CLASSIFICATION OF MAPPINGS OF AN (n+2)-COMPLEX INTO AN (n-1)-CONNECTED SPACE WITH VANISHING (n+1)-ST HOMOTOPY GROUP

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The present paper is concerned with the classification and corresponding extension theorem of mappings of the $(n+2)$-complex K^{n+2} $(n>2)$ into the space Y whose homotopy groups $\pi_i(Y)$ vanish for $i<n$ and $i=n+1$, and the n-th homotopy group $\pi_n(Y)$ of which has a finite number of generators. Our methods followed here are essentially analogous to those of Steenrod [2]. He introduced the important concept of the $\underset{i}{\cup}$-products of cocycles, which enables us to define $\mathscr{P}_i$-Square (refer to § 1), a certain type of a combination of $\underset{i}{\cup}$-products. This square is a modification of the so-called Pontrjagin square (Pontrjagin [1], Whitehead [4], and Whitney [3]). It induces a homomorphism of $H^n(K, I_m)$, the n-th cohomology group with integral coefficients reduced *mod.* m of a complex K, into $H^{2n-i}(K, I)$, the $(2n-i)$-th cohomology group with integral coefficients, when m is even and $n-i$ is odd. Together with squaring products we have a homomorphism (refer to § 5) of $H^n(K, \pi_n(Y))$ into $H^{n+3}(K, \pi_{n+2}(Y))$ in the case $i=n-3$. As its application, Eilenberg-MacLane's cohomology class K^{n+h+1} of the semi-simplicial complex $K(\pi_n(Y), n)$ with coefficients in $\pi_{n+h}(Y)$ is determined in case where $h=2$ and $n>2$ (Eilenberg-MacLane [7]).

Another information from the homomorphism may contribute partially to the homotopy type problem of A_n^3-complexes (J. H. C. Whitehead [5], Chang [12], Uehara [13]).

In § 1 the above mentioned product will be defined. In § 2 we shall sketch the computation of the homotopy groups of some elementary types of reduced A_n^3-complexes. In § 3 relations of products of cocycles in such complexes are discussed. The $(n+3)$-extension cocycle and the present classification of mappings will be embodied in § 4, § 5 respectively. The final section § 6 will contain some applications to related subjects.

§ 1. $\mathscr{P}_i$-square

Let K be a finite simplicial complex or a cell complex. Let us consider the n-dimensional integral cochain group C^n of K and its subgroup $Z^n(m)$ of all cocycles *mod.* m for an even integer m. If $u^n \in Z^n(m)$, then $\delta u^n \equiv 0$ (*mod.*

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m) and $\theta_m^{n+1}u^n=\dfrac{1}{m}\delta u^n$ is an $(n+1)$-integeral cocycle.

If we define

$$\mathscr{P}_i u^n = u^n \underset{i}{\cup} u^n + mu^n \underset{i+1}{\cup} \theta_m^{n+1}u^n + (-1)^n \frac{m^2}{2}\theta_m^{n+1}u^n \underset{i+2}{\cup} \theta_m^{n+1}u^n ,$$

for $u^n \in Z^n(m)$ ($m \geqq 0$ is even), straightforward calculations, by means of the coboundary formula of Steenrod [2], give the following

LEMMA 1. *If $n-i$ is odd, then we have*

1) *$\mathscr{P}_i u^n$ is a $(2n-i)$-dimensional integal cocycle,*
2) $2\mathscr{P}_i u^n \backsim 0$,
3) $\mathscr{P}_i(ku^n) = k^2\mathscr{P}_i u^n$,
4) $\mathscr{P}_i(u^n+v^n) \backsim \mathscr{P}_i u^n + \mathscr{P}_i v^n$ *for* $u^n,\ v^n \in Z^n(m)$,
5) $\mathscr{P}_i(mx^n) \backsim 0$ *for* $x^n \in C^n$,
6) $\mathscr{P}_i(\delta x^{n-1}) \backsim 0$ *for* $x^{n-1} \in C^{n-1}$.

Thus $\mathscr{P}_i$ induces a homomorphism such that:

$$\mathscr{P}_i : H^n(K, I_m) \longrightarrow {}_2H^{2n-i}(K, I),$$

where ${}_2H = \{g;\ g \in H,\ 2g=0\}$ for any abelian group H. We shall use this homomorphism in the following only when $i=n-3$.

§2. Some types of elementary A_n^2-complexes

We shall refer to the following types of polyhedra as elementary A_n^2-complexes;

i) $B^0 = S^n$, n-sphere,
ii) $B^1(m) = S^n \cup e^{n+1}$, where an $(n+1)$-element e^{n+1} is attached to S^n by a map $f: \partial e^{n+1} \to S^n$ of degree m,
iii) $B^2(0) = S^n \cup e^{n+2}$, where e^{n+2} is attached to S^n by an essential map $\eta: \partial e^{n+2} \to S^n$,
iv) $B^2(2r) = B^2(0) \cup e^{n+1}$, when e^{n+1} is attached to S^n of $B^2(0)$ by a map $f: \partial e^{n+1} \to S^n$ of degree $2r$.

Then we have

LEMMA 2.

α) $\pi_{n+1}(B^2(0)) = 0$,

β) $\pi_{n+1}(B^1(2r+1)) = 0$;

$\pi_{n+1}(B^1(2r)) = (2)$, *cyclic group of order* 2, *whose generator* ζ *is represented by an essential map of* S^{n+1} *onto* $S^n \subset B^1(2r)$,

γ) $\pi_{n+1}(B^2(2r)) = 0$.

LEMMA 3.

α) $\pi_{n+2}(B^2(0)) = I$, *free cyclic group, whose generator* ω *is represented by a* *map of degree* 2,

β) $\pi_{n+2}(B^1(2r+1))=0$;

$\pi_{n+2}(B^1(2r))=(2)+(2)$, *direct sum of two cyclic groups of order two, with generators* ξ *and* $\bar{\zeta}$, *where* ξ *is represented by a map covering* e^{n+1} *essentially and* $\bar{\zeta}$ *is represented by an essential map* $\eta: S^{n+2}\to S^n\subset B^1(2r)$,

γ) $\pi_{n+2}(B^2(2r))=I+(2)$: *direct sum of the free cyclic group with the generator* ω *and the cyclic group of order* 2 *with the generator* ξ.

Proof of Lemmas.

Some of these statements are easily deducible from known results of Freudenthal, J. H. C. Whitehead [6], G. W. Whitehead [9], Pontrjagin [10]. Thus we shall sketch here the proof of Lemma 3.

3, α) Any map which is homotopic to a map of S^{n+2} into S^n of $B^2(0)$, is contractible in $B^2(0)$ to a point, so that there is no essential map of degree 0. Next we prove that there is no essential map f of odd degree k. If we denote f^* the inverse homomorphism between cohomology groups of the two spaces, we obtain $f^*(S^n \underset{n-2}{\cup} S^n)=f^*S^n \underset{n-2}{\cup} f^*S^n \backsim 0$ in S^{n+2}, while in $B^2(0)$, $S^n \underset{n-2}{\cup} S^n=e^{n+2}$ (*mod* 2) and thereby $f^*(S^n \underset{n-2}{\cup} S^n)=f^*\ e^{n+2}=kS^{n+2}$ (*mod* 2). This is a contradiction.

Consider a map $\varphi: S^{n+2}\to B^2(0)$ such that $\varphi\,|\,V^{n+2}_{\geqq 0}$ represents twice of a suitably chosen generator of the relative homotopy group $\pi_{n+2}(B^2(0),\ S^n)$ and extend $\varphi\,|\,V^{n+2}_{\geqq 0}$ through the lower hemisphere $V^{n+2}_{\leqq 0}$ by contracting in S^n the resultant inessential map of the equator S^{n+1} into S^n to an point. φ has degree 2 and represents ω.

3, β) Let g be a map of S^{n+2} into $B^1(2r)$ such that $g\,|\,V^{n+2}_{\geqq 0}$ represents of a generator of $\pi_{n+2}(B^1(2r),\ S^n)$, and extend g through the lower hemisphere $V^{n+2}_{\leqq 0}$ by contracting the resulting inessential map of the equator S^{n+1} into S^n to a point in S^n. g represents ξ. $2\xi=0$. ξ is essential, for the superposition hg of g by the map h of $B^1(2r)$ onto S^{n+1}, is essential, where h maps S^n into a point p of S^{n+1} and e^{n+1} topologically to $S^{n+1}-p$.

3, γ) $\bar{\zeta}$ in $\pi_{n+2}(B^1(2r))$ vanishes by imbedding $B^1(2r)$ in $B^2(2r)$.

We add here some remarks which will be needed later.

Let $R^{n+1}=\sum_{\mu} B^1_\mu(n_\mu)$ be a cell complex consisting of a finite number of $B^1_\mu(n_\mu)$ (even n_μ) with a single common point belonging to each $S^n_\mu\subset B^1_\mu(n_\mu)$ and let $R^{n+2}=\sum_{\mu} B^2_\mu(n_\mu)$ be a cell complex constructed similarly. Let $\alpha_\mu\,\alpha_\nu$ denote the Whitehead product of α_μ and α_ν, where α_μ is a generator of $\pi_n(S^n_\mu)$, etc. Let $(\alpha_\mu\alpha_\nu)$ denote the subgroup of $\pi_{2n-1}(S^n_\mu \vee S^n_\nu)$ generated by $\alpha_\mu\alpha_\nu$.

Then we have

$$\pi_{n+1}(R^{n+1})=\sum_{\mu}\pi_{n+1}(B^1_\mu(n_\mu)),$$

$$\pi_{n+2}(R^{n+2})=\sum_{\mu}\pi_{n+2}(B^2_\mu(n_\mu)) \quad \text{for} \quad n>3,$$

and

$$\pi_{n+2}(R^{n+2}) = \sum_{\mu}\pi_{n+2}(B_{\mu}^{2}(n_{\mu})) + \sum_{\mu<\nu}(\alpha_{\mu}\alpha_{\nu}) \quad \text{for} \quad n=3,$$

by the recurrent usage of a result of G. W. Whitehead [8] or a slight generalization of lemma 5. 3. 2. of Blakers and Massay [11].

§ 3. Products in some types of elementary A_n^3-complexes

In § 2 we sketched elementary A_n^2-complexes whose $(n+1)$-*st* homotopy groups vanish but whose *n-th* homotopy groups do not vanish. Among them $B^2(0)$ and $B^2(2r)$ have non-trivial $(n+2)$-*nd* homotopy groups. Here we construct from $B^2(0)$ and $B^2(2r)$· A_n^3-complexes whose $(n+2)$-*nd* homotopy groups vanish.

Let $B^3(0, k) = B^2(0)\cup e^{n+3}$ and let $B^3(2r, k) = B^2(2r)\cup e_1^{n+3}\cup e_2^{n+3}$ where e^{n+3} and e_1^{n+3} are attached to $B^2(0)$ and to $B^2(2r)$ by maps of ∂e^{n+3}, ∂e_1^{n+3} representing $k\omega\in\pi_{n+2}(B^2(0))$, $\pi_{n+2}(B^2(2r))$ respectively and e_2^{n+3} is attached to $B^2(2r)$, by a map ∂e_2^{n+3} into $B^2(2r)$ representing $\xi\in\pi_{n+2}(B^2(2r))$.

THEOREM 1. *In $B^3(0, k)$ we have*

α) $$S^n \underset{n-3}{\cup} S^n = ke^{n+3}, \quad 2ke^{n+3}\backsim 0,$$

where S^n and e^{n+3} represent cocycles.

In $B^3(2r, k)$, we have

β) $\mathcal{P}_{n-3}S^n = ke_1^{n+3}$, $2ke_1^{n+3}\backsim 0$ *and*

γ) $\theta_{2r}^{n+1}S^n \underset{n-1}{\cup} \theta_{2r}^{n+1}S^n = e_2^{n+3}$ (*mod* 2),

where S^n represents itself as cocycle mod $2r$ [see § 1].

We denote $B^3(m, 1)$ simply by $B^3(m)$, ($m\geqq 0$ is even).

Proof of Theorem 1. In $B^3(0, k)$, by orienting e^{n+3} suitably, we can define $S^n \underset{n-2}{\cup} S^n = (-1)^n e^{n+2}$. By Lemma 3, α) in § 2, we have $\delta e^{n+2} = 2ke^{n+3}$. Since $\delta(S^n \underset{n-2}{\cup} S^n) = (-1)^n\, 2(S^n \underset{n-3}{\cup} S^n)$, we obtain α).

In $B^3(2r, k)$ S^n is a cocycle *mod* $2r$. Let $\kappa: B^3(0, k)\to B^3(2r, k)$ be the injection mapping, and let κ^* be its inverse homomorphism of cochain groups. Then $\kappa^*\mathcal{P}_{n-3}S^n = \mathcal{P}_{n-3}\kappa^*S^n = \kappa^*S^n \underset{n-3}{\cup} \kappa^*S^n = ke^{n+3} = \kappa^*ke_1^{n+3}$ in $B^3(0, k)$. We obtain therefore, $\mathcal{P}_{n-3}S^n = ke_1^{n+3} + le_2^{n+3}$, but $2\mathcal{P}_{n-3}S^n\backsim 0$. It follows that $l=0$ and β) is proved.

For the part of γ), set $M^{n+3} = S^{n+1}\cup e^{n+3}$, where e^{n+3} is attached to S^{n+1} by an essential map $f: \partial e^{n+3}\to S^{n+1}$. And let $\kappa: B^3(2r, k)\to M^{n+3}$ be such a map that κ maps $B^3(0, k)$ into a point p of S^{n+1} and maps e_2^{n+3} onto e^{n+3}, e^{n+1} onto $S^{n+1}-p$ topologically. Then, in M^{n+3}, $S^{n+1} \underset{n-1}{\cup} S^{n+1} = e^{n+3}$. It follows that

$$e_2^{n+3} = \kappa^*e^{n+3} = \kappa^*(S^{n+1} \underset{n-1}{\cup} S^{n+1}) = \kappa^*S^{n+1} \underset{n-1}{\cup} \kappa^*S^{n+1} = e^{n+1} \underset{n-1}{\cup} e^{n+1} \ (\mathit{mod}\ 2). \quad \text{q.e.d.}$$

§ 4. The $(n+3)$-extension cocycle

Let K be a finite complex, the r-skelton of which is denoted by K^r. Let Y be an arcwise connected topological space such that $\pi_i(Y)=0$ for each $i<n$ and for $i=n+1$, and $\pi_n(Y)$ has a finite number of generators α_μ $(\mu=1, 2, \dots, l)$.

Let $n_\mu \geqq 0$ be the order of α_μ. Define following reduced complexes:

$$R^n = \sum_\mu B^0_\mu(n_\mu) = \sum_\mu S^n_\mu ,$$

$$R^{n+2} = \sum_{n_\mu:\, even} B^2_\mu(n_\mu) + \sum_{n_\mu:\, odd} B^1_\mu(n_\mu) ,$$

$$R^{n+3} = \sum_{n_\mu:\, even} B^3_\mu(n_\mu) + \sum_{n_\mu:\, odd} B^1_\mu(n_\mu) \quad \text{for} \quad n>3 ,$$

and

$$R^{n+3} = \sum_{n_\mu:\, even} B^3_\mu(n_\mu) + \sum_{n_\mu:\, odd} B^1_\mu(n_\mu) + \sum_{\mu<\nu} e^6_{\mu,\nu} \quad \text{for} \quad n=3.$$

where $e^6_{\mu,\nu} = S^3_\mu \times S^3_\nu - S^3_\mu \vee S^3_\nu$ and $B^i(n_\mu)$'s and $e^6_{\mu,\nu}$'s in each reduced complex have only one point p in common. Then we can consider that $R^n \subset R^{n+2} \subset R^{n+3}$. (cf. § 2).

Let us define a map $\varphi: R^n \to Y$ such that $\varphi: S^n_\mu \to Y$ represents $\alpha_\mu \in \pi_n(Y)$. Then it is easily seen that φ is extended to a map $\varphi: R^{n+2} \to Y$. For a given normal map $f: K^n \to Y$, there exists a map $h: K^n \to R^n$ such that $h: K^{n-1} \to p$ and f is homotopic to φh. Thus it may be supposed that f and φh define the same map on K^n. If f is extensible to K^{n+1}, then f is also extensible to K^{n+2} from $\pi_{n+1}(Y)=0$. Then the secondary obstruction $c^{n+3}(f)$ is defined. Correspondingly, h can be extended to a map $h: K^{n+2} \to R^{n+2}$ such that φh and f are homotopic on K^{n+2} relative to K^n, Notice that h, moreover, can be extended to a map of K^{n+3} into R^{n+3}. It follows that $c^{n+3}(f) \backsim c^{n+3}(\varphi h) = h^* c^{n+3}(\varphi)$. If $\omega(\alpha_\mu)$ is such an element of $\pi_{n+2}(Y)$ as is represented by a map $\varphi\omega$, where ω is a map representing a generator of order 0 of $\pi_{n+2}(B^2_\mu(n_\mu))$ (n_μ *even*) (see § 2), and if $\xi(\alpha_\mu)$ is such an element of $\pi_{n+2}(Y)$ as is represented by a map $\varphi\xi$, where ξ is a map representing a generator of order 2 of $\pi_{n+2}(B^2_\mu(n_\mu))$, then, we have by theorem 1 in § 3,

$$\begin{aligned} c^{n+3}(\varphi h) = h^* c^{n+3}(\varphi) &= h^*\Big[\sum_{n_\mu \geqq 0,\, even} \omega(\alpha_\mu) e^{n+3}_{1,\mu} + \sum_{n_\mu>0,\, even} \xi(\alpha_\mu) e^{n+3}_{2,\mu} + \Big(\sum_{\mu<\nu} \alpha_\mu \alpha_\nu e^6_{\mu,\nu}\Big)\Big] \\ &= h^*\Big[\sum_{n_\mu \geqq 0,\, even} (\mathcal{G}_{n-3} S^n_\mu)\omega(\alpha_\mu) + \sum_{n_\mu>0,\, even} (S_{q_{n-1}} \theta^{n+1}_{n_\mu} S^n_\mu)\xi(\alpha_\mu) + \Big(\sum_{\mu<\nu} (S^3_\mu \cup S^3_\nu)\alpha_\mu \alpha_\nu\Big)\Big], \end{aligned}$$

where the last terms $\sum_{\mu<\nu} (S^3_\mu \cup S^3_\nu)\alpha_\mu\alpha_\nu$ are added only when $n=3$.

If we put $c^n_\mu = h^* S^n_\mu$, then the first obstruction $c^n(f)$ of f is expressible in the following form: $c^n(f) = \sum_\mu \alpha_\mu \cdot c^n_\mu$.

Thus we obtain the following

THEOREM 2. *Let K be a finite complex, and let K^r be its r-skeleton. Let Y be an $(n-1)$-connected topological space whose $(n+1)$-th homotopy group vanishes. Given a mapping $f: K^n \to Y$ such that f maps K^{n-1} into a point of Y.*

If the first obstruction $c^n(f)$ is a cocycle, then f is extensible to a map $\bar{f}: K^{n+2} \to Y$ and its $(n+3)$-extension cocycle $c^{n+3}(\bar{f})$ is determined from $c^n(f)$ in the following form: $(n \leqq 3)$

$$c^{n+3}(\bar{f}) \backsim \sum_{n_\mu \geqq 0,\ \text{even}} \Big(c_\mu^n \underset{n-3}{\cup} c_\mu^n + n_\mu c_\mu^n \underset{n-2}{\cup} \lambda_\mu^{n+1} + (-1)^n \frac{n_\mu^2}{2} \lambda_\mu^{n+1} \underset{n-1}{\cup} \lambda_\mu^{n+1}\Big)\omega(\alpha_\mu)$$
$$+ \sum_{n>0,\ \text{even}} \Big(\lambda_\mu^{n+1} \underset{n-1}{\cup} \lambda_\mu^{n+1}\Big)\hat{\varsigma}(\alpha_\mu) + \sum_{\mu<\nu} (c_\mu^3 \cup c_\nu^3)\alpha_\mu\alpha_\nu,$$

where the last terms is added only when $n=3$, and $c^n(f) = \sum_\mu \alpha_\mu c_\mu^n$, $\lambda_\mu^{n+1} = \theta_{n_\mu}^{n+1} \cdot c_\mu^n = \frac{1}{n_\mu}\delta c_\mu^n$ $(n_\mu > 0)$, and $\lambda_\mu^{n+1} = 0$ $(n_\mu = 0)$.

§ 5. Classification

We shall apply Theorem 2 in § 4 to the present classification problem in a usual way. Let Y be a space as was referred to above. It is our aim to classify all the classes of mappings of an $(n+2)$-dimensional complex K into the space Y. If we denote by $\mathscr{C}_{n-3}c^n(f)$ the first terms in the expression of $c^{n+3}(\bar{f})$ $(n>3)$ in Theorem 2 and if we denote the second terms by $S_{q_{n-1}}\theta^{n+1}c^n(f)$, then we have

$$c^{n+3}(\bar{f}) \backsim (\mathscr{C}_{n-3} + S_{q_{n-1}}\theta^{n+1})c^n(f).$$

We shall use this notation in the following.

Since $\mathscr{C}_{n-3} + S_{q_{n-1}}\theta^{n+1}$ is a homomorphism of $H^n(K, \pi_n(Y))$ into $H^{n+3}(K, \pi_{n+2}(Y))$, we have the classification theorem through analogous arguments of Steenrod [2].

THEOREM 3. $(n>3)$.

Let K be an $(n+2)$-dimensional finite complex, and let Y be a space with the same property in Theorem 2.

All the homotopy classes of mappings of K into Y, that are contained in one homotopy class of mappings of K^n into Y, are in one to one correspondence with the cosets of the factor group:

$$H^{n+2}(K, \pi_{n+2}(Y))/(\mathscr{C}_{n-4} + S_{q_{n-2}}\theta^n)H^{n-1}(K, \pi_n(Y)),$$

where $\mathscr{C}_{n-4} + S_{q_{n-2}}\theta^n$; $H^{n-1}(K, \pi_n(Y)) \to H^{n+2}(K, \pi_{n+2}(Y))$ is a homomorphism.

THEOREM 3′. (*The case $n=3$*). *All the homotopy classes of mappings of K^5 into Y, that are homotopic to each other on K^3, are in one to one correspondence with the cosets of the factor group:*

$$H^5(K^5, \pi_5(Y))/\Psi H^2(K^5, \pi_3(Y))$$

where Ψ: $H^2(K^5, \pi_3(Y)) \to H^5(K^5, \pi_5(Y))$ is a homomorphism defined in the following way.

Let $\{\lambda^2\} \in H^2(K^5, \pi_3(Y))$, and let $\lambda^2 = \sum_\mu \alpha_\mu \lambda_\mu^2$, where α_μ are generators of $\pi_3(Y)$. Then $\Psi\{\lambda^2\}$ is a cohomology class represented by

$$\sum_{n_\mu>0,\ even}(n_\mu\lambda_\mu^2\cup\theta_{n_\mu}^3\lambda_\mu^2-\frac{n_\mu^2}{2}\theta_{n_\mu}^2\lambda_\mu^2\underset{1}{\cup}\theta_{n_\mu}^3\lambda_\mu^2)\omega(\alpha_\mu)$$
$$+\sum_{n_\mu>0,\ even}(\theta_{n_\mu}^3\lambda_\mu^2\underset{1}{\cup}\theta_{n_\mu}^3\lambda_\mu^2)\xi(\alpha_\mu)+\sum_{\mu<\nu}(c_\mu^3\cup\lambda_\nu^2+\lambda_\mu^2\cup c_\nu^3)\alpha_\mu\alpha_\nu.$$

It is seen that $2\Psi\{\lambda^2\}=0$.

§6. Invariant cohomology class $\mathfrak{k}^{n+3}$

Eilenberg and MacLane [7] have introduced, for a space Y such that $\pi_i(Y)=0$ $(i<n<i<n+h)$, a cohomology class $\mathfrak{k}^{n+h+1}$ of an abstract complex $K(\pi_n(Y), n)$, and studied the influence of $\mathfrak{k}^{n+h+1}$ on homology groups of Y. We shall here deal with a space Y with the same property as in preceding sections. We consider the case $n>2$ and $h=2$.

THEOREM 4. *Let k^{n+3} be a cocycle belonging to $\mathfrak{k}^{n+3}$, then*

$$k^{n+3}\backsim(\mathcal{Y}_{n-3}+S_{q_{n-1}}\theta^{n+1})d^n \quad for \quad n>3,$$
$$k^{n+3}\backsim(\mathcal{Y}_0+S_{q_2}\theta^4)d^3+\sum_{\mu<\nu}(d_\mu^3\cup d_\nu^3)\alpha_\mu\alpha_\nu \quad for \quad n=3,$$

where d^n represents the element of $H^n(\pi_n(Y), n, \pi_n(Y))$ which acts as the identity endomorphism of $\pi_n(Y)$, and $d^n=\sum_\mu\alpha_\mu d_\mu^n$.

BIBLIOGRAPHY

[1] L. Pontrjagin, Mappings of the 3-dimensional sphere into an n-dimensional complex, C. R. Acad. Sci. URSS., **34** (1942), 35-37.

[2] N. E. Steenrod, Products of cocycles and extensions of mappings, Ann. of Math., **48** (1947), 290-320.

[3] H. Whitney, An extension theorem for mappings into simply connected spaces, Ann. of Math., **50** (1949), 285-296.

[4] J. H. C. Whitehead, On simply connected, 4-dimensional polyhedra, Comm. Math. Helv., **22** (1949), 48-92.

[5] J. H. C. Whitehead, The homotopy type of a special kind of polyhedron, Annals de la Soc. Polon. der Math., **21** (1948), 176-186. (This is inaccessible to us here.)

[6] J. H. C. Whitehead, On adding relations to homotopy groups, Ann of Math., **42** (1941), 409-428.

[7] S. Eilenberg and S. MacLane, Relations between homology and homotopy groups of spaces II, Ann. of Math., **51** (1950) 514-533.

[8] G. W. Whitehead, A generalization of the Hopf invariant, Ann. of Math., **51** (1950), 192-237.

[9] G. W. Whitehead, The $(n+2)$-nd homotopy group of the n-sphere, Ann. of Math., **52** (1950), 245-247.

[10] L. Pontrjagin, C. R. Acad. Sci. URSS, **70** (1950), 957-959.

[11] A. L. Blakers and W. S. Massay, Homotopy groups of a triad I, Ann. of Math., **53** (1951), 161-205.

[12] S. C. Chang, Homotopy invariants and continuous mappings, Proc. Roy. Soc. of London, A, **202** (1950), 253-263.

[13] H. Uehara, On homotopy type problems of special kind of polyhedra, to appear shortly.

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