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POSITIVENESS OF THE REPRODUCING KERNEL IN THE SPACE *PD*(*R*)

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An important problem in the study of the Hilbert space $PD(R)$ of Dirichlet finite solutions of $\Delta u = Pu$ on a Riemann surface R is to know the behavior of the reproducing kernel in $PD(R)$. The main result of this paper is that the reproducing kernel is strictly positive.

1. Let $P(z)dxdy\,(z = x + iy)$ be a nonnegative not identically zero α-Hölder continuous $(0 < \alpha \leq 1)$ second order differential on a Riemann surface R. We also assume that $R \notin O_{PD}$, i.e. there exists a nontrivial Dirichlet finite solution of

$$\Delta u(z) = P(z)u(z) \tag{1}$$

on R. If we mean by the scalar product of $u, v \in PD(R)$ the Dirichlet scalar product $(u, v) = D_R[u, v] = \int_R du \wedge *dv$ then $PD(R)$ is a Hilbert space; and as shown by Nakai [2], $PD(R)$ is then uniformly locally bounded on R. Hence there exists a unique reproducing kernel in $PD(R)$ which is a symmetric function on $R \times R$. Denote this kernel by $K(z, \zeta)$.

To show the positiveness of $K(z, \zeta)$ on $R \times R$ it will be enough to examine the kernel at a point z_0, i.e. the function $K(z, z_0)$, where $z_0 \in R$ is an arbitrary but fixed point. From now on, z_0 will be fixed and $K(z) = K(z, z_0)$.

Let Ω always be a regular subregion of R such that $z_0 \in \Omega$ and $P(z)dxdy \not\equiv 0$ on Ω. Then $\Omega \notin O_{PD}$ and since $P(z) \not\equiv 0$ on Ω, the Neumann's and Green's functions on Ω of (1) are well-defined; hence by Ozawa [6] their difference is 2π-times the reproducing kernel in the space $PE(\Omega)$, i.e. in the space of all energy finite solutions of (1) on Ω, while the

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scalar product of $u, v \in PE(\Omega)$ is the mixed energy integral $E_\Omega(u,v) = D_\Omega[u,v] + \int_R uPv$. Denote this kernel by

$$L_\Omega(z,\zeta) = \frac{1}{2\pi}(N_\Omega(z,\zeta) - G_\Omega(z,\zeta)) , \tag{2}$$

where N_Ω, resp. G_Ω is Neumann's, resp. Green's function of (1) on Ω. Making use of the joint finite continuity of N_Ω, G_Ω (cf. Nakai [1]) we can prove the *known fact* that if a function $f(z) \in L_P^\infty(\Omega)$ with the measure $P = P(z)dxdy$, then $\int_\Omega L_\Omega(z,\zeta)P(\zeta)f(\zeta)d\xi d\eta$ $(\zeta = \xi + i\eta)$ is a continuous function of z on $\bar{\Omega}$. We will extensively use this and also an important result of Nakai [3] that the vector space $PBD(R)$ of bounded Dirichlet finite solutions of (1) is dense in $PD(R)$ with respect to the CD-topology (for the notation cf. [7]).

2. For a regular subregion Ω, obviously $PE(\Omega) \subset PD(\Omega)$ but it may not be without interest to observe that the elements from the larger set PD are reproduced by the kernel $L_\Omega(z,\zeta)$. In particular, we have a simple but important lemma for our further work:

LEMMA 1. *If $u \in PD(\Omega)$ then*

$$u(z) = E_\Omega(u(\zeta), L_\Omega(z,\zeta)) \tag{3}$$

for all $z \in \Omega$.

Proof. By [2] $PD(\Omega)$ possesses a Riesz decomposition, thus $u = u^+ - u^-$ where u^+, u^- are positive elements of PD. Assuming that, say $u^+ \not\equiv 0$, we show (3) for u^+. According to [4] there exists a nondecreasing sequence $\{u_n^+\}$ of bounded PD-functions on Ω such that $u^+ = CD - \lim u_n^+$. Because $u_n^+ \in PE(\Omega)$ for each n, we may write

$$\begin{aligned} u_n^+(z) &= E_\Omega(u_n^+(\zeta), L_\Omega(z,\zeta)) \\ &= D_\Omega[u_n^+(\zeta), L_\Omega(z,\zeta)] + \int_\Omega u_n^+(\zeta)P(\zeta)L_\Omega(z,\zeta)d\xi d\eta . \end{aligned} \tag{4}$$

But since for a given $z \in \Omega$, $L_\Omega(z,\zeta) \in PD(\Omega)$ and $u_n^+ \geq 0$ on Ω, the Lebesgue convergence theorem yields (3). The same can be proved for u^-, and hence (3) is valid for u.

COROLLARY 1. *If $K_\Omega(z)$ is a reproducing kernel in $PD(\Omega)$ at the point z_0, then*

$$K_\Omega(z) = L_\Omega(z) + \int_\Omega L_\Omega(z,\zeta)P(\zeta)K_\Omega(\zeta)d\xi d\eta \tag{5}$$

where $L_\Omega(z) = L_\Omega(z, z_0)$.

COROLLARY 1′. $K_\Omega(z) \in C(\bar{\Omega})$.

Proof. Since for any Riesz decomposition of K_Ω, both K_Ω^+, K_Ω^- satisfy (3) we have

$$K_\Omega^\mp(z) = D_\Omega[K_\Omega^\mp(\cdot), L_\Omega(z,\cdot)] + \int_\Omega K_\Omega^\mp(\cdot)P(\cdot)L_\Omega(z,\cdot)\ .$$

For any $z \in \Omega$, $\inf_{\zeta\in\Omega} L_\Omega(z,\zeta) > 0$; thus K_Ω^+, K_Ω^- are in $L_P^1(\Omega)$ and consequently $K_\Omega \in L_P^1(\Omega)$. Then from (5) and by using Fubini's theorem we see that $K_\Omega \in L_P^2(\Omega)$; therefore by Schwarz's inequality, directly from (5) we obtain $K_\Omega \in L_P^\infty(\Omega)$. Thus by the remark in section 1, $K_\Omega(z) \in C(\bar{\Omega})$. The corollary is then proved.

We denote by $P(\Omega)$ the family of solutions of (1) on Ω. As far as a solution of the integral equation (5) is concerned we may state

LEMMA 2. *The integral equation*

$$f(z) - \int_\Omega f(\zeta)P(\zeta)L_\Omega(z,\zeta)d\xi d\eta = L_\Omega(z) \tag{6}$$

has a unique solution in the class $C(\bar{\Omega}) \cap P(\Omega)$.

Proof. Denote by $Q: C(\bar{\Omega}) \to C(\bar{\Omega})$ the operator defined by

$$Qf(z) = \int_\Omega f(\zeta)P(\zeta)L_\Omega(z,\zeta)d\xi d\eta \tag{7}$$

for every $f \in C(\bar{\Omega})$. Q is well-defined and $Q(C(\bar{\Omega})) \subset C(\bar{\Omega}) \cap P(\Omega)$. If we define the norm $\|f\| = \sup_\Omega |f|$ for $f \in C(\bar{\Omega})$ then

$$\begin{aligned} \|Qu\| &= \sup_{z\in\Omega}\left|\int_\Omega u(\zeta)P(\zeta)L_\Omega(z,\zeta)d\xi d\eta\right| \\ &\leq \|u\| \sup_\Omega q(z) \end{aligned} \tag{8}$$

for $u \in C(\bar{\Omega}) \cap P(\Omega)$, where

$$q(z) = \int_\Omega e_\Omega(\zeta)P(\zeta)L_\Omega(z,\zeta)d\xi d\eta \tag{9}$$

and e_Ω is the solution of (1) with constant boundary values 1. The function $q(z) \in C(\bar{\Omega}) \cap P(\Omega)$, and thus by the maximum principle $\sup_\Omega q(z) =$

$q(z') = q$, where $z' \in \partial\Omega$. From the construction of the Neumann's function N_Ω, using the double of Ω, we observe that

$$q(z') = \frac{1}{2\pi}\int_\Omega N_\Omega(z', \zeta)P(\zeta)e_\Omega(\zeta)d\xi d\eta \ , \tag{10}$$

and

$$\frac{1}{2\pi}\int_\Omega N_\Omega(z, \zeta)P(\zeta)d\xi d\eta = 1 \tag{11}$$

on $\bar{\Omega}$. Because from the maximum principle $e_\Omega < 1$ on Ω and as assumed $P(\zeta) \not\equiv 0$ on Ω, (10) and (11) give $q = q(z') < 1$. Thus by (8)

$$\sum_{n=1}^\infty Q^n u \in C(\bar{\Omega}) ;$$

and if $u(z) = L_\Omega(z)$, by Harnack's principle

$$\sum_{n=1}^\infty Q^n L_\Omega \in C(\bar{\Omega}) \cap P(\Omega) \ , \tag{12}$$

since $L_\Omega(z) \geq 0$ on $\bar{\Omega}$. Hence $\sum_0^\infty Q^n L_\Omega$ is a solution of (6) and obviously it is unique in the class $C(\bar{\Omega}) \cap P(\Omega)$. This completes the proof.

By Corollaries 1, 1′, and Lemma 2 we have the

LEMMA 3. *If $K_\Omega \in PD(\Omega)$ is the kernel at the point $z_0 \in \Omega$, then*

$$K_\Omega(z) = \sum_{n=0}^\infty Q^n L_\Omega(z) \ , \tag{13}$$

and $K_\Omega(z) > 0$ on Ω.

3. Finally we show that the kernel $K(z) \in PD(R)$ at the point z_0 can be obtained as $\lim_{\Omega \to R} K_\Omega(z)$ where Ω exhausts R. Then $K > 0$ on R.

Take a regular exhaustion $\{\Omega_n\}_1^\infty$ of R by regular subregions such that $z_0 \in \Omega_1$ and $P \not\equiv 0$ on Ω_1. By Lemma 3 for each $PD(\Omega_n)$ there exists a nonnegative reproducing kernel at z_0, say K_{Ω_n}. Since $\Omega_n \subset \Omega_{n+1}$, we have

$$D_{\Omega_n}[K_{\Omega_{n+1}}, K_{\Omega_n}] = K_{\Omega_{n+1}}(z_0) \ . \tag{14}$$

By Schwarz's inequality

$$(D_{\Omega_n}[K_{\Omega_{n+1}}, K_{\Omega_n}])^2 \leq K_{\Omega_{n+1}}(z_0)K_{\Omega_n}(z_0) ; \tag{15}$$

hence

$$K_{\Omega_{n+1}}(z_0) \leq K_{\Omega_n}(z_0) \tag{16}$$

and inductively

$$D_{\Omega_m}[K_{\Omega_m}] \leq D_{\Omega_n}[K_{\Omega_n}] \tag{17}$$

for $m \geq n$. Since $PD(\Omega_k)$ is a Hilbert space for each $k = 1, 2, \cdots,$ it follows from (16) and (17) that for any k there exists a subsequence $\{K_{\Omega_{k_i}}\} \subset \{K_{\Omega_n}\}, k_i \geq k,$ and a function $K_k \in PD(\Omega_k)$ such that

$$D_{\Omega_k}[K_{\Omega_{k_i}}, u] \to D_{\Omega_k}[K_k, u] \tag{18}$$

for each $u \in PD(\Omega_k)$ and thus for each $u \in PD(R)$. Moreover $\{K_{\Omega_{k_i}}\}$ can be chosen such that it converges to K_k uniformly on each compact subset of Ω_k. Using the diagonal process we obtain a subsequence $\{K_{\Omega_{n_i}}\} \subset \{K_{\Omega_n}\}$, converging to, say a function K, uniformly on any compact subset of R.

We show that K is in fact the kernel K at the point z_0. From the limiting process we know that $K \geq 0$ and K is a solution of (1) on R. It remains to prove the finiteness of the Dirichlet integral and the reproducing property at z_0 of K.

On $\Omega \in \{\Omega_{n_i}\}, K|_\Omega \in PD(\Omega)$ and $D_\Omega[K_{\Omega_{n_i}} - K] = D_\Omega[K_{\Omega_{n_i}} - K, K_{\Omega_{n_i}}] - D[K_{\Omega_{n_i}} - K, K]$. By (18)

$$\lim_{n_i} D_\Omega[K_{\Omega_{n_i}} - K, K] = 0 \tag{20}$$

and by (17)

$$\limsup_{n_i} D_\Omega[K_{\Omega_{n_i}} - K, K_{\Omega_{n_i}}] \leq K_{\Omega_1}(z_0) + \|K\|_\Omega (K_{\Omega_1}(z_0))^{1/2}\,, \tag{21}$$

where $\|\cdot\|_\Omega$ means Dirichlet norm. Also

$$D_\Omega[K] \leq D_\Omega[K_{\Omega_{n_i}} - K] + D_\Omega[K_{\Omega_{n_i}}] + 2\cdot\|K_{\Omega_{n_i}} - K\|_\Omega \cdot \|K_{\Omega_{n_i}}\|_\Omega\,.$$

Hence by (17), (20) and (21) we have for any $\Omega \in \{\Omega_{n_i}\}$ the estimate

$$\|K\|_\Omega^2 \leq 2a + b\|K\|_\Omega + c\sqrt{a + b\|K\|_\Omega} \tag{22}$$

where a, b and c are fixed positive constants. Therefore $\limsup_{n_i} D_{\Omega_{n_i}}[K] < \infty$.

Let $u \in PD(R)$. For $\varepsilon > 0$ choose an n_j such that $\|u\|_{R-\Omega_{n_j}} < \varepsilon/(K_{\Omega_1}(z_0))^{1/2}$. Then for $n_i \geq n_j$

$$\begin{aligned} |D_{\Omega_{n_j}}[K, u] - u(z_0)| &= |D_{\Omega_{n_j}}[K, u] - D_{\Omega_{n_j}}[K_{\Omega_{n_j}}, u]| \\ &\leq |D_{\Omega_{n_j}}[K - K_{\Omega_{n_i}}, u]| + |D_{\Omega_{n_j}}[K_{\Omega_{n_i}} - K_{\Omega_{n_j}}, u]|\,. \end{aligned} \tag{23}$$

Using the reproducing properties of $K_{\Omega_{n_j}}$ and $K_{\Omega_{n_i}}$ by (16) we obtain

$$|D_{\Omega_{n_j}}[K_{\Omega_{n_i}} - K_{\Omega_{n_j}}, u]| \leq |D_{\Omega_{n_i}-\Omega_{n_j}}[K_{\Omega_{n_i}}, u]| < \varepsilon ,$$

By (18) and (23), $|D_{\Omega_{n_j}}[K, u] - u(z_0)| < \varepsilon$, and since $D_R[K] < \infty$, $D_R[K, u] = u(z_0)$. Thus we have proved the following

THEOREM. *If $R \notin O_{PD}$ then there exists the reproducing kernel $K(z, \zeta)$ in the Hilbert space $PD(R)$ and it is a strictly positive symmetric function on $R \times R$.*

Unfortunately there is no such expression for K as (13), since by Nakai [5], $O_{PD} < O_{PE}$, i.e. there exists a Riemann surface which does not possess a nontrivial energy finite solution of (1); hence $L_R(z, \zeta) \equiv 0$ there, although if $R \in O_{PE} - O_{PD}$, the reproducing kernel $K \in PD(R)$ exists.

Still open questions remain as to whether or not the kernel $K(z, \zeta)$ as a function of one variable is bounded and if there exist more explicit expressions for K_Ω as it was introduced in (13).

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