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CORES OF POTENTIAL OPERATORS FOR PROCESSES WITH STATIONARY INDEPENDENT INCREMENTS

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1. Introduction.

Let $X_t(\omega)$ be a stochastic process with stationary independent increments on the N-dimensional Euclidean space R^N, right continuous in $t \geqq 0$ and starting at the origin. Let $C_0(R^N)$ be the Banach space of real-valued continuous functions on R^N vanishing at infinity with norm $\|f\| = \sup_x |f(x)|$. The process induces a transition semigroup of operators T_t on $C_0(R^N)$:

$$T_t f(x) = Ef(x + X_t) .$$

The semigroup is strongly continuous. Let A be the infinitesimal generator of the semigroup, and J_λ, $\lambda > 0$, be the resolvent. The potential operator V in Yosida's sense [7] is defined by $Vf = \lim_{\lambda \to 0+} J_\lambda f$ (limit in the strong topology) if and only if the set of f for which the limit exists is dense. If V is defined, then A is one-to-one, $V = -A^{-1}$, and hence V is a closed operator (see [7] or [4]). It is proved in [4] that the semigroup T_t admits a potential operator except if $X_t = 0$ with probability one. A subset $\mathfrak{M}$ of $\mathfrak{D}(V)$ is called a core of V, if for each $f \in \mathfrak{D}(V)$ there is a sequence $\{f_n\}$ in $\mathfrak{M}$ such that $f_n \to f$ and $Vf_n \to Vf$ strongly. The purpose of this paper is to describe cores of the potential operator V. An importance of finding cores of V lies in the fact that the operator V considered only on a core is enough to determine the semigroup. That is, if two strongly continuous semigroups $T_t^{(1)}$ and $T_t^{(2)}$ have potential operators $V^{(1)}$ and $V^{(2)}$, respectively, and if $V^{(1)}$ and $V^{(2)}$ coincide on a common core, then $T_t^{(1)}$ and $T_t^{(2)}$ are identical.

Let Σ be the collection of points x such that for each open neigh-

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borhood B of x there is a $t > 0$ satisfying $P(X_t \in B) > 0$. Let $\mathfrak{G}$ be the smallest closed subgroup which includes Σ. Let $\boldsymbol{M}$ be the collection of measures μ on the Borel sets in R^N such that μ is finite for compact sets and is invariant under translation by every $x \in \mathfrak{G}$. Let $C_K = C_K(R^N)$ denote the set of continuous functions on R^N with compact supports. We will prove the following (Theorem 4.1): *If the process is transient, then the set of functions* $f \in C_K$ *such that*

$$\int_{R^N} f(x)\mu(dx) = 0 \qquad \textit{for every } \mu \in \boldsymbol{M} \tag{1.1}$$

is a core of the potential operator V. Under the conditions $\mathfrak{G} = R^N$ and $E|X_t|^\alpha < \infty$, we will make refinement of the above result (Theorem 5.1). Namely, we will prove that certain smaller sets are cores of V. We will further obtain similar results in recurrent non-singular case (Theorems 6.1 and 6.2), using results of Port and Stone [2]. If a moment of higher order exists, we can choose a smaller set as a core. This is not unnatural considering the following fact obtained from Port and Stone [2]: Suppose $N = 1$ and $\mathfrak{G} = R^N$. Then $\mathfrak{D}(V) \cap C_K$ is related with the existence of the first or second order moment. More precisely, let $\mathfrak{M}_0$ be the set of functions $f \in C_K(R^1)$ such that $\int f(x)dx = 0$, and $\mathfrak{M}_1$ be the set of $f \in C_K(R^1)$ such that $\int f(x)dx = \int f(x)xdx = 0$. In transient case,

$$\mathfrak{D}(V) \cap C_K = \begin{matrix} \mathfrak{M}_0 & \text{if } E|X_t| < \infty\ , \\ C_K & \text{if } E|X_t| = \infty\ ; \end{matrix}$$

and in recurrent non-singular case,

$$\mathfrak{D}(V) \cap C_K = \begin{matrix} \mathfrak{M}_1 & \text{if } EX_t^2 < \infty\ , \\ \mathfrak{M}_0 & \text{if } EX_t^2 = \infty\ . \end{matrix}$$

The following notations are used throughout this paper: d is the dimension of $\mathfrak{G}$; m is a Haar measure of $\mathfrak{G}$; ν is the Lévy measure (see Theorem 2.1); $C_K^\infty = C_K^\infty(R^N)$ is the set of C^∞ functions on R^N with compact supports; $x = (x_1, \cdots, x_N)$ and $|x| = (x_1^2 + \cdots + x_N^2)^{1/2}$; $B_a = \{y : |y| < a\}$, the open ball in R^N with radius a and center at the origin; especially B_1 is the open unit ball; B_a^c is the complement of B_a; χ_B is the indicator function of a set B; $B + x$ is the set $\{y + x : y \in B\}$; $B - x = B + (-x)$; $B + C = \{y + z : y \in B \text{ and } z \in C\}$; and $B \backslash C$ is the intersection of B and the complement of C.

2. Infinitesimal generators.

An explicit expression of Au for nice functions u has been known essentially from 1930s. We need the following result.

THEOREM 2.1. *Let A be the infinitesimal generator in $C_0(R^N)$ of the transition semigroup of a right continuous process with stationary independent increments. Then, $C_K^\infty \subset \mathfrak{D}(A)$ and C_K^∞ is a core of A. For each $u \in C_K^\infty$, Au is of the form*

$$
(2.1)\quad \begin{aligned} Au(x) &= \sum_{i,j=1}^{N} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{N} b_i \frac{\partial u}{\partial x_i}(x) \\ &+ \int_{R^N\setminus\{0\}} \left[u(x+y) - u(x) - \chi_{B_1}(y)\sum_{i=1}^{N} y_i \frac{\partial u}{\partial x_i}(x)\right]\nu(dy)\ , \end{aligned}
$$

where a_{ij} and b_i are constants, (a_{ij}) is a symmetric nonnegative definite matrix, and ν is a measure on $R^N\setminus\{0\}$ satisfying

$$
\nu(R^N\setminus B_1) < \infty\ , \qquad \int_{B_1\setminus\{0\}} |y|^2\nu(dy) < \infty\ .
$$

The constants a_{ij}, b_i and the measure ν are uniquely determined by A. Conversely, for every choice of such a_{ij}, b_i and ν, we can find a corresponding A.

The measure ν is called Lévy measure. A proof of the above theorem is given in [3]. Another proof is as follows: Let C_0^∞ be the set of C^∞ functions whose derivatives of all orders belong to $C_0(R^N)$. By Theorems 1 and 2 of Courrège [1], C_0^∞ is included in $\mathfrak{D}(A)$ and, for each $u \in C_0^\infty$, $Au(x)$ is of the form (2.1). Since C_0^∞ is dense and mapped by T_t into itself, C_0^∞ is a core of A by Lemma 2.2 of Shinzo Watanabe [6]. For each $u \in C_0^\infty$, it is easy to find a sequence $u_n \in C_K^\infty$ such that $u_n \to u$, $\partial u_n/\partial x_i \to \partial u/\partial x_i$ and $\partial^2 u_n/\partial x_i x_j \to \partial^2 u/\partial x_i \partial x_j$ strongly for all i and j. It follows from (2.1) that $Au_n \to Au$ strongly. Hence C_K^∞ is a core of A. The converse part is obtained from Theorem 4 of [1].

As we pointed out in Introduction, a potential operator V is associated with A, unless $X_t = 0$ with probability one, that is, unless A is the zero operator. Since $V = -A^{-1}$, the following result is immediate.

COROLLARY 2.1. *The set $\{Au : u \in C_K^\infty\}$ is a core of V.*

3. General lemmas.

In this section $X_t(\omega)$ is the process described in Introduction and no further conditions are imposed. We will give lemmas which we need in the following sections.

LEMMA 3.1. *If $\mu \in \boldsymbol{M}$, $f \in C_0(R^N)$, and f is μ-integrable, then $J_\lambda f$ is μ-integrable and*

$$\lambda\int J_\lambda f(x)\mu(dx) = \int f(x)\mu(dx) . \tag{3.1}$$

Hence every $\mu \in \boldsymbol{M}$ is an invariant measure for the process.

Proof. It suffices to prove (3.1) for $f \geqq 0$. Let κ_λ be a probability measure defined by

$$\kappa_\lambda(B) = \lambda\int_0^\infty e^{-\lambda t}P(X_t \in B)dt .$$

Then κ_λ is supported in Σ, and

$$\lambda J_\lambda f(x) = \int f(x+y)\kappa_\lambda(dy) .$$

It follows from $\mu \in \boldsymbol{M}$ that

$$\int f(x + y)\mu(dx) = \int f(x)\mu(dx) \qquad \text{for } y \in \mathfrak{G} . \tag{3.2}$$

Hence we have (3.1) by Fubini's theorem.

LEMMA 3.2. *Let $\mu \in \boldsymbol{M}$ and $u \in \mathfrak{D}(A)$. If u and Au are μ-integrable, then Au has μ-integral null.*

Proof. We have

$$\int Au(x)\mu(dx) = \lambda\int J_\lambda Au(x)\mu(dx) = \lambda^2\int J_\lambda u(x)\mu(dx) - \lambda\int u(x)\mu(dx) = 0$$

by Lemma 3.1.

LEMMA 3.3. *The Lévy measure ν is supported in Σ.*

Proof. The set Σ obviously contains the origin. Suppose that x^0 is a point $\neq 0$ in the support of ν. Given ε such that $0 < \varepsilon < |x^0|$, let $\nu^{(1)}$ be the restriction of ν to $x^0 + B_\varepsilon$, and let

$$A^{(1)}u(x) = \int (u(x+y) - u(x))\nu^{(1)}(dy) \ , \qquad A^{(2)}u(x) = Au(x) - A^{(1)}u(x)$$

for $u \in C_K^\infty$. We can assume $X_t = X_t^{(1)} + X_t^{(2)}$, where $X_t^{(1)}$ and $X_t^{(2)}$ are independent processes generated by $A^{(1)}$ and $A^{(2)}$, respectively. Let β be the total mass of $\nu^{(1)}: \beta = \nu(x^0 + B_\varepsilon) > 0$. The process $X_t^{(1)}$ is a compound Poisson with jumping measure $\beta^{-1}\nu^{(1)}$, that is, $X_t^{(1)} = \sum_{n=1}^{Y_t} Z_n$, where $\{Z_n\}$ are independent identically distributed random variables, each Z_n has distribution $\beta^{-1}\nu^{(1)}$, and Y_t is a Poisson process with mean $EY_t = \beta t$, independent of $\{Z_n\}$. We have

$$P(|X_t - x^0| < 2\varepsilon) \geqq P(|X_t^{(1)} - x^0| < \varepsilon)P(|X_t^{(2)}| < \varepsilon) \ ,$$

$$P(|X_t^{(1)} - x^0| < \varepsilon) \geqq P(Y_t = 1)P(|Z_1 - x^0| < \varepsilon) > 0 \ ,$$

and also $P(|X_t^{(2)}| < \varepsilon) > 0$ for small $t > 0$. Hence $x^0 \in \Sigma$ and the lemma is proved.

LEMMA 3.4. *If u is in $C_K^\infty(R^N)$ with support in B_a, then*

$$Au(x) = 0 \qquad \text{for } x \notin \mathfrak{S} + B_a \tag{3.3}$$

$$|Au(x)| \leqq \|u\|\nu(B_a - x) \qquad \text{for } x \notin B_a \ , \tag{3.4}$$

and

$$\begin{aligned} &\int_{|x+y|\geqq b} |x+y|^\alpha |Au(x+y)|\mu(dy) \\ &\qquad \leqq \|u\| \int_{B_{b-a}^c} (a + |z|)^\alpha \nu(dz) \sup_{y \in \mathfrak{S}} \mu(B_a + y - x) \end{aligned} \tag{3.5}$$

for an arbitrary measure μ on R^N, $x \in R^N$, $b > a$, and $\alpha \geqq 0$.

Proof. The assertion (3.3) follows from (3.4) by Lemma 3.3. We have from Theorem 2.1

$$Au(x) = \int u(x+y)\nu(dy) \qquad \text{for } x \notin B_a \ , \tag{3.6}$$

which implies (3.4). Let us prove (3.5). We may assume $x = 0$, because for a general x we need only consider μ_x defined by $\mu_x(B) = \mu(B - x)$ instead of μ. We have

$$\begin{aligned} \int_{B_b^c} |y|^\alpha |Au(y)|\mu(dy) &\leqq \|u\| \int_{B_b^c} |y|^\alpha \nu(B_a - y)\mu(dy) \\ &= \|u\| \int_\Sigma \nu(dz) \int_{B_b^c} |y|^\alpha \chi_{B_a}(y+z)\mu(dy) \end{aligned} \tag{3.7}$$

by using (3.4), Lemma 3.3 and Fubini's theorem. If $y + z \in B_a$ and $y \in B_b^c$, then $|z| > b - a$ and $|y| < |z| + a$. Hence the last member in (3.7) is not larger than

$$\|u\| \int_{\Sigma \cap B_{b-a}^c} (|z| + a)^\alpha \mu(B_a - z)\nu(dz) ,$$

from which follows (3.5) for $x = 0$. The proof is complete.

LEMMA 3.5. *If $u \in C_K^\infty$ and $\mu \in \boldsymbol{M}$, then Au is μ-integrable and has μ-integral null.*

Proof. Suppose that u has support in B_a. We use the estimate (3.5) with $x = 0$ and $\alpha = 0$. Since

$$\sup_{y \in \mathfrak{G}} (B_a + y) = \mu(B_a) < \infty ,$$

the right-hand side of (3.5) is finite. Hence Au is μ-integrable. The μ-integral vanishes by Lemma 3.2.

LEMMA 3.6. *Let $f \in C_K(R^N)$. Then,* (1.1) *holds if and only if*

$$\int_{\mathfrak{G}} f(x + y)m(dy) = 0 \qquad \textit{for every } x \in R^N . \tag{3.8}$$

Proof. Since for every $x \in R^N$ a measure m_x defined by $m_x(B) = m((B - x) \cap \mathfrak{G})$ is a member of $\boldsymbol{M}$, (1.1) implies (3.8). Let us prove the converse. We can find a Borel set H such that every $z \in R^N$ is uniquely represented as $z = x + y$, $x \in \mathfrak{G}$, $y \in H$. Let $\mu \in \boldsymbol{M}$. Fix a Borel set B^0 in $\mathfrak{G}$ such that $0 < m(B^0) < \infty$ and define a measure μ' on H by

$$\mu'(C) = m(B^0)^{-1}\mu(B^0 + C) \qquad \text{for } C \subset H .$$

For Borel sets $B \subset \mathfrak{G}$ and $C \subset H$, we have

$$\mu(B + C) = m(B)\mu'(C) .$$

In fact, since $\mu(B + y + C) = \mu(B + C)$ for $y \in \mathfrak{G}$, we have $\mu(B + C) = \text{const}\, m(B)$ for a fixed C. The constant is no other than $\mu'(C)$. Therefore, we have

$$\int_{R^N} g(z)\mu(dz) = \int_H \int_{\mathfrak{G}} g(x + y)m(dx)\mu'(dy)$$

for every nonnegative measurable g. Hence, if (3.8) holds, then f has μ-integral null by Fubini's theorem. The proof is complete.

Let $h(\xi)$ be a continuous function on $[0, \infty)$ such that $h(\xi)$ is 1 for $0 \leqq \xi \leqq 1$, 0 for $\xi \geqq 4$, and $0 < h(\xi) < 1$ for $1 < \xi < 4$. Let $h_n(x) = h(|x|^2/n^2)$ for $n \geqq 1$.

LEMMA 3.7. *Given $u \in C_K^\infty(R^N)$ and $f = Au$, define*

$$g_n(x) = -\int_{\mathfrak{G}} f(x + y)h_n(x + y)m(dy)\Big/\int_{\mathfrak{G}} h_n(x + y)m(dy) \ , \tag{3.9}$$

$$f_n(x) = (f(x) + g_n(x))h_n(x) \ , \tag{3.10}$$

where we understand $g_n(x) = 0$ when the denominator in (3.9) vanishes. Then, $f_n \in C_K(R^N)$, f_n has μ-integral null for every $\mu \in \boldsymbol{M}$, and

$$\sup_{x \in R^N} |g_n(x)| = o(n^{-d}) \qquad as \ n \to \infty \ , \tag{3.11}$$

$$\|f_n - f\| \to 0 \qquad as \ n \to \infty \ . \tag{3.12}$$

Proof. The function $g_n(x)h_n(x)$ vanishes if $|x| > 2n$. If $|x| = 2n$ and $x' \to x$, then $h_n(x) = 0$ and $g_n(x')h_n(x') \to 0$ since $|g_n(x')| \leqq \|f\|$. If $|x| < 2n$, then the denominator in (3.9) is positive and $g_n(x)$ is continuous at x. Hence $f_n \in C_K$. We have

$$\int_{\mathfrak{G}} f_n(x + y)m(dy) = \int_{\mathfrak{G}} f(x + y)h_n(x + y)m(dy) + g_n(x)\int_{\mathfrak{G}} h_n(x + y)m(dy) = 0$$

for $x \in R^N$, since $g_n(x + y) = g_n(x)$ for $y \in \mathfrak{G}$. It follows that f_n has μ-integral null for $\mu \in \boldsymbol{M}$ by Lemma 3.6. Suppose that u has support in B_a. Let $D_a = \mathfrak{G} + B_a$. If $x \notin D_a$, then $x + y \notin D_a$ for $y \in \mathfrak{G}$ and $g_n(x) = 0$ by (3.3) in Lemma 3.4. Let $x \in D_a$ and let us give estimation of $g_n(x)$. We have $x = x^0 + x^1$ with $x^0 \in \mathfrak{G}$ and $|x^1| < a$, and hence

$$\int_{\mathfrak{G}} h_n(x + y)m(dy) \geqq m\{y \in \mathfrak{G} : |x + y| \leqq n\}$$
$$= m\{y \in \mathfrak{G} : |x^1 + y| \leqq n\} \geqq m\{y \in \mathfrak{G} : |y| \leqq n - a\} \geqq c(n - a)^d$$

with a positive constant c. Noting that f satisfies (3.8) by Lemma 3.5, we observe that

$$\left|\int_{\mathfrak{G}} f(x + y)h_n(x + y)m(dy)\right| = \left|\int_{\mathfrak{G}} f(x + y)(1 - h_n(x + y))m(dy)\right|$$
$$\leqq \int_{|x+y|>n} |f(x + y)|m(dy) \leqq \|u\|\nu(B_{n-a}^c)m(B_a - x)$$

by using Lemma 3.4. The last member tends to zero as $n \to \infty$ uniformly in $x \in D_a$. Thus we get (3.11). The assertion (3.12) follows from (3.11) and $f \in C_0$, since

$$\|f_n - f\| \leqq \sup_{|x|>n} |f(x)| + \sup_{x \in R^N} |g_n(x)| .$$

LEMMA 3.8. *If $u \in C_K^\infty(R^N)$, then Au is a C^∞ function.*

Proof. Using the expression (2.1) of Au in Theorem 2.1, we can see that Au is continuously differentiable and

$$\frac{\partial}{\partial x_i} Au = A\left(\frac{\partial u}{\partial x_i}\right).$$

Hence Au is a C^∞ function by induction.

LEMMA 3.9. *Let $\alpha > 0$. If*

$$E|X_t|^\alpha < \infty \tag{3.13}$$

holds for some $t > 0$, then it holds for every $t > 0$ and

$$\int_{R^N} |x|^\alpha |Au(x)| dx < \infty \tag{3.14}$$

for every $u \in C_K^\infty$. If

$$E|X_t| < \infty , \tag{3.15}$$

then

$$\int_{R^N} x_i Au(x) dx = -(EX_t^{(i)}) \int_{R^N} u(x) dx \tag{3.16}$$

for every $u \in C_K^\infty$, where $X_t^{(i)}$ is the i-th component of X_t.

Proof. Let $\phi_t(\xi)$ be the characteristic function of the distribution of X_t:

$$\phi_t(\xi) = E \exp\left(\sqrt{-1} \sum_{i=1}^N \xi_i X_t^{(i)}\right) \qquad \text{for } \xi = (\xi_1, \cdots, \xi_N) \in R^N .$$

Then, it is known that

$$\begin{aligned} \phi_t(\xi) = \exp \Bigg[t \Bigg(& - \sum_{i,j=1}^N a_{ij} \xi_i \xi_j + \sqrt{-1} \sum_{i=1}^N b_i \xi_i \\ & + \int_{R^N \setminus \{0\}} (e^{\sqrt{-1}\xi y} - 1 - \chi_{B_1}(y) \sqrt{-1}\, \xi y) \nu(dy) \Bigg) \Bigg] , \end{aligned} \tag{3.17}$$

where $\xi y = \sum_{i=1}^{N} \xi_i y_i$. Hence $E|X_t|^\alpha$ is finite if and only if

$$\int_{|y|>1} |y|^\alpha \nu(dy) < \infty \tag{3.18}$$

by the result of [5]. Therefore, if (3.13) holds for some $t > 0$, then it holds for every t and (3.14) holds by Lemma 3.4. If (3.15) holds, then we get on the one hand

$$\int_{R^N} x_i Au(x)dx = -\left(b_i + \int_{|y|\geqq 1} y_i \nu(dy)\right)\int_{R^N} u(x)dx$$

by elementary calculation from (2.1), and

$$EX_1^{(i)} = -\sqrt{-1}\frac{\partial \phi_1}{\partial \xi_i}(0) = b_i + \int_{|y|\geqq 1} y_i \nu(dy)$$

from (3.17) on the other hand. Hence (3.16).

4. Transient case.

We assume that X_t is transient. Let U be a measure defined by

$$U(B) = \int_0^\infty P(X_t \in B)dt .$$

This measure is finite for compact sets and concentrated on Σ. We need the following analogue of the Blackwell-Feller-Orey renewal theorem.

PROPOSITION 4.1. (Port-Stone [2]) (i) *Suppose that* $d \geqq 2$ *or suppose that* $d = 1$ *and* $E|X_t| = \infty$. *Then,*

$$\lim_{x \in \mathfrak{G}, |x| \to \infty} U(B + x) = 0 \tag{4.1}$$

for every bounded Borel set B. (ii) *Suppose that* $d = 1$ *and* $E|X_t| < \infty$. *Assume* $N = 1$ *for simplicity of statement. If* $\pm EX_t > 0$, *then*

$$\lim_{x \in \mathfrak{G}, x \to \pm\infty} U(B + x) = cm(B) , \qquad \lim_{x \in \mathfrak{G}, x \to \mp\infty} U(B + x) = 0 \tag{4.2}$$

with a finite positive constant c *for every bounded Borel subset* B *of* $\mathfrak{G}$ *such that the boundary of* B *in the relative topology of* $\mathfrak{G}$ *has zero* m*-measure.*

As a consequence, we have

$$\sup_{x \in R^N} U(B + x) < \infty \tag{4.3}$$

for every bounded Borel set B, if only transient. Note that if $d = 1$, $E|X_t| < \infty$ and $EX_t = 0$, then it is recurrent.

We will prove the following result.

THEOREM 4.1. *If X_t is transient, then the set $\mathfrak{M}$ of functions in $C_K(R^N)$ which have μ-integral null for every $\mu \in \boldsymbol{M}$ is a core of the potential operator V.*

LEMMA 4.1. *If $f \in \mathfrak{M}$, then $f \in \mathfrak{D}(V)$ and*

$$Vf(x) = \int f(x + y)U(dy) \ . \tag{4.4}$$

Proof. Suppose that f has support in B_a. Let $g(x)$ be the right-hand side of (4.4). This is a uniformly continuous function. In fact, for a given $\varepsilon < 0$, let δ be such that $0 < \delta < 1$ and $|f(x) - f(x')| < \varepsilon$ if $|x - x'| < \delta$. Then we have

$$|g(x) - g(x')| \leqq \varepsilon U(B_{a+1} - x) \leqq \text{const } \varepsilon$$

by (4.3). Suppose that

$$\lim_{|x| \to \infty} g(x) = 0 \tag{4.5}$$

is proven. Since we have

$$|J_\lambda f(x)| \leqq \int |f(x + y)| U(dy) \leqq \|f\| U(B_a - x) \ ,$$

which is bounded by (4.3), $J_\lambda f(x)$ tends to $g(x)$ boundedly and pointwise as $\lambda \to 0$; in other words $J_\lambda f$ tends weakly to g, and hence $f \in \mathfrak{D}(V)$ and $Vf = g$ by Theorem 2.4 of [4]. Let us prove (4.5). First, it follows from Proposition 4.1 and $f \in \mathfrak{M}$ that

$$\lim_{x \in \mathfrak{G}, |x| \to \infty} g(x + y) = 0 \tag{4.6}$$

for each fixed $y \in R^N$. Let $D_a = \mathfrak{G} + B_a$, the a-neighborhood of $\mathfrak{G}$. We can find a Borel set H such that every $z \in R^N$ is uniquely represented as $z = x + y$, $x \in \mathfrak{G}$, $y \in H$, and that $H \cap D_a \subset B_b$ for some $b > 0$. We claim that the convergence in (4.6) is uniform in $y \in H$. If $y \notin D_a$, then $g(x + y) = 0$ for $x \in \mathfrak{G}$. For a given $\varepsilon > 0$, we can find by the uniform continuity a $\delta > 0$ such that $|g(z) - g(z')| < \varepsilon$ if $|z - z'| < \delta$. Let $y^0 \in H \cap D_a$. If $x \in \mathfrak{G}$ and $|x|$ is large enough, then

$$|g(x+y)| < |g(x+y^0)| + \varepsilon < 2\varepsilon$$

for all y such that $|y - y^0| < \delta$. Since $H \cap D_a$ is a bounded set, it follows that (4.6) holds uniformly in $y \in H$. Given $\varepsilon > 0$, let $p > 0$ be such that if $x \in \mathfrak{G}$ and $|x| > p$, then $|g(x+y)| < \varepsilon$ for all $y \in H$. If $|z| > p + b$, then $z = x + y$, $x \in \mathfrak{G}$, $y \in H$, where $y \notin D_a$ or $|y| < b$. In either case we have $|g(z)| < \varepsilon$. Hence (4.5) is proved.

Proof of Theorem 4.1. We have $\mathfrak{M} \subset \mathfrak{D}(V)$ by the above lemma. Hence, by virtue of Corollary 2.1, it is enough to prove that for each $u \in C_K^\infty$ there are a sequence $\{f_n\}$ in $\mathfrak{M}$ and a g in C_0 such that $f_n \to Au$ and $Vf_n \to g$ strongly as $n \to \infty$. Let $f = Au$ and let f_n be the one defined by (3.10). Then, by Lemmas 3.7 and 4.1, we have $f_n \in \mathfrak{M}$, $f_n \to f$, and

$$Vf_n(x) = \int f_n(x+y)U(dy) . \tag{4.7}$$

Let

$$g(x) = \int f(x+y)U(dy) . \tag{4.8}$$

The integral exists by (4.3) and Lemma 3.4. We claim

$$\lim_{n\to\infty} Vf_n(x) = g(x) \qquad \text{uniformly in } x \in R^N . \tag{4.9}$$

It follows from (3.10) and (4.7) that

$$|Vf_n(x) - g(x)| \leqq \int_{|x+y|>n} |f(x+y)|U(dy) + \sup_z |g_n(z)| \int h_n(x+y)U(dy) .$$

The first term of the right-hand side tends to zero as $n \to \infty$ uniformly in x by (3.5) and (4.3), while the second term also tends to zero uniformly in x by (3.11), since we have

$$\begin{aligned} \sup_{x\in R^N} \int h_n(x+y)U(dy) &\leqq \sup_{x\in R^N} U(x+B_{2n}) = \sup_{x\in\mathfrak{G}_1} U(x+B_{2n}) \\ &\leqq cn^d \sup_{x\in\mathfrak{G}_1} U(x+B_1) \leqq c'n^d \end{aligned} \tag{4.10}$$

by (4.3), where $\mathfrak{G}_1$ is the d-dimensional Euclidean subspace including $\mathfrak{G}$, and c and c' are constants. Hence we get (4.9), which proves that $g \in C_0$ and $\|Vf_n - g\| \to 0$. The proof is complete.

5. Refinement in transient case.

We assume transience and $\mathfrak{S} = R^N$ in this section. We say that a function $\phi(x)$ is α order homogeneous outside a compact set, if there is a $b > 0$ such that

$$\phi(\lambda x) = \lambda^\alpha \phi(x) \qquad \text{for } |x| \geqq b,\ \lambda \geqq 1 .$$

For such a function ϕ we define the homogeneous modification

$$\tilde{\phi}(x) = \left(\frac{|x|}{b}\right)^\alpha \phi\left(\frac{bx}{|x|}\right).$$

Note that $\phi(x) = \tilde{\phi}(x)$ for $|x| \geqq b$.

THEOREM 5.1. *Suppose $E|X_t|^\alpha < \infty$ for a real number $\alpha > 0$. Let $\phi_i(x)$, $1 \leqq i \leqq l$, be an arbitrary number of continuous functions on R^N such that ϕ_i is α_i order homogeneous outside a compact set, $0 < \alpha_i \leqq \alpha$, and the set of the homogeneous modifications $\{\tilde{\phi}_i(x): 1 \leqq i \leqq l\}$ is linearly independent. Given real numbers a_i, $1 \leqq i \leqq l$, let $\mathfrak{M}$ be the set of functions $f \in C_K^\infty(R^N)$ such that*

$$\int_{R^N} f(x)dx = 0, \qquad \int_{R^N} f(x)\phi_i(x)dx = a_i \qquad \text{for } 1 \leqq i \leqq l . \tag{5.1}$$

Then, $\mathfrak{M}$ is a core of the potential operator V.

Proof. The set $\mathfrak{M}$ is included in $\mathfrak{D}(V)$, since M consists only of multiples of the Lebesgue measure of R^N in the present case. Using a C^∞ function $h(\xi)$, let $h_n(x)$ be the function given in Section 3. Let $u \in C_K^\infty$ and $f = Au$. By Lemma 3.8, f is a C^∞ function. Let $\psi_0(x) \equiv 1$ and let $\psi_i(x)$, $1 \leqq i \leqq l$, be C^∞ functions on R^N, α_i order homogeneous outside a compact set for each i. Let

$$f_n(x) = \left(f(x) + \sum_{j=0}^{l} b_{jn}\psi_j(x)\right) h_n(x) . \tag{5.2}$$

Surely f_n is in C_K^∞. We want to determine constants b_{jn} so that $f_n \in \mathfrak{M}$ and prove

$$\|f_n - f\| \to 0 , \tag{5.3}$$

$$\|Vf_n - g\| \to 0 \tag{5.4}$$

for g defined by (4.8). Let $a_0 = \alpha_0 = 0$. We have $f_n \in \mathfrak{M}$ if and only if

$$\int f(x)\phi_i(x)h_n(x)dx + \sum_{j=0}^{l} b_{jn}\int \phi_i(x)\psi_j(x)h_n(x)dx = a_i \,, \qquad 0 \leqq i \leqq l \,, \tag{5.5}$$

where $\phi_0 \equiv 1$. We have

$$\begin{aligned}\int \phi_i(x)\psi_j(x)h_n(x)dx &= n^N \int \phi_i(nx)\psi_j(nx)h_1(x)dx \\ &= n^N \int_{|x|\geqq b/n} \tilde{\phi}_i(nx)\tilde{\psi}_j(nx)h_1(x)dx \\ &\quad + n^N \int_{|x|<b/n} \phi_i(nx)\psi_j(nx)h_1(x)\, dx \,,\end{aligned}$$

hence

$$n^{-N-\alpha_i-\alpha_j}\int \phi_i(x)\psi_j(x)h_n(x)dx \to \int \tilde{\phi}_i(x)\tilde{\psi}_j(x)h_1(x)dx \tag{5.6}$$

as $n \to \infty$. It follows that

$$\begin{aligned}&n^{-N(l+1)-2\beta} \det\left(\int \phi_i(x)\psi_j(x)h_n(x)dx\right)_{i,j=0,\cdots,l} \\ &\longrightarrow c = \det\left(\int \tilde{\phi}_i(x)\tilde{\psi}_j(x)h_1(x)dx\right)_{i,j=0,\cdots,l}\end{aligned} \tag{5.7}$$

where $\beta = \sum_{i=1}^{l} \alpha_i$. Using Weierstrass' theorem, we choose the functions ψ_i in such a manner that $\max_{|x|=b} |\phi_i(x) - \psi_i(x)|$ $(1 \leqq i \leqq l)$ are so small that c is positive. This is possible because we have

$$\det\left(\int \tilde{\phi}_i(x)\tilde{\phi}_j(x)h_1(x)dx\right)_{i,j=0,\cdots,l} > 0$$

since it is the Gramian of $\{\tilde{\phi}_i(x)h_1(x)^{1/2}\}$ and the functions $\tilde{\phi}_i(x)$ restricted to $|x| < 2n$ are still linearly independent. Thus, for sufficiently large n, $\{b_{jn} : 0 \leqq j \leqq l\}$ which satisfies (5.5) uniquely exists. We have

$$\int f(x)h_n(x)dx = o(1) \quad \text{and} \quad \int f(x)\phi_i(x)h_n(x)dx = O(1) \tag{5.8}$$

as $n \to \infty$ by Lemma 3.5 and by

$$\int |x|^\alpha |f(x)|dx < \infty \,, \tag{5.9}$$

which follows from the assumption $E|X_t|^\alpha < \infty$ by Lemma 3.9. Hence we can easily check that

$$b_{jn} = o(n^{-N-\alpha_j}) \qquad \text{for } 0 \leqq j \leqq l \,, \tag{5.10}$$

solving the linear equations (5.5) and using (5.6) and (5.7). It follows that

$$\|f_n - f\| \leqq \sup_{|x|>n} |f(x)| + \sum_{j=0}^{l} |b_{jn}|(2n)^{\alpha_j} = \sup_{|x|>n} |f(x)| + o(n^{-N}) .$$

Further we have

$$|Vf_n(x) - g(x)| \leqq \int_{|x+y|>n} |f(x+y)|U(dy) + \text{const} \sum_{j=0}^{l} |b_{jn}| n^{\alpha_j} \int h_n(x+y)U(dy)$$

using (4.7) and (4.8), and see that the right-hand side tends to zero uniformly in x using (3.5) and (4.3) for the first term, and using (4.10) and (5.10) for the second term. Hence we get (5.3) and (5.4), completing the proof.

6. Recurrent case.

Let X_t be recurrent. In addition we assume that X_t is non-singular in the sense that for some t the distribution of X_t has non-trivial absolutely continuous part. We have necessarily $\mathfrak{G} = R^N$ and $N = 1$ or 2. Port and Stone give the following result.

PROPOSITION 6.1. (Port-Stone [2], Section 17) *If f is bounded, measurable, vanishes outside a compact set, and has null integral, then $\int_0^\infty e^{-\lambda t} Ef(x + X_t)dt$ is bounded uniformly in $\lambda > 0$ and tends to a function $g(x)$ as $\lambda \to 0$. The convergence is uniform on every compact set. There are a continuous function $a(x)$ and a finite measure μ_2 such that the following hold:* (i) *The function g is represented by*

$$g(x) = -\int f(x+y)a(y)dy - \int f(x+y)\mu_2(dy) . \tag{6.1}$$

(ii) *If $N = 2$ or if $N = 1$ and $E|X_t|^2 = \infty$, then*

$$\lim_{|x|\to\infty} (a(x+y) - a(x)) = 0 \tag{6.2}$$

uniformly in y on every compact set. (iii) *If $N = 1$ and $E|X_1|^2 = \sigma^2 < \infty$, then*

$$\lim_{x\to\pm\infty} (a(x+y) - a(x)) = \pm y/\sigma^2 \tag{6.3}$$

uniformly in y on every compact set

The following is a direct consequence of the above result. Noting that (6.1) is written as

$$g(x) = -\int f(y)(a(y-x) - a(-x))dy - \int f(x+y)\mu_2(dy) , \tag{6.4}$$

and recalling Theorem 2.4 of [4], we see that if $f \in C_K(R^N)$ and

$$\int f(x)dx = \int f(x)x_i dx = 0 \qquad \text{for } 1 \leqq i \leqq N , \tag{6.5}$$

then $g \in C_0(R^N)$, $f \in \mathfrak{D}(V)$ and $Vf = g$. Also, (6.2) as well as (6.3) imply

$$\sup_{x \in R^N} |a(x+y) - a(x)| \leqq \text{const} (|y| + 1). \tag{6.6}$$

THEOREM 6.1. *If $E|X_t| < \infty$, then the set of functions $f \in C_K^\infty$ satisfying* (6.5) *is a core of the potential operator V.*

The proof is obtained by a simplification of the proof of the following theorem with trivial changes.

THEOREM 6.2. *Suppose that $E|X_t|^\alpha < \infty$ for an $\alpha > 1$. Let $\phi_i(x)$, $N+1 \leqq i \leqq l$ be an arbitrary number of continuous functions such that ϕ_i is α_i order homogeneous outside a compact set for some α_i satisfying $1 < \alpha_i \leqq \alpha$ and the set of the homogeneous modifications $\{\tilde{\phi}_i : N+1 \leqq i \leqq l\}$ is linearly independent. Given real numbers a_i, $N+1 \leqq i \leqq l$, let $\mathfrak{M}$ be the set of functions $f \in C_K^\infty(R^N)$ which satisfy* (6.5) *and*

$$\int f(x)\phi_i(x)dx = a_i \qquad \textit{for } N+1 \leqq i \leqq l . \tag{6.7}$$

Then, $\mathfrak{M}$ is a core of V.

Proof. Let $\phi_0(x) \equiv 1$, $\alpha_0 = 0$, $\phi_i(x) = x_i$, $\alpha_i = 1$ for $1 \leqq i \leqq N$, and $a_i = 0$ for $0 \leqq i \leqq N$. Given $u \in C_K^\infty$, $f = Au$, define f_n by (5.2). By the same argument as in the proof of Theorem 5.1, we can determine for large n the constants b_{jn} in (5.2) in such a way that $f_n \in \mathfrak{M}$. We have also (5.8). This time we need a stronger result:

$$\left|\int f(x)h_n(x)dx\right| \leqq \int_{|x|>n} |f(x)|dx \leqq n^{-\alpha}\int_{|x|>n} |x|^\alpha |f(x)|dx = o(n^{-\alpha}) .$$

Noting that X_t has mean 0 by the recurrence and $E|X_t| < \infty$ and using Lemma 3.9, we have similarly

$$\int f(x)x_i h_n(x)dx = o(n^{1-\alpha}) .$$

Therefore we obtain

$$b_{jn} = o(n^{-N-1-\alpha_j}) , \qquad \text{for } 0 \leqq j \leqq l \tag{6.8}$$

from (5.5) in the same way as we get (5.10). Thus (5.3) is obvious. Define $g(x)$ by (6.4). Existence of the first integral in (6.4) follows from (5.9) and (6.6). Expressing Vf_n in the form of (6.4), we have

$$|Vf_n(x) - g(x)| \leqq \left| \int_{|y|>n} f(y)(a(y-x) - a(-x))dy \right|$$
$$+ \sum_{j=0}^{l} \left| b_{jn} \int_{|y|<2n} \psi_j(y)(a(y-x) - a(-x))dy \right| + \|f_n - f\| \mu_2(R^N) .$$

In the right side, the first term tends to zero uniformly in x by (5.9) and (6.6), and so does the second term by (6.8) and by

$$\int_{|y|<2n} \psi_j(y)(a(y-x) - a(-x))dy = O(n^{N+1+\alpha_j}) ,$$

which follows from (6.6). Hence we get (5.4), and the proof is complete.

Even if X_t is recurrent and non-singular, we do not know a core which can be explicitly described of the potential operator in the case $E|X_t| = \infty$. In order to find such, it is desirable to get information on the relation between behavior of $|a(y+x) - a(x)|$ for large $|x|$ and mass distribution of the Lévy measure ν in neighborhoods of infinity. An example is the Cauchy process on R^1 with or without drift, for which we have

$$|a(y+x) - a(x)| \leqq \text{const}\,(|\log|(1+y)/x|| + 1)$$

and $\nu(dy) = \text{const}\, y^{-2}dy$, and the set of functions in C_K^∞ with integral null is a core of the potential operator (Example 5.4 of [4]).

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