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PERIODIC ORBITS OF ISOMETRIC FLOWS

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1. Introduction

Let M be a compact C^∞ Riemannian manifold, X a Killing vector field on M, and φ_t its 1-parameter group of isometries of M. In this, paper, we obtain some basic properties of the set of periodic points of φ_t. We show that the set of least periods is always finite, and the set $P(X,t)$ of points of M having least period t for the vector field X is a totally geodesic submanifold, with possibly non-empty boundary. Moreover, we show there are at least m geometrically distinct closed geodesic orbits of φ_t, where m is the number of least periods which are not integral multiples of any other least period.

2. Finiteness of least periods

Let M be a complete Riemannian manifold of dimension n. Let $I^0(M)$ be the identity component of its isometry group, and $i(M)$ the Lie algebra of $I^0(M)$. $i(M)$ is naturally identified with the Lie algebra of Killing vector fields on M, and we will identify an element $X \in i(M)$ with the corresponding Killing vector field. If X is any vector field on M, let $\mathrm{Zero}\,(X) = \{p \in M \mid X_p = 0\}$. We use $\mathrm{Fix}\,(f)$ for the set of fixed points of a map $f: M \to M$.

LEMMA. *Let $X \in i(M)$ and φ_t its 1-parameter group. For each $p \in M$ there is a neighborhood U of p such that the set of least periods of periodic orbits of φ_t which intersect U is finite.*

Proof. (i) If $p \in \mathrm{Zero}\,(X)$ then $(\varphi_t)_*: T_pM \to T_pM$ is a 1-parameter subgroup of the orthogonal subgroup of the orthogonal group of T_pM, so there is a basis of T_pM in which $(\varphi_t)_* \mid T_pM$ has the form:

$$(\varphi_t)_* = \mathrm{diag}\,\{a_1(t), \cdots, a_k(t), I_{n-2k}\}$$

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where

$$a_i(t) = \begin{pmatrix} \cos \alpha_i t & \sin \alpha_i t \\ -\sin \alpha_i t & \cos \alpha_i t \end{pmatrix}$$

$\alpha_i \neq 0$ for $i = 1, \cdots, k$, and I_{n-2k} is the $(n-2k) \times (n-2k)$ identity matrix. The periodic points of φ_t in a spherical neighborhood of p correspond, by $\exp_p : T_pM \to M$, to periodic vectors in T_pM under $(\varphi_t)_*$. The least periods of such vectors are the least common multiples of subsets of the set of numbers $\{2\pi/\alpha_1, \cdots, 2\pi/\alpha_k\}$ and thus are finite in number.

(ii) If $p \notin \mathrm{Zero}(X)$, let t_0 be the least period of p. (Let $t_0 = +\infty$ if p is not periodic). If $t_0 = \infty$, then either p lies in a neighborhood of non-periodic points, or there is a sequence of periodic points $p_i \to p$. In the first case we're done, and in the second we replace p by a p_i sufficiently close to p so p lies in a convex normal neighborhood of p_i. Thus we assume p is periodic of least period $0 < t_0 < +\infty$. For each $\varepsilon > 0$ let $N_\varepsilon = \{Y \in T_pM \mid Y \perp X_p$ and $|Y| < \varepsilon\}$. If ε is sufficiently small, then $\varphi_t(\exp_p N_\varepsilon) \cap \varphi_{t'}(\exp_p N_\varepsilon) \neq \emptyset$ only when $t - t' \equiv 0 \pmod{t_0}$, and $\bigcup_t \varphi_t(\exp_p N_\varepsilon)$ is a tubular neighborhood of the closed orbit $\{\varphi_t(p) \mid t \in R\}$. Let $\delta \in (0, t_0/2)$ be so small that $U = \bigcup_t \{\varphi_t(\exp_p N_\varepsilon) \mid |t| < \delta\}$ is a normal neighborhood of p. We may assume the p_i we chose to replace p is close enough to p so they both lie in U. Now if we put $(\varphi_{t_0})_* \mid T_pM$ in the same normal form as in (i), the same argument shows there are only finitely many least periods of periodic points in U. q.e.d.

Remark. It follows from the proof that the number of least periods in each of the neighborhoods U is bounded above by the maximal number possible of least common multiples in a set of $[n/2]$ numbers; namely $2^{[n/2]}$. This is independent of the choice of X, but the neighborhood U does depend on X.

COROLLARY. *If M is compact, the set of least periods of φ_t is finite.*

This corollary thus follows from a simple geometrical argument. One can derive the same result using a theorem of Yang ([4]), which says that a compact Lie group acting differentiably on a compact manifold has only finitely many non-conjugate isotropy subgroups.

From now on, M will always be compact. The Lie algebra $i(M)$ has an $ad(I^0(M))$-invariant positive definite symmetric bilinear form, and we let S be the unit sphere in $i(M)$ with respect to this form.

LEMMA. *There is a number* $a > 0$ *such that for every* $X \in S$ *and every isotropy subgroup* G_p, *if* $0 < t_0 < a$ *and* $\exp(t_0X)$ *is in* G_p, *then* $\exp(tX) \in G_p$ *for all* $t \in R$.

Proof. $I^0(M)$ is a compact Lie group and therefore a compact symmetric space whose geodesics are the translates of 1-parameter subgroup $\exp(tX)$. If we assume $X \in S$ then t is arc-length. Let U_ε be the open ball of radius ε about 0 in $i(M)$, and assume $\varepsilon > 0$ is so small that $\exp(U_\varepsilon)$ lies in a normal neighborhood of e in $I^0(M)$. Suppose $0 < t_0 < \varepsilon$ and $\exp(t_0X) \in G_p =$ identity component of G_p. There is a minimizing geodesic $\exp(tY)$ from e to $\exp(t_0X)$ lying entirely in G_p^0, whose length is less than ε (since $\exp(U_\varepsilon) \cap G_p^0$ is the ε-ball about e in G_p^0). Now $\exp|U_\varepsilon$ is $1-1$ so we must have $\exp(tY) = \exp(tX)$ for all t.

If G_p is not connected, it has finitely many components and we let $r_p = \text{distance}(e, G_p - G_p^0)$. Clearly $r_p > 0$ since $G_p - G_p^0$ is compact; and r_p is constant on the conjugacy class of G_p since the metric in $I^0(M)$ is invariant by conjugation. There are only finitely many conjugacy classes so $r = \min r_p > 0$. Then any $0 < a < \min(\varepsilon, r)$ satisfies the requirements of the lemma. q.e.d.

From this lemma we can derive a "uniform" period bounding lemma for Killing vector fields:

COROLLARY. *The positive least periods of periodic orbits of* 1-*parameter groups of isometries are bounded away from zero uniformly if their generators* X *are taken from* S.

Proof. A point $p \in M$ is periodic of least period t_0 for the 1-parameter group $\exp(tX)$ if $\exp(t_0X) \in G_p$ but $\exp(tX) \notin G_p$ if $0 < t < t_0$. The number a of the previous lemma is then the required lower bound. q.e.d.

3. Submanifolds of periodic points

For each $X \in i(M) - \{0\}$ and each $0 < t < \infty$, let $P(X, t) = \{p \in M \mid p$ has least period t for the 1-parameter group $\exp(tX)\}$. Let $P(X, 0) = \text{Zero}(X)$, and $P(X, \infty) =$ set of non-periodic points of $\exp(tX)$. Then we know $P(X, t) = \emptyset$ except for a finite subset of $[0, \infty]$. Now assume $X \in i(M) - \{0\}$ is fixed, and φ_t is its 1-parameter group. It is well-known that for each $0 < t < \infty$, $\text{Fix}(\varphi_t)$ is a closed totally geodesic submanifold of M.

Let $0 < t_1 < t_2 < \cdots < t_N$ be the positive least periods for $X \in i(M)$. Let $\beta_1, \cdots, \beta_m$ be the subset of least periods which are not integral multiples of any other least period. Use the notation $t_i | t_j$ if $t_j = kt_i$ for some integer k, and call the β_i the *basic periods* for X. It is easy to see that for each $i = 1, \cdots, N$, $P(X, t_i) = \mathrm{Fix}(\varphi_{t_i}) - \cup\{\mathrm{Fix}(\varphi_{t_j}) | t_j | t_i\} - \mathrm{Zero}(X)$, the union consisting of finitely many closed totally geodesic submanifolds of $\mathrm{Fix}(\varphi_{t_i})$. Therefore we have:

PROPOSITION. *Each $P(X, t_i)$ is a totally geodesic submanifold of M (possibly with a finite number of closed submanifolds deleted).*

THEOREM. *If the Killing vector field X has m basic periods, there are at least m geometrically distinct smooth closed geodesics on M which are orbits of the* 1*-parameter group of X.*

Proof. For each $i = 1, \cdots, m$, $P(X, \beta_i) = \mathrm{Fix}(\varphi_{\beta_i}) - \mathrm{Zero}(X)$. Now $\mathrm{Fix}(\varphi_{\beta_i})$ is a closed totally geodesic submanifold of M, and X is tangent to it, so X is Killing vector field on $\mathrm{Fix}(\varphi_{\beta_i})$. Let $p \in \mathrm{Fix}(\varphi_{\beta_i})$ be a point at which $|X|$ achieves its maximum. Then $|X|^2$ has a critical point at p, so ([2]) the orbit of p is a geodesic. The orbit is non-trivial since $p \notin \mathrm{Zero}(X)$. q.e.d.

Remarks. (1) In fact, we get a closed geodesic for each component of $\mathrm{Fix}(\varphi_{\beta_i})$.

(2) The same argument as in Kobayashi ([1]), shows that the Euler numbers of the $\mathrm{Fix}(\varphi_{t_i})$ all equal that of M.

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