ON KRULL’S CONJECTURE CONCERNING VALUATION RINGS

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Introduction. Previously W. Krull conjectured\(^1\) that every completely integrally closed primary\(^2\) domain of integrity is a valuation ring. The main purpose of the present paper is to construct in §1 a counter example against this conjecture. In §2 we show a necessary and sufficient condition that a field is a quotient field of a suitable completely integrally closed primary domain of integrity which is not a valuation ring.

By a ring we mean a commutative ring with identity. We refer to the notations like \( \mathcal{O}_p \) as the ring of quotients of \( \mathfrak{p} \) with respect to \( \mathfrak{o} \) when \( \mathfrak{o} \) is a ring and \( \mathfrak{p} \) is a prime ideal of \( \mathfrak{o} \).

1. A counter example.

Let \( K \) be an algebraically closed field with a non-trivial special valuation \( w \) whose value group \( G \) does not fill up all real numbers. Take a positive number \( \alpha \) which is not in \( G \). Consider a rational function field \( K(x) \) of one variable \( x \) with constant field \( K \). Let us define the following two types of valuations of \( K(x) \) which are extensions of \( w \) to:

1. For every element \( e \) of \( K \) such that \( \alpha < w(e) < 2\alpha \), we define a valuation \( w_e \) (of \( K(x) \)) such that
   \[
   w_e \left( \sum_{i=0}^{n} a_i(x + e)^i \right) = \min \left( w(a_i) + 2\alpha i \right) \quad (a_i \in K). \]

2. For every real number \( \lambda \) such that \( \alpha < \lambda < 2\alpha \), we define a valuation \( w_\lambda \) such that
   \[
   w_\lambda \left( \sum_{i=0}^{n} a_i x^i \right) = \min \left( w(a_i) + \lambda i \right) \quad (a_i \in K).
   \]

**Theorem 1.** Let \( \mathfrak{c}_e \) and \( \mathfrak{c}_\lambda \) be the valuation rings determined by \( w_e \) and \( w_\lambda \), respectively (\( \alpha < w(e) < 2\alpha \), \( \alpha < \lambda < 2\alpha \)) and let \( \mathfrak{o} \) be the intersection of all such \( \mathfrak{c}_e \) and \( \mathfrak{c}_\lambda \). Then \( \mathfrak{o} \) is completely integrally closed and primary, but \( \mathfrak{o} \) is not a valuation ring.

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\(^2\) A ring is called primary if it has at most one proper prime ideal.

\(^3\) Observe the fact that \( 2\alpha \notin G \), because \( K \) is algebraically closed.

\(^4\) Since \( 2\alpha \notin G \), \( w_e \) is uniquely determined by the relation \( w_e(x + e) = 2\alpha \).
valuation ring.

Proof. Let \( c(\neq 0) \) be an element of \( \mathfrak{a} \). First we prove that (1) if \( w_\lambda(c) = 0 \) for some \( \lambda_0 \) (\( \alpha \leq \lambda_0 \leq 2\alpha \)), then \( w_\lambda(c) = 0 \) and \( w_\varepsilon(c) = 0 \) for every \( w_\lambda \) and \( w_\varepsilon \), and that (2) if \( w_\varepsilon(c) > 0 \), there exist the least and the largest values \( \varepsilon > 0 \) and \( \delta \) among values of \( c \) taken by \( w_\lambda \) and \( w_\varepsilon \) (\( \alpha \leq \lambda \leq 2\alpha, \alpha < w_\varepsilon(c) < 2\alpha \)).

Since \( K \) is algebraically closed, \( c \) is of the form

\[
c_0 \prod_{i=1}^n (x + a_i) / \prod_{j=1}^m (x + b_j) \quad (c_0, a_i, b_j \in K).
\]

Every factor \( x + d \) (\( d = a_i \) or \( b_j \)) such that \( w(d) > 2\alpha \) may be replaced by \( x \), since we only consider the values of \( c \) taken by \( w_\lambda \) and \( w_\varepsilon \). Similarly we may replace by \( d \) every factor \( x + d \) (\( d = a_i \) or \( b_j \)) such that \( w(d) > \alpha \). Therefore we may assume without loss of generality that (i) \( \alpha < w_\lambda(a_i) < 2\alpha \) or \( a_i = 0 \), \( \alpha < w_\varepsilon(b_j) < 2\alpha \) or \( b_j = 0 \) for each \( i \) and \( j \) (\( 1 \leq i \leq n, 1 \leq j \leq m \)), (ii) \( a_i \neq b_j \) for every pair \( (i, j) \) and (iii) \( w_\lambda(a_i) \leq w_\lambda(a_{i+1}), w_\lambda(b_j) \leq w_\lambda(b_{j+1}) \) (\( 1 \leq i < n, 1 \leq j < m \)).

First we assume that \( w_\lambda(c) = 0 \) for some \( \lambda_0 \) (\( \alpha \leq \lambda_0 \leq 2\alpha \)). If there exists one \( j \) such that \( w_\varepsilon(b_j) = \lambda_0 \), then we have \( w_{\lambda_0}(c) < 0 \), which is a contradiction. Therefore no \( w(b_j) \) is equal to \( \lambda_0 \). Assume that \( w(a_i) < \lambda_0 \) if \( i \leq i_0 \), \( w(a_i) = \lambda_0 \) if \( i > i_0 + s \), \( w(b_j) < \lambda_0 \) if \( j \leq j_0 \), \( w(b_j) > \lambda_0 \) if \( j > j_0 \). Set \( \lambda_1 = \max(\alpha, w(a_i), w(b_j)), \lambda_2 = \min(2\alpha, w(a_{i_0 + s + 1}), w(b_{j_0 + 1}). \)

Then

\[
\begin{align*}
w_\lambda(c) &= w_\lambda(c_0) + \sum_{i=1}^n w(a_i) - w_\lambda(b_j) + (n - i_0) \lambda_1 - (m - j_0) \lambda_1 \geq 0, \\
w_\varepsilon(c) &= w_\varepsilon(c_0) + \sum_{i=1}^n w(a_i) - w_\varepsilon(b_j) + (n - i_0) \lambda_0 - (m - j_0) \lambda_0 = 0, \\
w_{\varepsilon}(c) &= w_\varepsilon(c_0) + \sum_{i=1}^n w(a_i) - w_\varepsilon(b_j) + s \lambda_0 + (n - i_0 - s) \lambda_2 - (m - j_0) \lambda_2 \geq 0.
\end{align*}
\]

Hence we have

\[
w_\lambda(c) - w_\lambda(c) = (n - i_0)(\lambda_1 - \lambda_0) - (m - j_0)(\lambda_1 - \lambda_0) \geq 0,
\]

whence \( n - i_0 \leq m - j_0 \).

\[
(\text{If } \alpha = \lambda_0 \text{ or } 2\alpha = \lambda_0, \text{ we see easily that } n - i_0 = m - j_0 \text{ because } x \notin G. \text{ In this case, } s = 0 \text{ is also clear.})
\]

Similarly we have

\[
w_\lambda(c) - w_\lambda(c) = (n - i_0 - s)(\lambda_2 - \lambda_0) - (m - j_0)(\lambda_2 - \lambda_0) \geq 0,
\]

whence \( n - i_0 - s \leq m - j_0 \).

Thus we have \( s = 0 \) and \( n - i_0 = m - j_0 \). \( s = 0 \) shows that no \( w(a_i) \) is equal to \( \lambda_0 \). Further, \( n - i_0 = m - j_0 \) shows \( w_\lambda(c) = w_\lambda(c) = 0 \). Therefore neither \( w(a_i) \) nor \( w(b_j) \) are equal to \( \lambda_1 \) or \( \lambda_2 \), by the above observation. This means that \( \lambda_1 = \alpha \) and \( \lambda_2 = 2\alpha \). From \( \lambda_1 = \alpha \) we have that \( i_0 = j_0 = 0 \), whence \( m = n \); From \( \lambda_2 = 2\alpha \) we have that \( a_i = 0, b_j = 0 \) (\( 1 \leq i \leq n, 1 \leq j \leq m \)). By our assumption

\[
\text{if } \alpha = \lambda_0 \text{ or } 2\alpha = \lambda_0, \text{ we see easily that } n - i_0 = m - j_0 \text{ because } x \notin G. \text{ In this case, } s = 0 \text{ is also clear.}
\]
that \( a_i \neq b_j \), it follows that \( m = n = 0 \), i.e., \( c = c_0 \in K \). Since \( w_{v_0}(c) = 0 \), we have \( w(c) = 0 \). This proves (1). Next assume that \( w_v(c) > 0 \). Let us consider \( w_v(c) \) as a function of variable \( \lambda \) \( (\alpha \leq \lambda \leq 2\alpha) \). Then it is evidently continuous, and it takes the least and the largest values \( \varepsilon_1 \) and \( \delta_1 \) in \( \alpha \leq \lambda \leq 2\alpha \). By virtue of (1), we see that \( \varepsilon_1 \) is positive. Then (2) follows easily from the fact that \( w_v(c) \neq w_{v(e)}(c) \) holds only if \( e \) is one of \( a_i \) or \( b_j \) and in this case \( w_v(c) \neq 0 \), whence \( w_v(c) = 0 \).

These being proved, we see that \( \varpi \) is primary. Let \( a(\neq 0) \) and \( b(\neq 0) \) be two non-units in \( \varpi \). Then there exist positive numbers \( \varepsilon \) and \( \delta \) such that \( w_\lambda(a) \geq \varepsilon, w_\lambda(b) \leq \delta, w_\lambda(b) \leq \delta \) \( (\alpha \leq \lambda \leq 2\alpha, \alpha < w(e) < 2\alpha) \). Let \( k \) be an integer such that \( k\varepsilon > \delta \). Then we have \( w_\lambda(a^k/b) \geq 0, w_\lambda(a^k/b) \leq 0 \) \( (\alpha \leq \lambda \leq 2\alpha, \alpha < w(e) < 2\alpha) \), whence \( a^k/b \in \varpi \), i.e., \( a^k \in b\varpi \).

It is evident that \( \varpi \) is completely integrally closed, because \( \varpi \) is an intersection of special valuation rings. That \( \varpi \) is not a valuation ring follows from that \( c/\varpi \notin \varpi, c/\varpi \notin \varpi \) if \( \alpha < w(e) < 2\alpha \).

2. An existence theorem.

Lemma 1. Let \( r \) be an integrally closed integral domain which has only one maximal ideal \( \varpi_0 \). Let \( K \) be the quotient field of \( r \). If \( Z \) is a field containing \( K \), \( \varpi_0 \cap K = r \), where \( \varpi \) is the totality of \( r \)-integers in \( Z \) and \( \varpi_0 \) a maximal ideal of \( \varpi \).

Proof. We may assume without loss of generality that \( Z \) is algebraic over \( K \) because the quotient field of \( \varpi \) is algebraic over \( K \).

First we assume that \( Z \) is finite normal over \( K \). Let \( \{a_i, \ldots, a_n\} \) be the totality of automorphisms of \( Z \) over \( K \). We show that every maximal ideal of \( \varpi \) is one of \( \varpi_0^n \). Assume that a maximal ideal \( \varpi_0^n \) of \( \varpi \) is none of \( \varpi_0^n \). Then there exists an element \( c \) of \( \varpi_0^n \) such that \( c \notin \varpi_0^n \) for every \( i = 1, \ldots, h \). A power \( e \) of \( \Pi_i^n c^{a_i} \) is in \( K \), whence in \( r \). Since \( c \in \varpi_0^n \), we have \( e \in \varpi_0^n \), whence \( e \in \varpi_0^6 \). Therefore one of \( c^{a_i} \) must be in \( \varpi_0^n \), i.e., \( c \) is in some \( \varpi_0^n \), which is a contradiction. This being shown, we have \( \varpi = \bigcap_i^n (\varpi_0^n) \). Therefore \( \varpi_0 \cap K = (\varpi_0^n) \cap K = (\bigcap_i^n (\varpi_0^n)) \cap K = \varpi \cap K = r \).

Next we assume that \( Z \) is finite algebraic over \( K \). Let \( Z^* \) be a field containing \( Z \) which is finite normal over \( K \). Let \( \varpi^* \) be the totality of \( r \)-integers in \( Z^* \) and let \( \varpi^* \) be a maximal ideal of \( \varpi^* \) which contains \( \varpi_0^* \). Then evidently \( \varpi_0^* \supseteq \varpi_0 \). Since \( \varpi_0^* \cap K = r \), we have \( \varpi_0 \cap K = r \).

Making use of this, we prove the general case. Let \( c \) be an element of \( \varpi_0 \cap K \). \( c \) may be written in a form \( a/b \) \( (a, b \in \varpi, b \notin \varpi) \). We consider \( Z^* = K(a, b) \). We set \( \varpi^* = \varpi \cap Z^* \), and \( \varpi^* = \varpi \cap \varpi^* \). Then \( \varpi^* \) is a maximal ideal because \( \varpi \)
is integral over \( o^* \). It is clear that \( a, b \in o^*, b \in p^* \) whence \( o_p^* \subseteq c \). Since \( Z^* \) is finite over \( K \), we have \( o_p^* \cap K = c \), which proves our assertion.

**Lemma 2.** Let \( K \) be a field with a valuation ring \( v \) and let \( Z \) be a field containing \( K \) which is algebraic over \( K \). Let \( o \) be the totality of \( v \)-integers in \( Z \) and let \( \{v_{\lambda}; \lambda \in \Lambda \} \) be the totality of maximal ideals of \( o \). Then every valuation ring \( w \) of \( Z \), such that the valuation given by \( w \) is an extension of that given by \( v \), is one of \( o_{v_{\lambda}} (\lambda \in \Lambda) \). Conversely, every \( o_{v_{\lambda}} (\lambda \in \Lambda) \) is a valuation ring.

**Proof.** It is clear that any such valuation ring \( w \) contains one of \( o_{v_{\lambda}} \). Hence we have only to prove the converse part. But this follows immediately from the following facts:

1) An integrally closed domain \( m \) of integrity is a multiplication ring if and only if \( m_p \) is a valuation ring for every maximal ideal \( p \) of \( m \).

2) Let \( m \) be a multiplication ring with quotient field \( K \). If a field \( Z \) containing \( K \) is algebraic over \( K \), then the totality \( o \) of \( m \)-integers in \( Z \) is also a multiplication ring and \( Z \) is the quotient field of \( o \).

**Lemma 3.** Let \( r \) be a completely integrally closed integral domain with quotient field \( K \). If \( Z \) is a field containing \( K \), the totality \( o \) of \( r \)-integers in \( Z \) is also completely integrally closed.

**Proof.** Assume that \( Z \) is finite normal (algebraic) over \( K \). Let \( \{\sigma_1, \ldots, \sigma_n\} \) be the totality of automorphisms of \( Z \) over \( K \). Set \( r = [Z:K]/h \). Assume that \( (a/b)^n c \in o \) for every natural number \( n \), where \( a, b \) and \( c \) are non-zero elements of \( o \). Let \( f \) be an arbitrary elementary symmetric formula of \( [(a/b)^n] \), \( \ldots, [(a/b)^n] \), and set \( c' = (\prod_{i=1}^n c_i)^n. \) Then \( f^n c' \in o \), whence \( f^n c' \in r \) for every natural number \( n \). This shows that \( f \in r \), whence \( a/b \) satisfies a monic equation with coefficient in \( r \), i.e., \( a/b \in o \), which proves our assertion when \( Z \) is finite normal over \( K \). This being proved, we can reduce our problem to the general case by the same way as in the proof of Lemma 1.

**Theorem 2.** Let \( K \) be a field. Then there exists a completely integrally closed primary domain of integrity which is not a valuation ring such that its quotient field is \( K \) if and only if \( K \) satisfies one of the following two conditions:

1) \( K \) is of characteristic 0 and \( K \) is not algebraic over its prime field.

2) \( K \) is of characteristic \( p \) (\( \neq 0 \)) and \( K \) contains at least two algebraically independent elements over its prime field.

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5) W. Krull, Beiträge zur Arithmetik kommutativer Integritätsbereiche, Math. Zeit. 41 (1936), Theorem 7 (p. 554).

8) Früher, Untersuchungen über die Teilbarkeitseigenschaften in Körpern, Crelle 168, p. 31 or 1. c. note 6) Theorem 8 (p. 555).
Proof. (I) The case where $K$ satisfies neither of these conditions. Let $\mathfrak{o}$ be any integrally closed primary domain of integrity with quotient field $K$. When $K$ is algebraic over its prime field, let $K_0$ be its prime field. When $K$ is not algebraic over its prime field, let $K_0$ be its subfield which is isomorphic to the rational function field of one variable with its prime field as the constant field. Then evidently $\mathfrak{o} \cap K_0$ is a valuation ring. Then by Lemma 2 it follows that $\mathfrak{o}$ is also a valuation ring.

(II) Assume that $K$ satisfies one of the above two conditions. Then it is easy to see that there exists a subfield $K_0$ of $K$ such that $K_0$ has a non-trivial discrete special valuation and such that $K$ has transcendental degree 1 over $K_0$, that is, there exists an element $x$ of $K$ such that $x$ is not algebraic over $K_0$ and $K$ is algebraic over $K_0(x)$. Let $\overline{K}_0$ and $\overline{K}$ be the algebraic closures of $K_0$ and $K$ respectively. Then by Theorem 1 we can construct a completely integrally closed primary domain $\mathfrak{r}$ of integrity which is not a valuation ring and whose quotient field is $\overline{K}(x)$. Let $\overline{\mathfrak{o}}$ be the totality of $\mathfrak{r}$-integers in $\overline{K}$ and let $\overline{\mathfrak{b}}$ be a maximal ideal of $\overline{\mathfrak{o}}$. Set $\mathfrak{o} = \overline{\mathfrak{b}} \cap K$. Then since $\mathfrak{r}$ is completely integrally closed, $\overline{\mathfrak{o}}$ is so too by Lemma 3. Therefore $\mathfrak{o}$ is also completely integrally closed. Since $\mathfrak{r}$ is primary, so is $\overline{\mathfrak{b}}$ too, whence $\mathfrak{o}$ is primary. On the other hand, since $\overline{\mathfrak{b}} \cap K_0(x) = r$ by Lemma 1, $\overline{\mathfrak{b}}$ is not a valuation ring and therefore $\mathfrak{o}$ is not a valuation ring again by virtue of Lemma 2. Thus our proof is complete.

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5) We need not assume here that $\mathfrak{o}$ is "completely" integrally closed.