CHARACTERIZATION OF CERTAIN RIEMANN SPACES BY DEVELOPMENT

MINORU KURITA

The purpose of this paper is to characterize the Riemann space whose line element is given by

(1) $$ds^2 = a(x^i)^2 g_{\alpha\beta}(x^\gamma)dx^\alpha dx^\beta + g_{\lambda\mu}(x^i)dx^\lambda dx^\mu.$$

Through the whole description we use the indices $i, j, \alpha, \beta, \gamma, \lambda, \mu, \nu$ when they run as follows

$$i, j \quad = 1, 2, \ldots, n$$
$$\alpha, \beta, \gamma = 1, 2, \ldots, k$$
$$\lambda, \mu, \nu = k+1, k+2, \ldots, n$$

and $a(x^i)$ and $g_{\lambda\mu}(x^i)$ shall mean to be functions of $x^1, \ldots, x^n$, $g_{\alpha\beta}(x^\gamma)$ a function of $x^1, \ldots, x^k$ and $g_{\lambda\mu}(x^\nu)$ a function of $x^{k+1}, \ldots, x^n$. This Riemann space (1) contains as special cases many important spaces such as a directly decomposable space, a conformally separable space and a space with torse-forming vector field. K. Yano [1] characterized a conformally separable Riemann space by umbilical surfaces contained in it and the proof of Theorem 1 of [1] leads to the following:

THEOREM. *The necessary and sufficient condition for an n-dimensional Riemann space to have an arc-element given by (1) is that it has $n-k$-parametric family of k-dimensional totally umbilic surfaces and k-parametric family of $n-k$-dimensional surfaces which are orthogonal to the former.*

We characterize this space (1) from another point of view.

1. We begin by determining the Riemannian connection, namely the euclidean connection without torsion, attached to (1). We can take k Pfaffians

(2) $$\omega_\alpha = p_{\alpha\beta}(x^\gamma)dx^\beta$$

such that

$$g_{\alpha\beta}(x^\gamma)dx^\alpha dx^\beta = \sum(\omega_\alpha)^2.$$

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Then, according to E. Cartan, Pfaffians $\omega_{\alpha\beta}$ satisfying the relations

(3) $$\omega_\alpha' = [\omega_\beta \omega_{\beta\alpha}], \qquad \omega_{\alpha\beta} = -\omega_{\beta\alpha}$$

are uniquely determined. Next we take

(4) $$\pi_\lambda = p_{\lambda\mu}(x^i) dx^\mu$$

such that

$$g_{\lambda\mu}(x^i) dx^\lambda dx^\mu = \sum (\pi_\lambda)^2.$$

Putting

(5) $$\pi_\alpha = a(x^i)\omega_\alpha$$

we get from (1)

$$ds^2 = \sum (\pi_i)^2.$$

For these π_i there exist uniquely determined Pfaffians π_{ij} such that

(6) $$\pi_i' = [\pi_j, \pi_{ji}], \qquad \pi_{ij} = -\pi_{ji}.$$

We get by (3) and (5)

$$\pi_\alpha' = (a\omega_\alpha)' = [da, \omega_\alpha] + a\omega_\alpha' = [da, \omega_\alpha] + a[\omega_\beta, \omega_{\beta\alpha}].$$

Then by (6)

$$[da, \omega_\alpha] + a[\omega_\beta, \omega_{\beta\alpha}] = [\pi_\beta, \pi_{\beta\alpha}] + [\pi_\lambda, \pi_{\lambda\alpha}].$$

Taking (5) into consideration we have

$$\left[\delta_{\alpha\beta}\frac{da}{a} + \omega_{\alpha\beta} - \pi_{\alpha\beta}, \pi_\beta\right] + [-\pi_{\alpha\lambda}, \pi_\lambda] = 0.$$

As π_β's and π_λ's are linearly independent, we can put

(7) $$\delta_{\alpha\beta}\frac{da}{a} + \omega_{\alpha\beta} - \pi_{\alpha\beta} = C_{\alpha\beta i}\pi_i$$

(8) $$-\pi_{\alpha\lambda} = C_{\alpha\lambda i}\pi_i$$

where

(9) $$C_{\alpha ij} = C_{\alpha ji}.$$

As $\omega_{\alpha\beta}$'s and $\pi_{\alpha\beta}$'s are skew symmetric with respect to indices α and β we get for $\alpha \neq \beta$

(10) $$C_{\alpha\beta i} + C_{\beta\alpha i} = 0 \quad (\alpha \neq \beta).$$

For $\alpha \neq \beta \neq \gamma$ we get by (9) and (10)

$$C_{\alpha\beta\gamma} = -C_{\beta\alpha\gamma} = -C_{\beta\gamma\alpha} = C_{\gamma\beta\alpha} = C_{\gamma\alpha\beta} = -C_{\alpha\gamma\beta} = -C_{\alpha\beta\gamma}.$$

Hence

(11) $$C_{\alpha\beta\gamma} = 0 \quad (\alpha \neq \beta \neq \gamma).$$

For $\alpha = \beta$ we obtain by (7)

$$\frac{da}{a}=C_{\alpha\alpha i}\pi_i.$$

As π_i's are linearly independent da/a can be written uniquely as a linear combination of π_i's. So $C_{\alpha\alpha i}$'s depend only on i and not on α. We put $C_{\alpha\alpha i}=C_i$ and we have

$$\frac{da}{a}=C_i\pi_i. \tag{12}$$

By (9) and (10)

$$C_{\alpha\beta\beta}=-C_{\beta\alpha\beta}=-C_{\beta\beta\alpha}=-C_\alpha=-C_{\alpha\alpha\alpha} \quad (\alpha\neq\beta), \tag{13}$$

and by (7)

$$\pi_{\alpha\beta}=\omega_{\alpha\beta}-C_{\alpha\beta i}\pi_i.$$

Taking (11) and (13) into consideration we have

$$\pi_{\alpha\beta}=\omega_{\alpha\beta}-C_\beta\pi_\alpha+C_\alpha\pi_\beta-C_{\alpha\beta\lambda}\pi_\lambda. \tag{14}$$

Next by (6) and (8) we have

$$\begin{aligned}\pi_\lambda'&=[\pi_\alpha,\ \pi_{\alpha\lambda}]+[\pi_\mu,\ \pi_{\mu\lambda}]\\ &=[\pi_\alpha,\ -C_{\alpha\lambda\beta}\pi_\beta-C_{\alpha\lambda\nu}\pi_\nu]+[\pi_\mu,\ \pi_{\mu\lambda}]\\ &=-C_{\alpha\lambda\beta}[\pi_\alpha,\ \pi_\beta]-C_{\alpha\lambda\nu}[\pi_\alpha,\ \pi_\nu]+[\pi_\mu,\ \pi_{\mu\lambda}].\end{aligned}$$

Considering (2) and (4) and remarking that the terms $[dx^\alpha,\ dx^\beta]$ appear only in $C_{\alpha\lambda\beta}[\pi_\alpha,\ \pi_\beta]$ we get $C_{\alpha\lambda\beta}=C_{\beta\lambda\alpha}$. Then by (9) $C_{\alpha\beta\lambda}=C_{\beta\alpha\lambda}$, while according to (10) $C_{\alpha\beta\lambda}+C_{\beta\alpha\lambda}=0$, and we get

$$C_{\alpha\beta\lambda}=0 \quad (\alpha\neq\beta).$$

Then by (8)

$$\begin{aligned}\pi_{\alpha\lambda}&=-C_{\alpha\lambda i}\pi_i=-C_{\alpha\lambda\beta}\pi_\beta-C_{\alpha\lambda\mu}\pi_\mu\\ &=-C_{\alpha\lambda\alpha}\pi_\alpha-C_{\alpha\lambda\mu}\pi_\mu\\ &=-C_{\alpha\alpha\lambda}\pi_\alpha-C_{\alpha\lambda\mu}\pi_\mu.\end{aligned}$$

As $C_{\alpha\alpha\lambda}=C_\lambda$ we have

$$\pi_{\alpha\lambda}=-C_\lambda\pi_\alpha-C_{\alpha\lambda\mu}\pi_\mu. \tag{15}$$

Thus we obtain the following relations by (12), (14), (15) and $C_{\alpha\beta\lambda}=0$;

$$\text{(A)}\qquad \begin{cases}\pi_{\alpha\beta}=\omega_{\alpha\beta}-C_\beta\pi_\alpha+C_\alpha\pi_\beta\\ \pi_{\alpha\lambda}=-C_\lambda\pi_\alpha-C_{\alpha\lambda\mu}\pi_\mu\\ \dfrac{da}{a}=C_i\pi_i.\end{cases}$$

2. Now we introduce a Riemannian connection into our space by

$$d\mathbf{A}=\pi_i\mathbf{e}_i,\qquad d\mathbf{e}_i=\pi_{ij}\mathbf{e}_j$$

and develop the tangent euclidean spaces on a euclidean space. Then for a point $\bar{\mathbf{A}}=\mathbf{A}+t_\lambda \mathbf{e}_\lambda$ we have

$$d\bar{\mathbf{A}}=d\mathbf{A}+dt_\lambda \mathbf{e}_\lambda+t_\lambda d\mathbf{e}_\lambda=\pi_i \mathbf{e}_i+dt_\lambda \mathbf{e}_\lambda+t_\lambda \pi_{\lambda i}\mathbf{e}_i \tag{16}$$

$$=(\pi_\alpha+t_\lambda\pi_{\lambda\alpha})\mathbf{e}_\alpha+(\pi_\lambda+dt_\lambda+t_\mu\pi_{\mu\lambda})\mathbf{e}_\lambda.$$

When we develop along the k-dimensional surfaces $x^\lambda=$const, we have by (4) $\pi_\lambda=0$. Then $d\mathbf{A}=\pi_\alpha \mathbf{e}_\alpha$ and

$$\pi_\alpha+t_\lambda\pi_{\lambda\alpha}=\pi_\alpha+t_\lambda(C_\lambda\pi_\alpha+C_{\alpha\lambda\mu}\pi_\mu)=(1+C_\lambda t_\lambda)\pi_\alpha.$$

Now we assume $a(x^i)$ contains at least one x^λ. In this case, owing to (12), there exists among C_λ's at least one C_λ which is not zero, and consequently there exist t_λ's such that

$$1+C_\lambda t_\lambda=0. \tag{17}$$

When we develop along k-dimensional surfaces $x^\lambda=$const, we get for the point $\bar{\mathbf{A}}$ satisfying (17)

$$d\bar{\mathbf{A}}=(dt_\lambda+t_\mu\pi_{\mu\lambda})\mathbf{e}_\lambda. \tag{18}$$

The points $\bar{\mathbf{A}}=\mathbf{A}+t_\lambda \mathbf{e}_\lambda$ for t_λ's satisfying (17) generate $n-k-1$-dimensional plane P_0 in the tangent euclidean space at $\mathbf{A}$ of the Riemann space (1). (18) indicates that when we develop along the k-dimensional surfaces $x^\lambda=$const $\bar{\mathbf{A}}$ is a fixed point or describes a curve touching to the k-dimensional plane P spanned by the vectors $\mathbf{e}_{k+1}, \ldots, \mathbf{e}_n$ with $\mathbf{A}$ as their origin. Now we take $\mathbf{e}_n$ on the perpendicular from $\mathbf{A}$ to P_0. Then any point $\bar{\mathbf{A}}$ on P_0 can be represented by

$$\bar{\mathbf{A}}=\mathbf{A}+t_p\mathbf{e}_p+t\mathbf{e}_n$$

where p runs from $k+1$ to $n-1$ and t_p's are arbitrary numbers, while t is a fixed function of x^i's. For the frame thus chosen the relations (A) hold and along surfaces $x^\lambda=$const we have (18) and (17), and so

$$1+C_\lambda t_\lambda=1+C_p t_p+C_n t=0$$

holds for any t_p's ($p=k+1, \ldots, n-1$). Thus

$$C_p=0, \qquad C_n=-\frac{1}{t}.$$

Hence we get from (A)

$$\text{(B)}\qquad \begin{cases} \pi_{\alpha p}=-C_{\alpha p\mu}\pi_\mu, & \pi_{\alpha n}=\dfrac{\pi_\alpha}{t}-C_{\alpha n\mu}\pi_\mu \\ \dfrac{da}{a}=-\dfrac{\pi_n}{t}+C_\alpha\pi_\alpha. \end{cases}$$

If we put $x^\alpha=$const, we get $\pi_\alpha=0$ and so $d\mathbf{A}=\pi_\lambda \mathbf{e}_\lambda$ and $\dfrac{da}{a}=-\dfrac{\pi_n}{t}$. Thus we obtain

THEOREM 1. *Let the square of arc-element of a Riemann space be given by (1), where $a(x^i)$ actually contains at least one x^λ. Then there exist in the tangent euclidean space at any point* A *of the Riemann space an $n-k$-dimensional plane P through* A *and an $n-k-1$-dimensional plane P_0 lying on P and not passing through* A, *which have the following property;*

1°. If we develop along the k-dimensional surfaces $x^\lambda = const$, A *describes an arc perpendicular to P at every instant and any point on P_0 is either a fixed point or describes an arc touching to P.*

2°. If we develop along $n-k$-dimensional surfaces $x^\alpha = const$, A *describes an arc touching to P, and when we denote the length of the normal from* A *to P_0 by t and the orthogonal component of an arc-element of* A *in the direction of this normal by π_n, π_n/t is a total differential of a certain function of x^λ's.*

3. Before stating the inverse of theorem 1 we prove a lemma.

LEMMA. Let ω_α $(\alpha=1, \ldots, k)$ be linearly independent Pfaffians with n variables. If there exist Pfaffians $\omega_{\alpha\beta}$ such that

$$\omega'_\alpha = [\omega_\beta, \omega_{\beta\alpha}], \qquad \omega_{\alpha\beta} = -\omega_{\beta\alpha},$$

then by a suitable choice of variables x^i $(i=1, \ldots, n)$ we have $\omega_\alpha = p_{\alpha\beta}(x^i)dx^\beta$, and for these variables $\sum(\omega_\alpha)^2$ does not contain x^λ $(\lambda=k+1, \ldots, n)$, namely $\sum(\omega_\alpha)^2 = g_{\alpha\beta}(x^\gamma)dx^\alpha dx^\beta$.

The proof, which is quite natural, runs as follows. By Frobenius's theorem [2] p. 193 variables stated in the lemma exist. Writing the condition $\omega'_\alpha = [\omega_\beta, \omega_{\beta\alpha}]$ more fully we have

$$d\omega_\alpha(\delta) - \delta\omega_\alpha(d) = \omega_\beta(d)\omega_{\beta\alpha}(\delta) - \omega_\beta(\delta)\omega_{\beta\alpha}(d).$$

Let the symbols d and δ be such that

$$(dx^1, \ldots, dx^k, 0, \ldots, 0) \quad \text{and} \quad (0, \ldots, 0, \delta x^{k+1}, \ldots, \delta x^n).$$

Then

$$-\delta\omega_\alpha(d) = \omega_\beta(d)\omega_{\beta\alpha}(\delta).$$

And we have

$$\delta(\sum(\omega_\alpha)^2) = 2\sum\omega_\alpha\delta\omega_\alpha = -2\sum\omega_\alpha\omega_\beta\omega_{\beta\alpha} = 0.$$

This shows $\sum(\omega_\alpha)^2$ does not contain x^λ.

Now the inverse of theorem 1 is

THEOREM 2. *If an n-dimensional Riemann space has coordinate system x^i in which the conditions 1° and 2° in theorem 1 hold, then the square of the arc-element of the space is of the form (1), $a(x^i)$ containing at least one of x^λ's.*

Proof. In the tangent euclidean space at $\mathbf{A}$ of the Riemann space we take a component $\mathbf{e}_n$ of an orthonormal system of vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ on the perpendicular from $\mathbf{A}$ to P_0, and components $\mathbf{e}_\lambda$ $(\lambda=k+1, \ldots, n)$ on P. Let the Riemannian connection be given by $d\mathbf{A}=\pi_i\mathbf{e}_i$, $d\mathbf{e}_i=\pi_{ij}\mathbf{e}_j$. By the condition 1° we have $d\mathbf{A}=\pi_\alpha\mathbf{e}_\alpha$, which leads to $\pi_\lambda=0$. So in general

$$\pi_\lambda=p_{\lambda\mu}(x^i)dx^\mu. \tag{19}$$

For $\bar{\mathbf{A}}=\mathbf{A}+t_\lambda\mathbf{e}_\lambda$ we have by (16)

$$d\bar{\mathbf{A}}=(\pi_\alpha+t_\lambda\pi_{\lambda\alpha})\mathbf{e}_\alpha+(\pi_\lambda+dt_\lambda+t_\mu\pi_{\mu\lambda})\mathbf{e}_\lambda.$$

The latter half of 1° shows that $d\bar{\mathbf{A}}$ does not contain $\mathbf{e}_\alpha$ for $x^\lambda=\text{const}$ and so

$$\pi_\alpha+t_\lambda\pi_{\lambda\alpha}=0.$$

As $\bar{\mathbf{A}}$ moves on P_0, $t_n=t$ is constant, while $t_{k+1}, \ldots, t_{n-1}$ vary freely. So $\pi_\alpha+t\pi_{n\alpha}=0$, $\pi_{\alpha p}=0$ $(p=k+1, \ldots, n-1)$. In general owing to (19) we obtain for suitable $C_{\alpha\lambda\mu}$

$$\pi_{\alpha\lambda}=\delta_{n\lambda}\frac{\pi_\alpha}{t}-C_{\alpha\lambda\mu}\pi_\mu. \tag{20}$$

From the condition 2° $d\mathbf{A}=\pi_\lambda\mathbf{e}_\lambda$ along the surfaces $x^\alpha=\text{const}$, and so $\pi_\alpha=0$. In general

$$\pi_\alpha=p_{\alpha\beta}(x^i)dx^\beta. \tag{21}$$

By the latter half of 2° for $x^\alpha=\text{const}$ there exists a function $a=a(x^i)$ such that $\pi_n/t=-da/a$. So in general

$$\frac{\pi_n}{t}=-\frac{da}{a}+C_\beta\pi_\beta \tag{22}$$

where $a=a(x^i)$ and C_β's are suitable functions.

Let $\omega_\alpha=\pi_\alpha/a$. Then

$$\pi'_\alpha=[da,\ \omega_\alpha]+a\omega'_\alpha=\left[da,\ \frac{\pi_\alpha}{a}\right]+a\omega'_\alpha$$
$$=\left[\frac{da}{a},\ \pi_\alpha\right]+a\omega'_\alpha=\left[-\frac{\pi_n}{t}+C_\beta\pi_\beta,\ \pi_\alpha\right]+a\omega'_\alpha.$$

By (20)

$$[\pi_i,\ \pi_{i\alpha}]=[\pi_\beta,\ \pi_{\beta\alpha}]+[\pi_\lambda,\ \pi_{\lambda\alpha}]$$
$$=[\pi_\beta,\ \pi_{\beta\alpha}]+\left[\pi_\lambda,\ C_{\alpha\lambda\mu}\pi_\mu-\delta_{n\lambda}\frac{\pi_\alpha}{t}\right]$$
$$=[\pi_\beta,\ \pi_{\beta\alpha}]+C_{\alpha\lambda\mu}[\pi_\lambda,\ \pi_\mu]-\frac{1}{t}[\pi_n,\ \pi_\alpha].$$

As $\pi'_\alpha=[\pi_i\pi_{i\alpha}]$, we have

$$a\omega'_\alpha=[\pi_\beta,\ \pi_{\beta\alpha}]-C_\beta[\pi_\beta,\ \pi_\alpha]+C_{\alpha\lambda\mu}[\pi_\lambda,\ \pi_\mu]. \tag{23}$$

Here by (21) and $\pi_\alpha = a\omega_\alpha$ there are not terms $[dx^\lambda, dx^\mu]$ on the left side $a\omega'_\alpha = q_{ij}[dx^i, dx^j]$, while on the right side $C_{\alpha\lambda\mu}[\pi_\lambda, \pi_\mu] = r_{\lambda\mu}[dx^\lambda, dx^\mu]$. So the last terms must be zero, namely $C_{\alpha\lambda\mu} = C_{\alpha\mu\lambda}$. We have by (23)

$$\omega'_\alpha = [\omega_\beta, \pi_{\beta\alpha}] - C_\beta[\omega_\beta, \pi_\alpha].$$

Then putting

$$\omega_{\beta\alpha} = \pi_{\beta\alpha} - C_\beta \pi_\alpha + C_\alpha \pi_\beta \tag{24}$$

we have

$$\omega'_\alpha = [\omega_\beta, \omega_{\beta\alpha}].$$

As $\omega_{\alpha\beta}$ are shew symmetric with respect to indices α and β, we have by lemma $\sum(\omega_\alpha)^2 = g_{\alpha\beta}(x^\gamma)dx^\alpha dx^\beta$. Thus

$$\begin{aligned} ds^2 = \sum(\pi_i)^2 = \sum(\pi_\alpha)^2 + \sum(\pi_\lambda)^2 = \sum(a\omega_\alpha)^2 + \sum(\pi_\lambda)^2 \\ = a(x^i)^2 g_{\alpha\beta}(x^\gamma)dx^\alpha dx^\beta + g_{\lambda\mu}(x^i)dx^\lambda dx^\mu. \end{aligned}$$

By (22) $a(x^i)$ contains at least one of x^λ's.

4. Now we return to **2** and treat the case in which $a(x^i)$ in (1) does not contain any x^λ. This case reduces to the case $a(x^i)=1$. When $a(x^i)=1$, we have by (12) $C_i=0$ and by (15) $\pi_{\alpha\lambda} = -C_{\alpha\lambda\mu}\pi_\mu$. Along surfaces x^λ=const we have $\pi_{\alpha\lambda}=0$ and consequently $d\mathbf{e}_\alpha = \pi_{\alpha\beta}\mathbf{e}_\beta$. Hence

THEOREM 3. *If the square of arc element of an n-dimensional Riemann space is for a suitably chosen coordinate system* x^i

$$ds^2 = g_{\alpha\beta}(x^\gamma)dx^\alpha dx^\beta + g_{\lambda\mu}(x^i)dx^\lambda dx^\mu,$$

then there exists in a tangent euclidean space at any point **A** *of the Riemann space an* $n-k$*-dimensional plane P satisfying the following conditions:*

1°. If we develop along k-dimensional surfaces x^λ=const, **A** *describes an arc which is perpendicular to P at any instant, and P moves parallel to a fiixed* $n-k$*-dimensional plane.*

2°. If we develop along $n-k$*-dimensional surfaces x^α=const,* **A** *describes an arc which is parallel to P.*

The inverse of this theorem is also true.

THEOREM 4. *If two conditions 1° and 2° in theorem 3 hold for suitably chosen coordinates x^i of an n-dimensional Riemann space, then the square of arc element of our space is of the form*

$$ds^2 = g_{\alpha\beta}(x^\gamma)dx^\alpha dx^\beta + g_{\lambda\mu}(x^i)dx^\lambda dx^\mu.$$

Proof. In the tangent euclidean space at **A** of our Riemann space we take $\mathbf{e}_\lambda$ $(\lambda=k+1, \ldots, n)$ on the plane P. Then by the condition 1° we have $\pi_\lambda=0$

along surfaces $x^\lambda=$const, and in general

(25) $$\pi_\lambda=p_{\lambda\mu}(x^i)dx^\mu.$$

The latter half of 1° indicates $de_\lambda=\pi_{\lambda\mu}e_\mu$ along $x^\lambda=$const and consequently $\pi_{\alpha\lambda}=0$. So in general

(26) $$\pi_{\alpha\lambda}=-C_{\alpha\lambda\mu}\pi_\mu.$$

By the condition 2°

(27) $$\pi_\alpha=p_{\alpha\beta}(x^i)dx^\beta.$$

By (26) and (6)

$$\pi'_\alpha=[\pi_\beta,\ \pi_{\beta\alpha}]+[\pi_\lambda,\ \pi_{\lambda\alpha}]=[\pi_\beta,\ \pi_{\beta\alpha}]+C_{\alpha\lambda\mu}[\pi_\lambda,\ \pi_\mu].$$

Taking (25) and (27) into consideration we get $C_{\alpha\lambda\mu}[\pi_\lambda,\ \pi_\mu]=0$ and we have $\pi'_\alpha=[\pi_\beta,\ \pi_{\beta\alpha}]$ and by lemma $\sum(\pi_\alpha)^2=g_{\alpha\beta}(x^\gamma)dx^\alpha dx^\beta$.

Remark. In place of 1° and 2° in theorem 4 we take the following two conditions:

1°. If we develop along k-dimensional surfaces $x^\lambda=$const, P moves parallel to a fixed $n-k$-dimensional plane.

2°. If we develop along $n-k$-dimensional surfaces $x^\alpha=$const, P moves parallel to a fixed $n-k$-dimensional plane.

Then by 1° $\pi_{\alpha\lambda}=0$ for $x^\lambda=$const and by 2° $\pi_{\alpha\lambda}=0$ for $x^\alpha=$ const. So in general $\pi_{\alpha\lambda}=0$, and by (6)

$$\pi'_\alpha=[\pi_i,\ \pi_{i\alpha}]=[\pi_\beta,\ \pi_{\beta\alpha}]+[\pi_\lambda,\ \pi_{\lambda\alpha}]=[\pi_\beta,\ \pi_{\beta\alpha}]$$
$$\pi'_\lambda=[\pi_i,\ \pi_{i\lambda}]=[\pi_\alpha,\ \pi_{\alpha\lambda}]+[\pi_\mu,\ \pi_{\mu\lambda}]=[\pi_\mu,\ \pi_{\mu\lambda}].$$

Then by lemma $\sum(\pi_\alpha)^2=g_{\alpha\beta}(y^\gamma)dy^\alpha dy^\beta$ for suitably chosen coordinates $y^i=y^i(x^j)$ and $\sum(\pi_\lambda)^2=g_{\lambda\mu}(z^\nu)dz^\lambda dz^\mu$ for suitably chosen coordinates $z^i=z^i(x^j)$. If we take $u^\alpha=y^\alpha(x^j)$, $u^\lambda=z^\lambda(x^j)$ as coordinates we get

$$ds^2=\sum(\pi_i)^2=g_{\alpha\beta}(u^\gamma)du^\alpha du^\beta+g_{\lambda\mu}(u^\nu)du^\lambda du^\mu.$$

This space has been treated by many authors.

5. Now we treat the special case $k=n-1$. In this case the condition 2° in theorem 2 is unnecessary. The second part of 2°, namely the condition that π_n/t is a total differential for $x^\alpha=$const, is always satisfied. We will prove the first part of 2° is also satisfied under the condition 1°. By 1 the equations (20) hold, which in this case reduce to $\pi_{\alpha n}=\pi_\alpha/t-C_{\alpha nn}\pi_n$. Consequently

$$\pi'_\alpha=[\pi_\beta,\ \pi_{\beta\alpha}]+[\pi_n,\ \pi_{n\alpha}]=[\pi_\beta,\ \pi_{\beta\alpha}]+\left[\pi_n,\ -\frac{\pi_\alpha}{t}+C_{\alpha nn}\pi_n\right]$$
$$=\left[\pi_\beta,\ \pi_{\beta\alpha}+\delta_{\beta\alpha}\frac{\pi_n}{t}\right]$$

$$\pi'_n=[\pi_\alpha,\ \pi_{\alpha n}]=\left[\pi_\alpha,\ \frac{\pi_\alpha}{t}-C_{\alpha nn}\pi_n\right]=C_{\alpha nn}[\pi_n,\ \pi_\alpha].$$

By Frobenius's theorem [2] p. 193 for suitably chosen coordinates $y^\alpha=y^\alpha(x^j)$ and $y^n=y^n(x^j)$ we have

$$\pi_\alpha=p_{\alpha\beta}(y^i)dy^\beta, \qquad \pi_n=p(y^i)dy^n.$$

As $\pi_n=0$ for $x^n=$const by (19), y^n is a function of only one variable x^n. For coordinate system y^i thus chosen the first part of 2° is satisfied, and we have

THEOREM 5. *The necessary and sufficient condition for an n-dimensional Riemann space to have an arc element which can be writen in the form*

$$ds^2=a(x^i)^2g_{\alpha\beta}(x^\gamma)dx^\alpha dx^\beta+g_{nn}(x^i)(dx^n)^2$$
$$(\alpha,\ \beta,\ \gamma=1,\ 2,\ \ldots,\ n-1)$$

is that there exists a point P_0 in the tangent euclidean space at any point **A** *of the Riemann space which has the property that, if we develop along any hypersurface of one-parametric family of hypersurfaces,* **A** *describes an arc perpendicular to* $\mathbf{A}P_0$ *and* P_0 *describes an arc which touches the straight line* $\mathbf{A}P_0$ *or is a fixed point.*

Especially we treat the case $g_{nn}(x^i)=g_{nn}(x^n)$, which reduces to $g_{nn}(x^i)=1$. Then in 2 we have $\pi_n=dx^n$ and consequently $\pi'_n=[\pi_\alpha,\ \pi_{\alpha n}]=0$.
By (B) we have $\pi_{\alpha n}=\pi_\alpha/t-C_{\alpha nn}\pi_n$. So $-C_{\alpha nn}[\pi_\alpha,\ \pi_n]=0$ and hence $C_{\alpha nn}=0$. Thus

$$\pi_{\alpha n}=\frac{\pi_\alpha}{t}. \tag{28}$$

By (16) we have

$$d(\mathbf{A}+t\mathbf{e}_n)=(\pi_\alpha-t\pi_{\alpha n})\mathbf{e}_\alpha+(\pi_n+dt)\mathbf{e}_n. \tag{29}$$

So on account of (28)

$$d(\mathbf{A}+t\mathbf{e}_n)=(\pi_n+dt)\mathbf{e}_n. \tag{30}$$

Thus by development the point $P_0=\mathbf{A}+t\mathbf{e}_n$ is a fixed point or describes an arc touching to $\mathbf{A}P_0$.

Conversely we assume there is a point P_0 in the tangent euclidean space of any point **A** of Riemann space which has the property that by development P_0 is a fixed point or describes an arc touching to $\mathbf{A}P_0$. Then (30) holds and (28) holds by virtue of (29). Consequently $\pi'_n=[\pi_{n\alpha},\ \pi_\alpha]=0$ and π_n is a total differential of a certain function. Putting $\pi_n=dx^n$ we proceed as in the proof of theorem 2. Hence

THEOREM 6. *The necessary and sufficient condition for a Riemann space to have an arc-element such that*

$$ds^2 = a(x^i)^2 g_{\alpha\beta}(x^\gamma)dx^\alpha dx^\beta + (dx^n)^2 \qquad (\alpha, \beta, \gamma = 1, 2, \ldots, n-1)$$

is that there is a point P_0 in the tangent euclidean space at any point **A** *of the Riemann space such that by development P_0 is a fixed point or describes an arc touching to* $\mathbf{A}P_0$.

This case has been proved by K. Yano (the so-colled Riemann space with torse-forming vector field). If P_0 is a fixed point we have by (30) $\pi_n + dt = 0$ and then by 2 (B) $da/a = dt/t + C_\alpha \pi_\alpha$, namely $a = ct$, c being a function of x^α. So this case reduces to the case $a(x^i) = x^n$. The inverse is also true.

If $k = n-1$ in theorem 4, the condition 2° is also unnecessary. Hence

THEOREM 7. *The necessary and sufficient condition for the Riemann space to have an arc-element given by*

$$ds^2 = g_{\alpha\beta}(x^\gamma)dx^\alpha dx^\beta + g_{nn}(x^i)(dx^n)^2 \qquad (\alpha, \beta, \gamma = 1, 2, \ldots, n-1)$$

is that there exists a direction P in the tangent euclidean space at any point of the Riemann space such that if we develop along any hypersurface of one-parametric family of hypersurfaces P moves parallel to a fixed direction.

The special case in which $g_{nn}(x^i) = g_{nn}(x^n)$ holds reduces to the case $g_{nn}(x^i) = 1$, which is treated in the remark of theorem 4.

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Mathematical Institute,
Nagoya University