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SATURATED IDEALS IN BOOLEAN EXTENSIONS

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0. Introduction. Let κ be an uncountable cardinal, and let λ be a regular cardinal less than κ. Let I be a λ-saturated non-trivial ideal on κ. Prikry, in his thesis, showed that, in certain Boolean extensions, κ has a λ-saturated non-trivial ideal on κ. More precisely,

THEOREM (Prikry [8]). *Let κ, λ and I be as above. Let $\mathscr{B}$ be a λ-saturated complete Bollean algebra. Let $J \in V^{(\mathscr{B})}$ such that, with probability* **1**, *J is the ideal on $\check{\kappa}$ generated by $\check{I}$. Then, it is $\mathscr{B}$-valid that J is a $\check{\lambda}$-saturated non-trivial ideal on $\check{\kappa}$.*

The following questions naturally arise; 1) If I is κ-saturated (κ^+-saturated), does J remain κ-saturated (κ^+-saturated)? 2) If $\mathrm{sat}(\mathscr{B}) = \kappa$, what is the saturatedness of J?

For 1), we obtain the following theorem.

THEOREM 1. *Let κ and λ be as above. Let γ be a regular cardinal such that $\lambda \leq \gamma \leq \kappa^+$, and let I be a γ-saturated non-trivial ideal on κ. Let $\mathscr{B}$ be a λ-saturated complete Boolean algebra. Then, it is $\mathscr{B}$-valid that J is γ-saturated.*

For 2), we get the following theorems.

THEOREM 2. *Let κ be an uncountable cardinal, and I be a κ-saturated non-trivial ideal on κ. Let $\mathscr{B}$ be a homogeneous complete Boolean algebra such that $\mathrm{sat}(\mathscr{B}) = \kappa$. Then, it is $\mathscr{B}$-valid that J is not κ-saturated.*

THEOREM 3. *Let κ be a measurable cardinal, and I be a non-trivial prime ideal on κ. Let $\mathscr{B}$ be a homogeneous complete Boolean algebra such tnat $\mathrm{sat}(\mathscr{B}) = \kappa$. Then, it is $\mathscr{B}$-valid that J is not κ^+-saturated.*

We will prove the above theorems as applications of a certain useful lemma, which will be proved in § 4.

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We assume that the reader is familiar with the Scott-Solovay Boolean-valued models for set theory.

1. Saturated ideals.

1.1. Let λ be a cardinal. Let $\mathscr{B}$ be a Boolean algebra. We say that $\mathscr{B}$ is λ-saturated if, for any pairwise disjoint family $\{b_\alpha\}_{\alpha<\lambda}$ of $\mathscr{B}$, there exists some $\alpha < \lambda$ such that $b_\alpha = 0$. Clearly, if $\lambda < \gamma$ and $\mathscr{B}$ is λ-saturated, then $\mathscr{B}$ is γ-saturated. $\operatorname{sat}(\mathscr{B})$ denotes the least cardinal λ such that $\mathscr{B}$ is λ-saturated.

The following lemma is well-known.

LEMMA. *If* $\operatorname{sat}(\mathscr{B}) \geq \aleph_0$ *then* $\operatorname{sat}(\mathscr{B})$ *is an uncountable regular cardinal.*

1.2. Let κ be an uncountable cardinal. Let I be an ideal on κ. I is called non-trivial if;

1) I is non-principal, that is, $\{\alpha\} \in I$ for all $\alpha < \kappa$.

2) I is κ-complete, that is, if whenever $\eta < \kappa$, and $\{A_\alpha, \alpha < \eta\}$ is a family such that $A_\alpha \in I$ for each $\alpha < \eta$, then $\bigcup_{\alpha<\eta} A_\alpha \in I$.

Let I be an non-trivial ideal on κ. We can form the quotient algebra $\mathscr{A} = P(\kappa)/I$. If $\mathscr{A}$ is λ-saturated, we say that I is λ-saturated.

Solovay proved the following theorem.

THEOREM (Solovay [5]). *Suppose that* κ *has* κ*-saturated non-trivial ideal on* κ*. Then,* κ *is the* κ*-th weakly inaccessible.*

For more informations about saturated ideals, the reader may refer to Kunen [1], Kunen-Paris [2] and Solovay [5].

2. The ultrapowers inside $V^{(\mathscr{A})}$.

In this section, we restate the necessary results from Solovay [5].

From 2.1 to 2.3, we fix a transitive model M of ZFC, and an ordinal ρ in M.

2.1. Let $\mathscr{U}$ be a subset of $P(\rho) \cap M$. We say that $\mathscr{U}$ is an M-ultrafilter on ρ if:

(1) $\mathscr{U}$ contains no singletons.

(2) If $A \in \mathscr{U}$, $B \in P(\rho) \cap M$, and $A \subseteq B$, then $B \in \mathscr{U}$.

(3) If $A \in P(\rho) \cap M$, then either $A \in \mathscr{U}$ or $\rho - A \in \mathscr{U}$.

(4) Let $\eta < \rho$. Let $\langle A_\xi, \xi < \eta \rangle$ be a sequence such that $A_\xi \in \mathscr{U}$ for each $\xi < \eta$ and $\langle A_\xi : \xi < \eta \rangle \in M$. Then, $\bigcap_{\xi<\eta} A_\xi \in \mathscr{U}$.

The concept of M-ultrafilter is due to Kunen [1]. The reader should note that this definition somewhat differs from that of Kunen.

2.2. Let $\mathscr{U}$ be an M-ultrafilter on ρ. We define an equivalence relation $\simeq$ on $M \cap M^\rho$ as follows; for $f, g \in M \cap M^\rho$ let

$$f \simeq g \qquad \text{iff } \{\alpha < \rho ; f(\alpha) = g(\alpha)\} \in \mathscr{U} .$$

We denote by $[f]$ the Scott equivalence class of f with respect to $\simeq$.

Next, we put $N = \{[f] ; f \in M \cap M^\rho\}$. We define a binary relation E on N as follows; Let $f, g \in M \cap M^\rho$.

$$[f]E[g] \qquad \text{iff } \{\alpha < \rho ; f(\alpha) \in g(\alpha)\} \in \mathscr{U} .$$

It is clear that the definition of E does not depend on the choice of f and g. The relational structure $\langle N, E \rangle$ is denoted by $\mathrm{Ult}(M, \mathscr{U})$.

2.3. LEMMA 1 (Los). *Let $\phi(v_0, \cdots, v_{n-1})$ be a set-theoretical formula, and let $f_0, \cdots, f_{n-1}$ be elements of $M \cap M^\rho$. Then,*

$$N \models \phi([f_0], \cdots, [f_{n-1}]) \qquad \textit{iff } \{\alpha < \rho ; M \models \phi(f_0(\alpha), \cdots, f_{n-1}(\alpha))\} \in \mathscr{U} .$$

Let x be in M. We define $c_x \in M \cap M^\rho$ by $c_x(\alpha) = x$ for all $\alpha < \rho$, and define $c : M \to N$ by $c(x) = [c_x]$.

LEMMA 2. *c is an elementary embedding.*

In the remainder of this section, κ will be uncountable cardinal, and I a κ^+-saturated non-trivial ideal on κ.

2.4. We form the quotient algebra $\mathscr{A} = P(\kappa)/I$. Let $A \in P(\kappa)$. We denote by $[A]$ the element of $\mathscr{A}$ represented by A.

LEMMA 3.[1] *$\mathscr{A}$ is complete.*

Let $V^{(\mathscr{A})}$ be the Scott-Solovay $\mathscr{A}$-valued model. We assume that $V^{(\mathscr{A})}$ is separated.

2.5. We define an element $\mathscr{U}$ of $V^{(\mathscr{A})}$ as follows;

$$\|\check{A} \in \mathscr{U}\| = [A] \qquad \text{for each } A \in P(\kappa) .$$

1) See Sikorski, Boolean algebras, Springer-Verlag, Berlin, 1960 p.65, 21.3.

LEMMA 4. *With probability* **1**. $\mathscr{U}$ *is a* $\check{V}$*-ultrafilter on* $\check{\kappa}$.

By Lemma 4, we can form $\mathrm{Ult}(\check{V},\mathscr{U})$ inside $V^{(\mathscr{A})}$.

LEMMA 5. *Let* $f_0,\cdots,f_{n-1}\in V^{\kappa}$. *Let* $\phi(v_0,\cdots,v_{n-1})$ *be a set-theoretical formula. Then,*

$$\|\mathrm{Ult}(\check{V},\mathscr{U})\models\phi([\check{f}_0],\cdots,[\check{f}_{n-1}])\|=[\{\alpha<\kappa\,;\,\phi(f_0(\alpha),\cdots,f_{n-1}(\alpha)\}]\ .$$

The lemma is easily proved by using Lemma 1 and the following sublemma.

SUBLEMMA. *Let* $x_0,\cdots,x_{n-1}\in V$. *Let* $\phi(v_0,\cdots,v_{n-1})$ *be a set-theoretical formula. Then,*

$$\|\check{V}\models\phi(\check{x}_0,\cdots,\check{x}_{n-1})\|=\mathbf{1}\qquad\textit{iff}\ \ \phi(x_0,\cdots,x_{n-1})\ .$$

LEMMA 6. *Let* $x\in V^{(\mathscr{A})}$. *Suppose that* $\|x\in\mathrm{Ult}(\check{V},\mathscr{U})\|=\mathbf{1}$. *Then, for some* $f\in V^{\kappa}$, $\|x=[\check{f}]\|=\mathbf{1}$.

LEMMA 7. *With probability* **1**, $\mathrm{Ult}(\check{V},\mathscr{U})$ *is well-founded.*

2.6. By Lemma 7, there exists a transitive class of $V^{(\mathscr{A})}, N$, and an isomorphism $\psi:\mathrm{Ult}(\check{V},\mathscr{U})\to N$ inside $V^{(\mathscr{A})}$. Let $f\in V^{(\kappa)}$. Let $\psi(f)$ be the element of $V^{(\mathscr{A})}$ such that $\|\psi(f)=\psi([\check{f}])\|=\mathbf{1}$. We put $x^*=\psi(c_x)$.

LEMMA 8. (1) *With probability* $\mathbf{1}, N$ *is a transitive class containing all ordinals.*

(2) *Let* $f_0,\cdots,f_{n-1}\in V^{\kappa}$. *Let* $\phi(v_0,\cdots,v_{n-1})$ *be a set-theoretical formula. Then,*

$$\|N\models\phi(\psi(f_0),\cdots,\psi(f_{n-1}))\|=[\{\alpha<\kappa\,;\,\phi(f_0(\alpha),\cdots,f_{n-1}(\alpha))\}]\ .$$

(3) *Let* $\|x\in N\|=\mathbf{1}$. *Then,* $x=\psi(f)$ *for some* $f\in V^{\kappa}$.

(4) *If* $\alpha<\kappa$, $\alpha^*=\check{\alpha}$.

(5) $\|\kappa^*>\check{\kappa}\|=\mathbf{1}$.

LEMMA 9. *With probability* $\mathbf{1}, N$ *contains all* $\check{\kappa}$*-sequences of* N *in* $V^{(\mathscr{A})}$.

Proof. Let $s\in V^{(\mathscr{A})}$ be such that $\|s\,;\,\check{\kappa}\to N\|=\mathbf{1}$. For each $\alpha<\kappa$, we can choose $f_\alpha\in V^{\kappa}$ such that $\|s(\check{\alpha})=\psi(f_\alpha)\|=\mathbf{1}$. Let $\psi(g)=\kappa$. We define $f\in V^{\kappa}$ by $f(\alpha)=\langle f_\beta(\alpha):\beta<g(\alpha)\rangle$.

Clearly, $\|N\models\psi(f)$ is a $\check{\kappa}$-sequence$\|=\mathbf{1}$. We claim that $\|\psi(f)=s\|=\mathbf{1}$. Now, choose $h_\alpha\in V^{\kappa}$ so that $\|(\psi(f))(\check{\alpha})=\psi(h_\alpha)\|=\mathbf{1}$ for each $\alpha<\kappa$.

Then, $\|N \models \psi(h_\alpha)$ is the value of $\check{\alpha}$ by $\psi(f)\| = \mathbf{1}$. By Lemma 8, for almost all $\beta < \kappa$, $h_\alpha(\beta)$ is the value of α by $f(\beta)$. Then, $\|\psi(h_\alpha) = \psi(f_\alpha)\| = \mathbf{1}$. We have just proven that $\|(\forall \alpha < \check{\kappa})((\psi(f))(\alpha) = s(\alpha))\| = \mathbf{1}$. Since $\psi(f)$ and s are $\check{\kappa}$-sequences, $\|\psi(f) = s\| = \mathbf{1}$.

3. Boolean algebras in Boolean extensions.

Let $\mathscr{B}$ be a complete Boolean algebra. Let $\mathscr{D} \in V^{(\mathscr{B})}$ such that $\|\mathscr{D}$ is a Boolean algebra$\| = \mathbf{1}$. We put $\mathscr{D}_{[\mathscr{B}]} = \{x \in V^{(\mathscr{B})} : \|x \in \mathscr{D}\| = \mathbf{1}\}$. We can make $\mathscr{D}_{[\mathscr{B}]}$ into a Boolean algebra, by defining Boolean operations as follows;

Let $x, y \in \mathscr{D}_{[\mathscr{B}]}$. Then, there exist uniquely z_1 and z_2 such that the followings are $\mathscr{B}$-valid respectively.

1) $z_1 \in \mathscr{D}$ and $x +_{\mathscr{D}} y = z_1$

2) $z_2 \in \mathscr{D}$ and $-_{\mathscr{D}} x = z_2$

Put $z_1 = x +_{\mathscr{D}[\mathscr{B}]} y$ and $z_2 = -_{\mathscr{D}[\mathscr{B}]} x$.

The following lemma is due to Solovay-Tennenbaum [7]

LEMMA 1. *$\mathscr{D}_{[\mathscr{B}]}$ is complete iff it is $\mathscr{B}$-valid that $\mathscr{D}$ is complete.*

The proof of the following lemma is similar to Lemma 5.2.6 of Solovay-Tennenbaum [7]. So we omit the proof.

LEMMA 2. *Let λ bea regular cardinal. Then the following are equivalent:*

1) *$\mathscr{B}$ is λ-saturated, and it is $\mathscr{B}$-valid that $\mathscr{D}$ is λ-saturated*

2) *$\mathscr{D}_{[\mathscr{B}]}$ is λ-saturated.*

LEMMA 3.[1)] *If there is a surjection Φ form $\mathscr{B}$ to $\mathscr{D}_{[\mathscr{B}]}$ such that $\|\Phi(b) = \mathbf{1}_{\mathscr{D}}\| = b$ and $\|\Phi(b) = \mathbf{0}_{\mathscr{D}}\| = -b$ for all $b \in \mathscr{B}$, then $\mathscr{D} = \mathbf{2}$ in $V_{(\mathscr{B})}$.*

4. The basic lemma and proof of Theorem 1.

4.1. Let κ, I and $\mathscr{A}$ be as in § 2. Let $\mathscr{B}$ be a complete Boolean algebra. Let $J \in V^{(\mathscr{B})}$ such that J is the ideal on $\check{\kappa}$ generated by $\check{I}$ in $V^{(\mathscr{B})}$. Clearly $\|A \in J\| = \sum_{B \in I} \|A \subseteq B\|$.

LEMMA 1. *If $\mathscr{B}$ is κ-saturated, then it is $\mathscr{B}$-valid that J is non-trivial.*

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1) cf. Solovay-Tennenbaum [7], p.214.

Proof. Trivially, J is non-principal. The fact that J is κ-complete is easily proved by using the following sublemma.

SUBLEMMA. *If $\mathscr{B}$ is κ-saurated, then $\|A \in J\| = \|A \subseteq B\|$ for some $B \in I$.*

4.2. Let $\mathscr{D} \in V^{(\mathscr{B})}$ such that $\|\mathscr{D} = P(\check{\kappa})/J\|^{(\mathscr{B})} = \mathbf{1}$.

BASIC LEMMA. *If $\mathscr{B}$ is κ-saturated, then $\mathscr{D}_{[\mathscr{B}]}$ is isomorphic to $\mathscr{B}^*_{[\mathscr{A}]}$*

Proof. Let $x \in \mathscr{D}_{[\mathscr{B}]}$. Then, there exists $A \in V^{(\mathscr{B})}$ such that $\|x = [A]\|^{(\mathscr{B})} = \mathbf{1}$ and $\|A \subseteq \check{\kappa}\|^{(\mathscr{B})} = \mathbf{1}$. We define $f_A ; \kappa \to \mathscr{B}$ by $f_A(\alpha) = \|\check{\alpha} \in A\|^{(\mathscr{B})}$. Then, $\|\psi(f_A) \in \mathscr{B}^*\|^{(\mathscr{A})} = \mathbf{1}$. Put $\Phi(x) = \psi(f_A)$. We must show that the definition of $\Phi(x)$ does not depent on the choice of A. So let, $A, B \in P^{(\mathscr{B})}(\kappa)$ such that $\|[A] = [B]\|^{(\mathscr{B})} = \mathbf{1}$. Then, $\|A \Delta B \in J\|^{(\mathscr{B})} = \mathbf{1}$. ($A \Delta B$ denotes the symmetric difference of A and B.) By the sublemma of Lemma 1, for some $N \in I$, $\|A \Delta B \subseteq \check{N}\|^{(\mathscr{B})} = \mathbf{1}$. It follows that if $\alpha \notin N$, then $\|\check{\alpha} \in A\|^{(\mathscr{B})} = \|\check{\alpha} \in B\|^{(\mathscr{B})}$. Since $N \in I$, for almost all $\alpha < \kappa$, $f_A(\alpha) = f_B(\alpha)$. By Lemma 8 of § 2, we have $\|\psi(f_A) = \psi(f_B)\|^{(\mathscr{B})} = \mathbf{1}$. Since $V^{(\mathscr{B})}$ is sepatate $\psi(f_A) = \psi(f_B)$.

Φ is surjective: Let $y \in \mathscr{B}^*_{[\mathscr{A}]}$. By Lemma 8 of § 2, for some $f \in V^{\kappa}$, $\psi(f) = y$. We may suppose that $f ; \kappa \to \mathscr{B}$. We define $A \in V^{(\mathscr{B})}$ by $\|\check{\alpha} \in A\|^{(\mathscr{B})} = f(\alpha)$ for $\alpha < \kappa$. Clearly, $\|A \subseteq \check{\kappa}\|^{(\mathscr{B})} = \mathbf{1}$. Let $\|x = [A]\|^{(\mathscr{B})} = \mathbf{1}$. Then, $x \in \mathscr{D}_{[\mathscr{B}]}$. By the definition of $\Phi, \Phi(x) = y$.

Φ is injective: Let $x, y \in \mathscr{D}_{[\mathscr{B}]}$ such that $\Phi(x) = \Phi(y)$. Let $A, B \in V^{(\mathscr{B})}$ be such that $\|x = [A]\|^{(\mathscr{B})} = \|y = [B]\|^{(\mathscr{B})} = \mathbf{1}$. Then, $\psi(f_A) = \Phi(x) = \Phi(y) = \psi(f_B)$. Thus, $f_A(\alpha) = f_B(\alpha)$ for almost all $\alpha < \kappa$, that is, $\{\alpha < \kappa ; \|\check{\alpha} \in A\| = \|\check{\alpha} \in B\|\} \in I$. By the definition of J, we have $\|A \Delta B \in J\|^{(\mathscr{B})} = \mathbf{1}$. It follows that $\|x = y\|^{(\mathscr{B})} = \mathbf{1}$.

Φ is an isomorphism: Let $x, y \in \mathscr{D}_{[\mathscr{B}]}$ be such that $x \leqq y$. Let $A, B \in P^{(\mathscr{B})}(\kappa)$ such that $\|x = [A]\|^{(\mathscr{B})} = \|x = [B]\|^{(\mathscr{B})} = \mathbf{1}$. Since $x \leqq y$, we have $\|A - B \in J\|^{(\mathscr{B})} = \mathbf{1}$. By the sublemma of Lemma 1, for some $N \in I$, $\|A - B \subseteq \check{N}\|^{(\mathscr{B})} = \mathbf{1}$. Thus, if $\alpha \notin N$, then $\|\check{\alpha} \in A\|^{(\mathscr{B})} \leqq \|\check{\alpha} \in B\|^{(\mathscr{B})}$. That is, for almost all $\alpha < \kappa$, $f_A(\alpha) \leqq f_B(\alpha)$. It follows that $\psi(f_A) \leqq \psi(f_B)$. So, $\Phi(x) \leqq \Phi(y)$.

4.3. Now, we prove Theorem 1. Let λ be a regular cardinal less than κ, and γ be a regular cardinal $\lambda \leqq \gamma \leqq \kappa^+$. Suppose that I is γ-saturated and $\mathscr{B}$ is λ-saturated. Since $\mathscr{B}$ is λ-saturated and $\lambda < \kappa$, we have $\|N \models \mathscr{B}^*$ is γ-saturated$\|^{(\mathscr{A})} = \mathbf{1}$. Since $\mathscr{A}$ is γ-saturated, $\|\check{\gamma}$ is a cardinal$\|^{(\mathscr{A})} = \mathbf{1}$.

By Lemma 9 of §2 and the fact that $\lambda \leqq \gamma$, we have $\|\mathscr{B}^*$ is γ-saturated$\|^{(\mathscr{A})} = \mathbf{1}$. By Lemma 2 of §3, we have $\mathscr{B}^*_{[\mathscr{A}]}$ is $\check{\gamma}$-saturated. By the basic lemma, $\mathscr{D}_{[\mathscr{B}]}$ is γ-saturated.

Again, by Lemma 2 of §3, $\|\mathscr{D}$ is $\check{\gamma}$-saturated$\|^{(\mathscr{B})} = \mathbf{1}$. That is, $\|J$ is $\check{\gamma}$-saturated$\|^{(\mathscr{B})} = \mathbf{1}$.

Remark. In the case when κ is measurable and I is a non-trivial prime ideal on $\kappa, \mathscr{A} = P(\kappa)/I = 2$. So we may consider N as a transitive class in the real world.

The following theorem can be proved by using the basic lemma.

THEOREM (Lévy-Solovay [3]). *Let κ be a measurable cardinal and I be a non-trivial prime ideal on κ. Let $\mathscr{B}$ be a complete Boolean algebra such that* card$(\mathscr{B}) < \kappa$. *Then, it is $\mathscr{B}$-valid that J is a non-trivial prime ideal on κ.*

Proof. By the basic lemma, $\mathscr{D}_{[\mathscr{B}]}$ is isomorphic to $\mathscr{B}^*$. Let Φ be an isomorphism from $\mathscr{D}_{[\mathscr{B}]}$ to $\mathscr{B}^*$. Define $\Psi ; \mathscr{B} \to \mathscr{B}^*$ by $\psi(b) = b^*$. Trivially Ψ is injective. Let $\psi(f) \in \mathscr{B}^*$. We may suppose that $f ; \kappa \to \mathscr{B}$. Since card$(\mathscr{B}) < \kappa$, there is the unique $b \in \mathscr{B}$ such that $f(\alpha) = b$ for almost all $\alpha < \kappa$. Thus, $\psi(f) = \Psi(b)$. It follows that Ψ is bijective. Let $i = \Phi^{-1} \circ \Psi$. Let $b \in \mathscr{B}$. By easy computations, we have $\|(\Phi^{-1} \circ \Psi)(b) = \mathbf{1}_{\mathscr{D}}\| = b$ and $\|(\Phi^{-1} \circ \Psi)(b) = \mathbf{0}_{\mathscr{D}}\| = -b$. By Lemma 3 of §3, we have $\|\mathscr{D} = \mathbf{2}\| = \mathbf{1}$. That is, $\|J$ is prime$\|^{(\mathscr{B})} = \mathbf{1}$.

5. Proofs of Theorem 2 and 3.

5.1. Let $\mathscr{B}$ a complete Boolean algebra, and π be an automorphism of $\mathscr{B}$. Then, π induces the automorphism π_* of $V^{(\mathscr{B})}$.

LEMMA 1. *Let $\phi(v_0, \cdots, v_{n-1})$ be a set-theoretical formula, and let $x_0, \cdots, x_{n-1}$ be elements of $V^{(\mathscr{B})}$. Then,,*

$$\|\phi(\pi_*(x_0), \cdots, \pi_*(x_{n-1}))\| = \pi(\|\phi(x_0, \cdots, x_{n-1})\|) .$$

Proof. The lemma is easily proved by induction on the length of ϕ.

An element x of $V^{(\mathscr{B})}$ is called π-invariant if $x = \pi_*(x)$. x is called invariant if x is π-invariant for all automorphisms π of $\mathscr{B}$. For example, $\check{x}$ is invariant for each $x \in V$.

By using Lemma 1, the following lemma is trivial.

LEMMA 2. *Let $\phi(v_0, \cdots, v_{n-1})$ be a set-theoretical formula, and let $x_0, \cdots, x_{n-1}$ be invariant elements of $V^{(\mathscr{B})}$. Then,* $\|\phi(x_0, \cdots, x_{n-1})\| = \pi(\|\phi(x_0, \cdots, x_{n-1})\|)$.

5.2. Let $\mathscr{B}$ be a Boolean algebra. We consider the following condition (*).

(*) **0** and **1** are the only invariant elements of $\mathscr{B}$.

We say that a Boolean algebra $\mathscr{B}$ is homogeneous if: for every $\mathbf{0} < b$, $c < \mathbf{1}$, there exists an automorphism π such that $\pi(b) = c$. Clearly, if $\mathscr{B}$ is homogeneous, then $\mathscr{B}$ satisfies the condition (*).

LEMMA 3. *Let $\phi(v_0, \cdots, v_{n-1})$ be a set-theoretical formula, and $\mathscr{B}$ be a complete Boolean algebra satisfying the condition (*). Let $x_0, \cdots, x_{n-1}$ be invariant elements of $V^{(\mathscr{B})}$. Then, $\|\phi(x_0, \cdots, x_{n-1})\| = \mathbf{0}$ or $\mathbf{1}$.*

Proof. Suppose not. Put $\|\phi(x_0, \cdots, x_{n-1})\| = b$. Then, $\mathbf{0} < b < \mathbf{1}$. Since $\mathscr{B}$ satisfies the condition (*), there exists an automorphism π such that $\pi(b) \neq b$. Then,

$$\pi(\|\phi(x_0, \cdots, x_{n-1})\|) \neq \|\phi(x_0, \cdots, x_{n-1})\| .$$

This contradicts to Lemma 2.

Let $\mathscr{P}$ be a partially ordered set. We make $\mathscr{P}$ into a topological space by taking sets of the form

$$U_p = \{q \in \mathscr{P} ; q \leq p\}$$

as a basis for the open sets. Let $\mathscr{B}_{\mathscr{P}}$ be the complete Boolean algebra of regular open sets of $\mathscr{P}$. Let π be an automorphism of $\mathscr{P}$. Then, π induces the automorphism $\bar{\pi}$ of $\mathscr{B}_{\mathscr{P}}$ by $\bar{\pi}(U) = \{\pi(p) ; p \in U\}$.

LEMMA 4. *Let $\mathscr{P}$ be a partially ordered set satisfying the condition* (**).

(**) *Let p and q be elements of $\mathscr{P}$. Then, there is an automorphism π of $\mathscr{P}$ such that $\pi(p)$ and q are compatible.*

Then, $\mathscr{B}_{\mathscr{P}}$ satisfies the condition (*).

Proof. Suppose not. Then, there exists an element $\mathbf{0} < U < \mathbf{1}$ of such that $\pi(U) = U$ for all automorphisms π of $\mathscr{B}_{\mathscr{P}}$. Let p and q be elements of $\mathscr{P}$ such that $p \in U$ and $q \in \text{interior}(-U)$. Since $\mathscr{P}$ satisfies the condition (**) there exists an automorphism π of $\mathscr{P}$ such that $\pi(p)$ and q are compatible. Then, there exists an element r of $\mathscr{P}$ such that

$r \leq \pi(p)$ and $r \leq q$. Since $\bar{\pi}(U) = U$, $\pi(p) \in U$. By the fact that U is open, $r \in U$. Since $q \in \text{interior}(-U)$, $r \in -U$. This is a contradiction.

5.3. Let κ be an uncountable cardinal, and let I be a non-trivial ideal on κ. Let $J \in V^{(\mathscr{B})}$ be the ideal generated by $\check{I}$ inside $V^{(\mathscr{B})}$.

LEMMA 5. *J is invariant.*

Proof. Let π be an automorphism of $\mathscr{B}$. By Lemma 1, $\|\pi_*(J)$ is the ideal on $\pi_*(\check{\kappa})$ generated by $\pi_*(\check{I})\| = \mathbf{1}$. Since $\check{\kappa}$ and $\check{I}$ are invariant, $\|\pi_*(J)$ is the ideal on $\check{\kappa}$ generated by $\check{I}\| = \mathbf{1}$. Hence, $\|\pi_*(J) = J\| = \mathbf{1}$. Since $V^{(\mathscr{B})}$ is separate, $\pi_*(J) = J$.

5.4. Let κ and I be as in 5.3. Suppose that I is κ-saturated.

LEMMA 6. *Let $\mathscr{B}$ be a complete Boolean algebra satisfying the condition $(*)$. Suppose that* $\text{sat}(\mathscr{B}) = \kappa$. *Then, it is $\mathscr{B}$-valid that J is not κ-saturated.*

Proof. Suppose not. Since $\mathscr{B}$ satisfies the condition $(*)$, $\|J$ is $\check{\kappa}$-saturated$\|^{(\mathscr{B})} = \mathbf{1}$ by Lemma 3 and Lemma 5. Let $\mathscr{D} \in V^{(\mathscr{B})}$ such that $\|\mathscr{D} = P(\kappa)/J\|^{(\mathscr{B})} = \mathbf{1}$. By Lemma 2 of § 3, $\mathscr{D}_{[\mathscr{B}]}$ is κ-saturated. By the basic lemma, $\mathscr{B}^*_{[\mathscr{A}]}$ is κ-saturated. Then, $\|\mathscr{B}^*$ is κ-saturated$\|^{(\mathscr{A})} = \mathbf{1}$. Clearly, $\|N \models \mathscr{B}^*$ is κ-saturated$\|^{(\mathscr{A})} = \mathbf{1}$. Choose $f \in V^{\kappa}$ so that $\psi(f) = \check{\kappa}$. We may suppose that $f : \kappa \to \kappa$. The, for almost all $\alpha < \kappa$, $\mathscr{B}$ is $f(\alpha)$-saturated. Thus, $\text{sat}(\mathscr{B}) < \kappa$. This contradicts to the assumption of $\mathscr{B}$.

Now Theorem 2 is a corollary of Lemma 6.

5.5. Let κ be a measurable cardinal, and I be a non-trivial prime ideal on κ.

LEMMA 7. $2^{\kappa} < \kappa^*$.

Proof. Since $P(\kappa) = P(\kappa) \cap N$, $2^{\kappa} \leq 2^{\kappa(N)}$. On the other hand κ^* is measurable in N, so κ^* is strongly inaccessible in N. Hence, $2^{\kappa(N)} < \kappa^*$. Thus, $2^{\kappa} < \kappa^*$.

Theorem 3 is a corollary of the following lemma.

LEMMA 8. *Let $\mathscr{B}$ be a complete Boolean algebra satisfying the condition $(*)$. Assume that* $\text{sat}(\mathscr{B}) = \kappa$. *Let $J \in V^{(\mathscr{B})}$ be the ideal on $\check{\kappa}$ generated by $\check{I}$ inside $V^{(\mathscr{B})}$. Then, it is $\mathscr{B}$-valid that J is not κ^+-saturated.*

Proof. By using Lemma 7, the proof can be carried out analogously to the proof of Lemma 6. (Note that $\kappa^+ < \kappa^*$ by Lemma 7.).

5.6. We give an application of Lemma 8. Let κ and I be as in 5.5. We consider the following partially ordered set $\mathscr{P}$; $p \in \mathscr{P}$ if

1) p is a function
2) the domain of p is a finite subset of $\kappa \times \omega$
3) the range of $p \subseteq \kappa$
4) $p(\langle\alpha, n\rangle) < \alpha$ whenever $\langle\alpha, n\rangle \in$ domain (p).

The ordering of $\mathscr{P}$ is $\subseteq$. Clearly, $\mathscr{P}$ satisfies the condition $(**)$.

LEMMA 9.[1] $\mathrm{Sat}(\mathscr{B}_{\mathscr{P}}) = \kappa$. $\|\kappa = \aleph_1^{(\mathscr{B}_{\mathscr{P}})}\| = \mathbf{1}$.

By the theorem of § 2 and Lemma 9, $\|\check{\kappa}$ has no $\check{\kappa}$-saturated non-trivial ideal on $\kappa\| = \mathbf{1}$. On the other hand, by Lemma 8 we have $\|J$ is not an $\aleph_2^{(\mathscr{B}_{\mathscr{P}})}$-saturated ideal on $\check{\kappa} = \aleph_1^{(\mathscr{B}_{\mathscr{P}})}\| = \mathbf{1}$, where J is the ideal on κ generated by $\check{I}$ inside $V^{(\mathscr{B}_{\mathscr{P}})}$.

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1) See Solovay [6], p.15.