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# UMBILICAL SUBMANIFOLDS AND MORSE FUNCTIONS

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Let $M^n$ be a differentiable manifold (of class $C^\infty$). By a Morse function on $M^n$ we mean a differentiable function whose critical points are all non-degenerate. If $f$ is an immersion of $M^n$ into a Euclidean space $R^m$, we may obtain Morse functions on $M^n$ in the following way. Let $p$ be a point of $R^m$ and define a differentiable function $L_p$ on $M^n$ by

$$L_p(x) = d(p, f(x))^2\,, \qquad x \in M^n$$

where $d$ denotes the Euclidean distance in $R^m$. Then, for almost all $p \in R^m$, $L_p$ is a Morse function on $M^n$ (see [2], p. 36).

It is a well-known theorem of Reeb that if a compact differentiable manifold $M^n$ admits a Morse function with exactly two critical points, then $M^n$ is a topological sphere (see [2], p. 25). In the present note we shall prove the following results of a geometric nature (in contrast to a topological nature).

THEOREM A. *Let $M^n$ be a connected compact differentiable manifold ($n \geqq 2$) immersed in a Euclidean space $R^m$. If every Morse function on $M^n$ of the form $L_p, p \in R^m$, has exactly two critical points, then $M^n$ is imbedded as a Euclidean n-sphere.*

Of course, a Euclidean $n$-sphere in $R^m$ means a hypersphere in a Euclidean $(n+1)$-subspace $R^{n+1}$ of $R^m$. As a matter of fact, Theorem A follows from the following more general result.

THEOREM B. *Let $M^n, n \geqq 2$, be a connected, complete Riemannian manifold isometrically immersed in a Euclidean space $R^m$. If every Morse function on $M^n$ of the form $L_p, p \in R^m$, has index 0 or $n$ at any of its critical points, then $M^n$ is imbedded as a Euclidean n-subspace or a Euclidean n-sphere in $R^m$.*

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As another corollary, we obtain

THEOREM C. *Under the assumptions of Theorem* B, *if the index is always* 0, *then* $M^n$ *is imbedded as a Euclidean n-subspace of* $R^m$.

## 1. Preliminaries.

It is necessary to recall certain concepts and results on focal points, which can be found in [2, pp. 32–38]. Although this reference treats submanifolds imbedded in a Euclidean space, the same results hold for immersed submanifolds.

Let $f$ be an immersion of a differentiable manifold $M^n$ into a Euclidean space $R^m$. A point of the normal bundle $N$ of $M^n$ is denoted by $(x, \xi)$, where $x$ is a point of $M^n$ and $\xi$ is a vector normal to $f(M^n)$ at $f(x)$. Let $F$ be a differentiable mapping of $N$ into $R^m$ given by $F(x, \xi) = f(x) + \xi$. A point $p \in R^m$ is called a focal point of $M$ if $p = F(x, \xi)$, where $(x, \xi)$ is a point of $N$ where the Jacobian $F_*$ of $F$ is degenerate. In this case, we also say that $p$ is a focal point of $(M, x)$. By virtue of Sard's theorem, the set of focal points of $M$ has measure 0.

It is known that a point $p = F(x, \xi)$, where $(x, \xi) \in N$, is a focal point of $(M, x)$ if and only if the endomorphism $I - A_\xi$ on the tangent space $T_x(M^n)$ is degenerate. Here $I$ is the identity transformation of $T_x(M^n)$ and $A_\xi$ is the symmetric endomorphism corresponding to the second fundamental form of $M$ at $x$ in the direction of $\xi$.

On the other hand, let $p \in R^m$ and consider the function $L_p(x) = d(f(x), p)^2$ on $M^n$. A point $x \in M^n$ is a critical point of $L_p$ if and only if the vector $\xi$ from $f(x)$ to $p$ is normal to $f(M^n)$. In this case, the Hessian $H$ of $L_p$ at $x$, which is a bilinear symmetric function on $T_x(M) \times T_x(M)$, is given by

$$H(X, Y) = 2\langle I - A_\xi(X), Y\rangle , \qquad X, Y \in T_x(M^n) ,$$

where $\langle\ ,\ \rangle$ is the inner product on $T_x(M)$ induced from the Euclidean metric in $R^m$ through the immersion $f$. Thus $H$ is degenerate at $x$ (i.e., $x$ is a degenerate critical point of $L_p$) if and only if $I - A_\xi$ is degenerate (i.e., $p$ is a focal point of $(M, x)$). If $x$ is a nondegenerate critical point of $L_p$, the index at $x$ is equal to the number of negative eigenvalues of $I - A_\xi$, counting multiplicities, in other words, the number of eigenvalues of $A_\xi$ that are larger than 1, counting multiplicities.

Finally, let $(x, \xi) \in N$, where $\xi$ is a *unit* vector. For $t > 0$, let $p = F(x, t\xi)$. Then $p$ is a focal point of $(M, x)$ if and only if $1/t$ is an eigenvalue of $A_\xi$. Suppose $1/t$ is not an eigenvalue of $A_\xi$. Then the function $L_p$ has $x$ as a nondegenerate critical point and the index at $x$ is equal to the number of positive eigenvalues (counting multiplicities) that are greater than $1/t$.

We now prove a lemma which is crucial in the proof of our results.

LEMMA. *Let $p \in R^m$ and assume that the function $L_p$ has a nondegenerate critical point $x \in M^n$ of index $k$. Then there exists a point $q \in R^m$ such that $L_q$ is a Morse function which has a critical point $z$ of index $k$. ($q$ and $z$ may be chosen as close to $p$ and $x$, respectively, as we want.)*

*Proof.* Let $p = F(x, \xi)$, where $\xi$ is a normal vector at $f(x)$. By assumption, $p$ is not a focal point of $(M, x)$, that is, the Jacobian $F_*$ is nondegenerate at $(x, \xi)$. Thus there exists a neighborhood $U$ of $(x, \xi)$ in the normal bundle $N$ such that $F$ gives a diffeomorphism of $U$ onto a neighborhood $V = F(U)$ of $p$ in $R^m$. (Of course, $U$ and $V$ may be chosen as small as we like.) Now $V$ has a point $q$ such that $L_q$ is a Morse function (i.e., $q$ is not a focal point of $M$), because the set of focal points of $M$ has measure 0. We have $q = F(z, \zeta)$ for some $(z, \zeta) \in U$. We show that the index of $L_q$ at $z$ is equal to $k$.

Consider a differentiable family of symmetric endomorphisms $I - A_\eta$ on $T_y(M^n)$, where $(y, \eta)$ runs over $U$. If we denote the eigenvalues by

$$\lambda_1(y, \eta) \geqq \lambda_2(y, \eta) \geqq \cdots \geqq \lambda_n(y, \eta) ,$$

then it can be shown that each $\lambda_i$ is a continuous function on $U$. Since $F_*$ is nondegenerate at each point of $U$, none of these functions takes value 1 on $U$. The index of $L_p$ at $x$ being $k$ by assumption, we have that $\lambda_1, \cdots, \lambda_k$ are greater than 1 at $(x, \xi)$ and $\lambda_{k+1}, \cdots, \lambda_n$ are less than 1 at $(x, \xi)$. It follows that the same arrangement holds at $(z, \zeta)$. This means that the index of $L_q$ at $z$ is equal to $k$. We have thus proved the lemma.

## 2. Proof of Theorem B.

Under the assumptions of Theorem B, we shall show the following fact. If $x \in M^n$ and if $\xi$ is a unit vector normal to $f(M^n)$ at $f(x)$, then

$A_\xi = cI$ for some constant $c$, that is, $A_\xi$ has only one eigenvalue (of multiplicity $n$). Suppose $A_\xi$ has a non-zero eigenvalue, say, $a$. We may assume that $a > 0$, because if $a < 0$, then $A_{-\xi}$ has $-a > 0$ as eigenvalue; if we can show that $A_{-\xi} = (-a)I$, then we know that $A_\xi = -A_{-\xi} = aI$.

Assuming thus that $a$ is the largest positive eigenvalue of $A_\xi$, take $t > 0$ such that $1/a < t < 1/b$, where $b$ is the next largest positive eigenvalue if any (if $a$ is the only positive eigenvalue, just consider $1/a < t$). Then $p = F(x, t\xi)$ is not a focal point of $(M, x)$ and the function $L_p$ has $x$ as a nondegenerate critical point. The index at $x$ is equal to the multiplicity, say, $k$, of the eigenvalue $a$. If $L_p$ is a Morse function, the assumption in Theorem B implies $k = n$, since $k$ cannot be 0. Now $L_p$ may not be a Morse function (it can have a degenerate critical point elsewhere). By the lemma in Section 1, however, we know that there must exist a Morse function of the form $L_q$, $q \in R^m$, which has a critical point $z$ of index $k$. Thus we may conclude that $k = n$. This means that $a$ is an eigenvalue of $A_\xi$ with multiplicity $n$ so that $A_\xi = aI$.

What we have just shown implies that $M^n$ is umbilical, that is, if $\eta$ denotes the mean curvature vector field, then for any normal vector $\xi$ at $x$ we have

$$A_\xi = \langle \xi, \eta \rangle I .$$

Equivalently, every $X \in T_x(M^n)$ is a principal vector in the sense that there exists a 1-form $\omega$ on the normal space $N_x$ such that

$$A_\xi(X) = \omega(\xi)X \quad \text{for all} \quad \xi \in N_x \quad \text{and} \quad X \in T_x(M) .$$

It is known (see [1, p. 231]) that a complete Riemannian manifold isometrically and umbilically immersed in $R^m$ is actually imbedded as a Euclidean $n$-subspace or a Euclidean $n$-sphere. This completes the proof of Theorem B.

It is quite easy to derive Theorem A from Theorem B. If a Morse function $L_p$ has exactly two critical points, then one is where $L_p$ has a maximum (hence of index $n$) and the other is where $L_p$ has a minimum (hence of index 0). Thus every Morse function $L_p$ has index $n$ or 0 at a critical point.

Suppose $S^n$ is a Euclidean $n$-sphere in $R^m$ and assume we have taken a rectangular coordinate system $x_1, \cdots, x_m$ in $R^m$ so that

$$S^n = \left\{(x_1, \cdots, x_{n+1}, 0, \cdots, 0)\,;\, \sum_{k=1}^{n+1} x_k^2 = r^2\right\} .$$

Then we can see that the set of focal points of $S^n$ is the Euclidean $(m - (n + 1))$-subspace defined by $x_1 = \cdots = x_{n+1} = 0$. If $p$ is not a focal point, the Morse function $L_p$ has exactly two critical points, one of index $n$ and the other of index 0.

What we have just said is sufficient to derive Theorem C from Theorem B.

## 3. Remarks.

Our main results may be formulated without explicitly involving the notion of Morse functions and, indeed, under a weaker assumption. *Let $D$ be a dense subset of $R^m$. In Theorems* A, B *and* C, *we may replace "every Morse function on $M^n$ of the form $L_p, p \in R^m$" by "every function on $M^n$ of the form $L_p, p \in D$".*

The proof of Theorem B under this weaker assumption remains almost the same as before except for a corresponding change in the lemma, namely, the conclusion of the lemma should be modified as follows: "*Then there exists a point $q \in D$ such that $L_q$ has a critical point $z$ of index $k$.*"

Finally, we note that if $M^2$ immersed in $R^m$ is topologically a 2-sphere, then our original assumption in Theorem A is equivalent to the spherical two-piece property studied by T. F. Banchoff: *The spherical two-piece property and tight surfaces in spheres*, J. Differential Geometry 4 (1970), 193–205 (see, in particular, Theorem 3).

## REFERENCES

[1] É. Cartan, Leçons sur la géométrie des espaces de Riemann, deuxième édition, Gauthier-Villars, Paris, 1946.
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