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SOME EXAMPLES OF ONE DIMENSIONAL GORENSTEIN DOMAINS

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Introduction.

In this paper, I will prove the following theorems;

THEOREM 1. *For given integers n and m such that $m \geqq 2^n$, there exist 1-dimensional local domains which are complete intersections and have embedding dimension $n + 1$ and multiplicity m.*

THEOREM 2. *For given integers n and m such that $4 \leqq n \leqq m - 1$, there exist 1-dimensional local domains which are Gorenstein with multiplicity m and embedding dimension n and which are not complete intersections.*

To give these examples I heavily use the theory of the value-semigroups of 1-dimensional local domains by Kunz and Herzog ([1], [3]).

§1. Review of the theory of value-semigroups of 1-dimensional local domains ([1], [3]).

In the following, a 'semigroup' always means an additive subsemigroup of N, the additive semigroup of non-negative integers.

(1) A *'numerical semigroup'* is a semigroup H which satisfies two conditions;

1. $0 \in H$
2. There exists an integer c such that any integer $n \geqq c$ is in H.

(2) The *conductor* of a numerical semigroup H, denoted by $c(H)$, is the smallest integer c such that all integers n satisfying $n \geqq c$ belong to H.

(3) We denote by $\langle n_1, \cdots, n_k \rangle$ the semigroup generated by $n_1, \cdots, n_k$; $\langle n_1, \cdots, n_k \rangle = \{\sum_{i=1}^k a_i n_i \mid a_i \in N\}$.

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(4) We say that $\{n_1, \cdots, n_k\}$ is the *minimal generator system* of a semigroup H if $H = \langle n_1, \cdots, n_k \rangle$ and any proper subset of $\{n_1, \cdots, n_k\}$ does not generate H. If we suppose that $n_1 < n_2 < \cdots < n_k$, this is equivalent to say that $n_i \notin \langle n_1, \cdots, n_{i-1} \rangle$ for $2 \leqq i \leqq k$.

When we write $H = \langle n_1, \cdots, n_k \rangle$, we agree that $\{n_1, \cdots, n_k\}$ is the minimal generator system. Minimal generator system of an arbitrary subsemigroup of N uniquely exists.

(5) A numerical semigroup H is *symmetric* if for any integer n, $n \in H \Leftrightarrow c - 1 - n \notin H$ $(c = c(H))$.

(6) $K[H] = K[T^h; h \in H] \subset K[T]$ (K is a field and T is an indeterminant). $K[H]_{\text{loc}}$ the localization of $K[H]$ 'at the origin'. If H is a numerical semigroup, the integral closure of $K[H]$ in the quotient field of $K[H]$ is $K[T]$.

If $H = \langle n_1, \cdots, n_k \rangle$, then $K[H] = K[T^{n_1}, \cdots, T^{n_k}]$.

(7) We say that a numerical semigroup H is a *complete intersection* if the ring $K[H]$ is a complete intersection. When $H = \langle n_1, \cdots, n_k \rangle$, and if we consider the homomorphism $\Phi_H\colon K[X_1, \cdots, X_k] \to K[H]$, $\Phi_H(X_i) = T^{n_i}$, H is a complete intersection if and only if $\operatorname{Ker}(\Phi_H)$ is generated by $k - 1$ elements.

(8) The *multiplicity* of H, denoted by $m(H)$ is the least positive integer in H. If $H = \langle n_1, \cdots, n_k \rangle$ with $n_1 < n_2 < \cdots < n_k$, then $m(H) = n_1$.

(9) The *embedding dimension* of H, denoted by $\operatorname{emb}(H)$, is the number of the minimal generators of H. If $H = \langle n_1, \cdots, n_k \rangle$, then $\operatorname{emb}(H) = k$ (recall that $\{n_1, \cdots, n_k\}$ is the minimal generator system).

(10) Let $H = \langle n_1, \cdots, n_k \rangle$ and $h \in H$. If h has different expressions as linear combinations of n_i's, then we say that h is a *relation* of H. For example, if $H = \langle 3, 4, 5 \rangle$, $8 = 2 \cdot 4 = 3 + 5$ and $9 = 3 \cdot 3 = 4 + 5$ are relations of H.

(11) For a relation h in $H = \langle n_1, \cdots, n_k \rangle$, we associate to h a vector $v_h \in \boldsymbol{Z}^k$ in the following way. If $h = \sum_{i=1}^k a_i n_i = \sum_{i=1}^k b_i n_i$, then $v_h = (a_1 - b_1, a_2 - b_2, \cdots, a_k - b_k)$. In the example in (10), $v_8 = (-1, 2, -1)$, and $v_9 = (3, -1, -1)$. Of course v_h is not determined uniquely by v_h. But as it is not important in our following arguments, we agree to fix one such v_h.

(12) For $H = \langle n_1, \cdots, n_k \rangle$, we define;

$$M(H) = \min\left\{ h_1 + h_2 + \cdots + h_{k-1} \,\middle|\, \begin{array}{l} h_1, \cdots, h_{k-1} \text{ are relations in } H \text{ and} \\ v_{h_1}, \cdots, v_{h_{k-1}} \text{ are linearly} \\ \quad \text{independent in } \boldsymbol{Z}^k. \end{array} \right\}$$

For example, if $H = \langle 3, 4, 5 \rangle$, $M(H) = 8 + 9 = 17$.

Let R be an analytically irreducible 1-dimensional Noetherian local domain. Then the integral closure V of R in the quotient field of R is a discrete valuation ring. We assume that R and V has the same residue class field. (Which is true if $R = K[H]_{\text{loc}}$.) If we denote by v the valuation attached to V, then $H_R = v(R)$ is a numerical semigroup and we have the following propositions.

PROPOSITION 1. (1) *Multiplicity of* $R = m(H_R)$.

(2) *Embedding dimension of* $K[H]_{\text{loc}} = \text{emb}(H)$,

(3) *R is Gorenstein if and only if H_R is symmetric.*

(4) *If H_R is a complete intersection, then R is a complete intersection.*

(4′) *If $R = K[H]_{\text{loc}}$, then the converse of* (4) *holds.*

PROPOSITION 2. ([1], *Satz* 5.10) *If $H = \langle n_1, \cdots, n_k \rangle$, then we have that $M(H) - \sum_{i=1}^{k} n_i + 1 \geqq c(H)$, and the equality holds if and only if H is a complete intersection.*

§ 2. Examples of 1-dimensional local domains which are complete intersections and have given embedding dimension and multiplicity.

LEMMA 1. *Let $H_1 = \langle n_1, \cdots, n_k \rangle$, a and b be positive integers such that;*

(i) *$a \in H_1$ and $a \neq n_i$ $(i = 1, \cdots, k)$.*

(ii) *a and b are relatively prime.*

Then, if we put $H = \langle a, bn_1, \cdots, bn_k \rangle$ (which we will denote by $H = \langle a, bH_1 \rangle$), we have;

(1) *H is a complete intersection if and only if H_1 is a complete intersection.*

(2) *H is symmetric if and only if H_1 is symmetric.*

Proof. We consider the canonical homomorphisms $\Phi_1 \colon K[Y_1, \cdots, Y_k] \to K[H_1]$ and $\Phi \colon K[Y_1, \cdots, Y_k, X] \to K[H]$ defined by $\Phi_1(Y_i) = T^{n_i}, \Phi(X) = T^a, \Phi(Y_i) = T^{bn_i}$ $(i = 1, 2, \cdots, k)$. We put $\operatorname{Ker}(\Phi_1) = A_1$ and $A = (A_1, X^b - Y_1^{e_1} Y_2^{e_2} \cdots Y_k^{e_k})$ where $e_1, \cdots, e_k$ are defined by $a = \sum_{i=1}^{k} e_i n_i$ (we fix one such expression). We claim $\mathrm{Ker}(\Phi) = A$. $\mathrm{Ker}(\Phi) \supset A$ is obvious. Conversely, if $f(X, Y_1, \cdots, Y_k) \in \mathrm{Ker}(\Phi)$, we can find $f_0(Y), \cdots, f_{b-1}(Y) \in K[Y_1, \cdots, Y_k]$ such that $f \equiv f_0 + Xf_1 + \cdots + X^{b-1}f_{b-1} \pmod{A}$. Hence $\Phi(f_i) \in K[T^b]$. As $\Phi(X^i) = T^{ai}$ and $(a, b) = 1$, it follows from $\Phi(f) = 0$, that $\Phi(f_i) = 0$, i.e. $f_i \in \mathrm{Ker}(\Phi_1) = A_1$ $(i = 1, \cdots, b-1)$.

Now, we have $K[H] = K[H_1][X]/(g)$, where $g(X, Y) = X^b - Y_1^{e_1} \cdots Y_k^{e_k}$. Thus we have; $K[H]$ is a complete intersection (resp. Gorenstein) $\Leftrightarrow K[H_1][X]$ is a complete intersection (resp. Gorenstein) $\Leftrightarrow K[H_1]$ is a complete intersection (resp. Gorenstein). By Proposition 1 we are done.

By Lemma 1, we have semigroups which are complete intersections and have arbitrarily high embedding dimensions. When embedding dimension is $\leqq 3$, the converse holds.

PROPOSITION 3. *If H is a semigroup which is a complete intersection and if $emb(H) = 3$, then $H = \langle a, bH_1 \rangle$ where H_1 is a semigroup of $emb(H_1) = 2$ (which is necessarily a complete intersection) and a and b are integers satisfying the conditions of Lemma* 1.

(This proposition is proved by Herzog [2]. But as his proof is considerably long, I give a shorter proof.)

To prove the proposition, we need a lemma.

LEMMA 2. *Let $H = \langle n_1, \cdots, n_k \rangle$ be a semigroup which is a complete intersection, $\Phi_H\colon K[X_1, \cdots, X_k] \to K[H]$ be the canonical homomorphism and $(f_1, \cdots, f_{k-1})$ the generators of* $\mathrm{Ker}(\Phi_H)$. *If we denote by J_p the ideal generated by p variables $X_{i_1}, \cdots, X_{i_p}$, then there exists at most $p-1$ f_i's which belong to J_p.*

COROLLARY. *For every variable X_j $(1 \leqq j \leqq k)$, one of the f_i's includes a monomial of the type X_j^s.*

Proof of Lemma 2. We consider the ideal $A = (J_p, f_1, \cdots, f_{k-1})$. If $f_1, \cdots, f_p \in J_p$ (for simplicity, we renumber f_i's), then $A = (J_p, f_{p+1}, \cdots, f_{k-1})$ and A is generated by $k-1$ elements and $ht(A)$ must be $\leqq k-1$. But on the other hand, $\dim(K[H]) = 1$ and $\Phi_H(J_p) \neq 0$. So, we must have $ht(A) = k$. Contradiction!

The corollary is a special case of the lemma when $p = k - 1$.

Proof of Proposition 3. Let $H = \langle n_1, n_2, n_3 \rangle$, $\Phi_H\colon K[X_1, X_2, X_3] \to K[H]$, $\mathrm{Ker}(\Phi_H) = (f_1, f_2)$. By the definition of Φ_H, each f_i is of the form (monomial)-(monomial). Then, by the corollary of Lemma 2, after renumbering X_i's and f_i's, we may assume, $f_1 = X_2^n - X_3^m$, $f_2 = X_1^b - X_2^e X_3^f$. As (f_1, f_2) is a prime ideal of height 2, f_1 and f_2 must be irreducible and we have $(m, n) = 1, n \cdot n_2 = m \cdot n_3, bn_1 = en_2 + fn_3$. We put $H_1 = \langle m, n \rangle$, $n_1 = a$, $n_2 = dm$, $n_3 = dn$. Then $ab = d(em + fn)$. From $(n_1, n_2, n_3) = 1$, we have $(a, d) = 1$ and $b = db'$. We claim that $d = b$ and $a = em + fn$. Let us assume $d \neq b, b' \neq 1$. If $a \notin H_1$, take an integer s such that $sa \in H_1$ and s is not a multiple of b'. Then, writing $sa = e'm + f'n$, $g = X_1^{sd} - X_2^{e'} X_3^{f'} \in \operatorname{Ker}(\Phi_H)$. But it is easy to assure that $g \notin (f_1, f_2)$. This contradicts the fact that $(f_1, f_2) = \operatorname{Ker}(\Phi_H)$. If $a \in H_1$, $a = e'm + f'n$, then $X_1^d - X_2^{e'} X_3^{f'} \in \operatorname{Ker}(\Phi_H)$. From $\operatorname{Ker}(\Phi_H) = (f_1, f_2)$, we get $d = b$.

Remark 1. Proposition 3 is not true if $emb(H) \geqq 4$. For example, If we put $H = \langle 14, 21, 15, 20 \rangle$, H is a complete intersection with $c(H) = 68$, $\operatorname{Ker}(\Phi_H) = (X_1^3 - X_2^2, X_1 X_2 - X_3 X_4, X_3^4 - X_4^3)$ and clearly H can not be written in the form $H = \langle a, bH_1 \rangle$.

Remark 2. By Proposition 3, we can determine the types of H's which are complete intersections and $emb(H) \leqq 3$. For example, if $emb(H) = 3$ and $m(H) = 5$ and if H is a complete intersection, (this is equivalent to say that H is symmetric, in this case) then $H = \langle 5, 2p, 3p \rangle$, $p \geqq 3$, $(p, 5) = 1$.

LEMMA 3. *Let a be an odd integer. Then the semigroup $H = \langle 2^n, 2^n + a, 2^n + 2a, \cdots, 2^n + 2^i a, \cdots, 2^n + 2^{n-1} a \rangle$ is a complete intersection for $n \geqq 1$.*

Proof. Easy by induction and applying Lemma 1.

THEOREM 1. *Let m and n be given positive integers such that $m \geqq 2^n$. Then there exists a 1-dimensional local domain R which is a complete intersection with embedding dimension $n + 1$ and multiplicity m.*

Proof. We find a semigroup H which is a complete intersection and $m(H) = m$, $emb(H) = n + 1$.

(i) If m is odd, we put $m = 2^n + a$. Then, by Lemma 3, $H_1 = \langle 2^{n-1}, 2^{n-1} + a, \cdots, 2^{n-1} + 2^{n-2} a \rangle$ is a complete intersection and $m \in H_1$. If we take an integer b, such that $(b, m) = 1$ and $2^{n-1} b \geqq m$, then $H = \langle m, bH_1 \rangle$ is the desired semigroup.

(ii) If m is even, using induction on n, we may assume that there exists a semigroup H_1 which is a complete intersection and $m(H_1) = m/2$, $emb(H_1) = n$. Then, if we take an odd integer $a \in H_1$ such that $a > m$ and a is not a generator of $H_1, H = \langle a, 2H_1 \rangle$ is the desired semigroup by Lemma 1.

Remark. If (R, M) is a regular local ring and if $(x_1, \cdots, x_n)$ is a regular sequence of R contained in M^2, then the multiplicity of $R/(x_1, \cdots, x_n)$ is at least 2^n. So the condition $m \geqq 2^n$ is necessary.

§ 3. Examples of 1-dimensional Gorenstein local domains which are not complete intersections.

LEMMA 4. *Let m and n be positive integers such that $m - 1 \geqq n \geqq 4$. If there exist integers a, b, e such that*

(i) *$a, b \geqq 0$ and $e > 0$,*

(ii) *if $b > 0$, then e is even,*

(iii) *$ea + (e/2)b + 2 = m$ (if e is odd, then $b = 0$),*

(iv) *$n = a + b + 1$.*

Then there exists a symmetric semigroup H with $m(H) = m$ and $emb(H) = n$ and H is not a complete intersection. Actually,

$$H = \langle m, m + 1, \cdots, m + a, 2m - b, 2m - b + 1, \cdots, 2m - 1 \rangle .$$

Proof. We have $c(H) = e(m + a) + 2$. It is easy to see that H is symmetric. To prove that H is not a complete intersection, we restrict ourselves to the case $a > 0$. (The case $a = 0$ can be proved similarly. But as the case $a = 0$ is not used later, we omit the proof.) We give two different proofs, the first one using Proposition 2 and the second one using Lemma 2,

First proof. We compute $M(H)$. In the notation of § 1, (12), we have;

$$h_1 = 2m + 2 = m + (m + 2) = 2(m + 1)$$
$$h_2 = 2m + 3 = m + (m + 3) = (m + 1) + (m + 2)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$h_{a-1} = 2m + a = m + (m + a) = (m + 1) + (m + a - 1)$$
$$h_a = 3m - b + 1 = m + (2m - b + 1) = (m + 1) + (2m - b)$$
$$h_{a+1} = 3m - b + 2 = m + (2m - b + 2) = (m + 1) + (2m - b + 1)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$h_{a+b-1} = 3m = (m + 1) + (2m - 1)$
$h_{a+b} = c(H) + m - b = (e/2 + 1)(2m - b)$.

$M(H) - \sum_{i=1}^{n} n_i + 1 - c(H) = mb + m(a - 2) = (n - 3)m > 0$. By Proposition 2, H is not a complete intersection.

Second proof. We consider the canonical homomorphism $\Phi_H: K[X_0, X_1, \cdots, X_a, X_{a+1}, \cdots, X_{a+b}] \to K[H]$, defined by $\Phi_H(X_i) = T^{m+i} (0 \leqq i \leqq a)$, $\Phi_H(X_j) = T^{2m-a-b-1+j} (a + 1 \leqq j \leqq a + b)$. We assume that Ker$(\Phi_H)$ is generated by $a + b$ elements and lead to a contradiction. By the definition of Φ_H, it is clear that $f_1 = X_0X_2 - X_1^2$, $f_2 = X_0X_3 - X_1X_2, \cdots, f_{a-1} = X_0X_a - X_1X_{a-1}, f_a = X_0X_{a+2} - X_1X_{a+1}, \cdots, f_{a+b-1} = X_1X_{a+b} - X_0^3$ are $a + b - 1$ members of minimal generators of Ker(Φ_H). As Ker(Φ_H) is generated by $f_1, \cdots, f_{a+b-1}$ and one more polynomial g, and as g can include at most 2 monomials of the form X_i^s, we have $a + b + 1 \leqq 4$. It remains to show that $a + b + 1 = 4$ does not occur. If $a = 1, b = 2$, then $f_1 = X_0X_3 - X_1X_2$ and $f_2 = X_1X_3 - X_0^3$. So it is impossible to find f_3 satisfying the condition of the corollary of Lemma 2. If $a = 2$, $b = 1$, then $f_1 = X_0X_2 - X_1^2$ and $f_2 = X_1X_3 - X_0^3$. But in this case, $f_1, f_2 \in (X_0, X_1)$ and this contradicts Lemma 2 ($p = 2$). If $a = 3$, $b = 0$, $f_1 = X_0X_2 - X_1^2$ and $f_2 = X_0X_3 - X_1X_2$ and it is impossible to find f_3 satisfying the condition of the corollary of Lemma 2. This concludes the proof of Lemma 4.

LEMMA 5. *If $m - 1 \geqq n \geqq m/2$, there exist a, b and e satisfying the conditions of Lemma 4. Furthermore, we can take $a > 0$.*

Proof. Put $e = 2, b = 2n - m, a = m - n - 1$ if $n \neq m - 1$. When $n = m - 1$, we put $e = 1$, $b = 0$, $a = n - 1 = m - 2$.

LEMMA 6. *If $m \geqq 5$, there exists a symmetric semigroup H, which is not a complete intersection and with $m(H) = m$, $emb(H) = 4$.*

Proof. Case I $m \equiv 1 \pmod 4$. Writing $m = 4m' + 1$, we put

$$H = \langle m, m + 1, m + 2, m'(m + 2) + 1 \rangle .$$

Then H is symmetric with;

$$c(H) = 2m'm , \qquad M(H) = h_1 + h_2 + h_3 ,$$

where

$h_1 = 2m + 2 = 2(m + 1) = m + (m + 2)$

$$h_2 = (m'(m+2)+1) + m = m'(m+2) + (m+1)$$
$$h_3 = c(H) + m = (2m'+1)m = (m'(m+2)+1) + m'(m+2)\ .$$

$M(H) - m - (m+1) - (m+2) - (m'(m+2)+1) + 1 - c(H) = m > 0$, H is not a complete intersection.

Case II $m \equiv 2 \pmod 4$. In the Lemma 4, put $a = 1$, $b = 2$, $e = (m-2)/2$.

Case III $m \equiv 3 \pmod 4$. We put

$$H = \langle m, m+1, 2m+3, 2m+4 \rangle\ .$$

Then H is symmetric with;

$$c(H) = \frac{m(m+1)}{2}\ , \qquad M(H) = h_1 + h_2 + h_3\ ,$$

where

$$h_1 = 3m + 3 = 3(m+1) = m + (2m+3)$$
$$h_2 = 3m + 4 = (m+1) + (2m+3) = m + (2m+4)$$
$$h_3 = c(H) + m = (2m+3) + \frac{m-3}{4}(2m+4) = \frac{m+3}{2}m\ .$$

$M(H) - m - (m+1) - (2m+3) - (2m+4) + 1 - c(H) = m > 0$. Hence H is not a complete intersection.

Case IV $m \equiv 0 \pmod 4$. We put

$$H = \left\langle m, m+1, n_3 = \frac{(m-4)(m+1)}{2} + 1, n_4 = \frac{(m-2)(m+1)}{2} + 2 \right\rangle\ .$$

Then H is symmetric with;

$$c(H) = m(m-3)\ , \qquad M(H) = h_1 + h_2 + h_3\ ,$$

where

$$h_1 = m + n_3 = (m+1)\frac{m-2}{2}$$
$$h_2 = m + n_4 = 2(m+1) + n_3$$
$$h_3 = c(H) + m = m(m-2) = n_3 + n_4\ .$$

$M(H) - m - (m+1) - n_3 - n_4 + 1 - c(H) = m > 0$. Hence H is not a complete intersection.

THEOREM 2. *For given positive integers m and n, such that $m-1 \geqq n \geqq 4$, there exists 1-dimensional local domain R which is Gorenstein with $emb(R)=n, m(R)=m$ and which is not a complete intersection.*

Proof. It suffices to find a symmetric semigroup H with $emb(H)=n$ and $m(H)=m$ and which is not a complete intersection.

(i) It is done for $n=4$ by Lemma 6.

(ii) If $n \geqq m/2$, this is true by Lemma 5.

(iii) If $m/2 \geqq n \geqq 4$, let $H_1=\langle n, n+1, \cdots, 2n-2\rangle$. By Lemma 4, H_1 is symmetric with $c(H_1)=2n$ and $emb(H_1)=n-1$ which is not a complete intersection and $m \in H_1$. If we choose an integer b so that $(b, m)=1$ and $bn>m$, then $H=\langle m, bH_1\rangle$ is the desired example by Lemma 1.

Remark. The condition $m-1 \geqq n \geqq 4$ is necessary. If $n \geqq m$, we can choose $x \in R$ such that $m=m(R)=$ length (R/xR). But as length $(R/xR) \geqq emb(R)=n$, the only possibility is the case when $m=$ length $(R/xR)=emb(R)$. But in this case the principal ideal xR can not be irreducible and R is not Gorenstein.

If $n=3$, then it is known by Serre that if R is Gorenstein, then R is a complete intersection.

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