, we have $b = na + ua + av + \sum u_i av_i$, where n is an integer and u 's and v 's are elements of R . Since a is strongly regular, there exists an element $x \in a$ such that $a^2x = a$. Consequently, $b = na + (ua)ax + a(axv) + \sum (u_i a) a(xv_i)$, $ua, axv, u_i a, xv_i \in a$. Hence we have $b \in (a)'$, where $(a)'$ denotes an ideal generated by a in a . Therefore, b is strongly regular, and the element a is properly strongly regular in R . The converse part is clear.

From Lemma 3, we have immediately

THEOREM 3. *If a is a two-sided ideal in R , then $N(a) = a \cap N(R)$.*

2. Some relations between $N(R)$ and $M(R)$.

Let us consider some properties of elements in $N(R)$.

LEMMA 4. *$N(R)$ has no non-zero nilpotent element.*

Proof. Let $a \in N(R)$, and $a^n = 0$. Then $a^2x = a$, and so $a = a^2x = \dots = a^n x^{n-1} = 0$.

LEMMA 5. *Let $a \in N(R)$ and x be an element in R such that $a^2x = a$. Then,*
 (i) $a^2x = axa = xa^2 = a$, and a is regular. (ii) $ax = xa$, and ax is an idempotent.
 (iii) $e = ax$ belongs to the center of R .

Proof. From $a^2x = a$, we have easily $(a - axa)^2 = 0$. Since $a - axa \in N(R)$. Lemma 4 implies $a = axa$, and similarly $axa = xa^2$, so we have (i). From $ax = (xa^2)x = x(a^2x) = xa$ we have (ii). As for (iii), let u be any element of R . By analogous argument as (i), we have $ue = eue$, $ue = eue$, and therefore $ue = eu$.

The above lemma shows that each element of $N(R)$ is regular, so we have

THEOREM 4. $N(R) \subset M(R)$.

While $M(R)$ satisfies $M(R_n) = (M(R))_n$ (cf. [1]), $N(R)$ does not possess this property, which is shown by the following theorem:

THEOREM 5. *Let R_n be the full matrix ring of order $n > 1$ over R . Then, $N(R_n) = 0$.*

Proof. First, let us suppose that R has a unit element. Let $A \in N(R_n)$. Then there exists a matrix X such that $A^2X = A$, and AX belongs to the center of R_n . Hence, $AX = aE$, where a is an element belonging to the center of R , and E is the unit matrix of R_n . So, we have $A = aA$, and

$$B = \begin{pmatrix} 0 & a & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} = aE \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \in (aE) \subset (A) \subset N(R_n).$$

Therefore B is strongly regular: $B = B^2Y$. But since $B^2 = 0$, we have $B = 0$, $a = 0$, and $A = 0$.

When R does not possess a unit element, we can obtain a ring \hat{R} in the usual way by adjoining a unit element to R . Then R_n is an ideal of \hat{R}_n , and $N(R_n) = R_n \cap N(\hat{R}_n) = 0$.

The above theorem shows that there exists a ring R such that $M(R) \subsetneq N(R)$.

THEOREM 6. *$M(R) = N(R)$ if and only if $M(R)$ has no non-zero nilpotent element.*

Proof. Suppose that $M(R)$ has no non-zero nilpotent element. Then, since for every $a \in M(R)$ there is an x such that $a = axa$ whence $(a - a^2x)^2 = 0$, we have $a = a^2x$, $a \in N(R)$. This means $M(R) = N(R)$. The converse follows from Lemma 4.

COROLLARY. *If R is either commutative or has no non-zero nilpotent element, then $M(R) = N(R)$.*

REFERENCES

[1] B. Brown and N. H. McCoy, The maximal regular ideal of a ring. Proc. of the Amer. Math. Soc. **1** (1950), pp. 165-171.
 [2] R. F. Arens and I. Kaplansky, Topological representation of algebras. Trans. Amer. Math. Soc. **63** (1948), pp. 457-481.

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