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COMPARISON THEOREMS ON REGULAR POINTS FOR MULTI-DIMENSIONAL MARKOV PROCESSES OF TRANSIENT TYPE[1)]

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§ 1. Introduction

The study of regular points for the Dirichlet problem has a long history. The probabilistic approach to regular points is originated by Doob [2] and [3] for Brownian motion and the heat process. The extension to general Markov processes is discussed in Dynkin [4] and [5]. They also clarified the relation between the fine topology and regular points.

Regular points are by definition reflected in the behaviour of sample paths of Markov processes. Further the inclusion relation of collections of regular points for open sets determines the strength and the weakness of fine topologies between two processes. Hence it is meaningful to compare the collections of regular points for compact or open sets between two Markov processes apart from the Dirichlet problem.

Our aim of this article is to give a certain answer to the following problem. *Given two Markov processes. Can we give any characteristic quantities which determine whether a point is regular or not for one process provided that it is regular for the other process?* This type of problem has been studied for a certain class of uniformly elliptic differential operators of second order in $R^n (n \geqq 3)$ by many authors. They have shown that regular points for operators of such a class are the same as those for the Laplace operator by proving that there exist Green functions with singularity r^{2-n}. The relation between singularities of Green functions and regular points plays main roles in this article, too. Here we note that certain answer to the above problem has been given for diffusion processes by N.V. Krylov [17], [18], [19] and Markov processes having Green functions with monotone and isotropic singularities by the author [13], [14], [15].

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Now we state the outline of our results.

In § 2 we will establish the basic notations and give some elementary remarks.

In § 3 and § 4 we will show that a certain kind of order of singularities of Green functions for two Markov processes is reflected in the inclusion relation of sets of regular points for such processes. For example it will be proved that a collection of regular points for one process coincides with that for the other process if Green functions of two processes have the same singularity. The results in § 3 includes the result of Theorem 5 in [14].

The converse of the above result will be discussed in § 4 for a class of Markov processes having Green functions with monotone and isotropic singularities. As a result of § 4 we have the following. The singularity of a Green function for a Markov process of the above class is $r^{\alpha-n}$, $0<\alpha\leqq 2$, if and only if regular points coincide with those of an isotropic stable process of index α. This has been established in the previous paper [15] in case $1<\alpha\leqq 2$.

In sections 5~8 we will deal with more concrete Markov processes on R^n. Using the results in § 3 and § 4, we will study another quantity which decides whether a point is regular for one process or not provided that it is regular for the other process.

In § 5 we will consider diffusion processes corresponding to uniformly elliptic differential operators of second order on $R^n (n\geqq 3)$ which are not of divergence form. As mentioned before it is known that regular points for the above processes coincide with those for Brownian motion provided that the coefficients are smooth. We will prove in this section that a point is regular for diffusion processes with continuous coefficients if it is regular for some isotropic stable process of index α, $0<\alpha<2$. We will also show the known result by another method that regular points coincide with those for Brownian motion if the coefficients are uniformly Dini continuous.

The object of § 6 is a class of Markov processes subordinate to diffusion processes with uniformly Hölder continuous coefficients. Singularities of Green functions for Markov processes of this class are monotone and isotropic, but fairly abound in variety. We will introduce some inclusion relations of collections of regular points by comparing singularities at infinity of exponents of subordinators.

In § 7 and § 8 we will deal with Markov processes with homogeneity.

Our object in § 7 is the class of Lévy processes with mixed homogeneous exponents. It will be shown that, for two processes of the above class, regular points for the one are also regular for the other provided that exponents are sufficiently smooth and that they have same degree of mixed homogeneity. If exponents are not smooth, there arises certain difficulty.

In § 8 we will consider Markov processes with C^∞-homogeneous Lévy measure $n(x, y)dy$ of degree α, $0<\alpha<1$ or $1<\alpha<2$. (That is, $n(x, y)$ is C^∞-homogeneous function of y of degree α for each fixed x). Under certain regularity condition on $n(x, y)$, we will show that there exists Green functions with singularity $r^{\alpha-n}$ for the above processes. From this fact it follows that regular points are the same as those for an isotropic stable process of index α. For the construction of Green functions, the theory of pseudo-differential operators plays essential roles.

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§ 2 Preliminaries

This section contains some preliminary materials that will appear in this article. We will denote a Markov process[2] with state space E by $X=(\Omega, \mathcal{M}, \mathcal{M}_t, x_t, \theta_t, P_x)$ on E or simply by X on E, where E is a locally compact separable Hausdorff space. Throughout this paper Markov processes are assumed to satisfy Hunt's Hypothesis (A) (G. A. Hunt [9]) without special mentioning. In other words they are Hunt processes in the sense of [1]. For a subset A of E we define two functions

$$\sigma_A(\omega)=\inf\{t>0,\ x_t(\omega)\in A\},\quad \tau_A(\omega)=\inf\{t\geqq 0,\ x_t(\omega)\in A^c\},$$

where the infimum of the empty set is understood to be $+\infty$. A point x is called a *regular point* (an *irregular point*) of a nearly Borel set A for X provided that $P_x(\sigma_A=0)=1$ (resp. $P_x(\sigma_A=0)=0$). If A is simply a subset of E, we say that x is a regular point of A (an irregular point of A) for X provided that x is a regular point of B (an irregular point of B) for X for every nearly Borel set B containing A (resp. some nearly Borel set B contained in

2) We use the terminology in Blumenthal-Getoor [1].

A). We denote the collection of all regular points of A (the collection of all irregular points of A) for X by A_X^r (resp. A_X^{ir}). A set $A \subset E$ is called finely open if A^c is thin at each x in A. In other words, for each $x \in A$ there is a nearly Borel set D such that $A^c \subset D$ and $x \in D_X^{ir}$. Let $\mathcal{O}$ be the collection of all finely open subset of E. One checks that $\mathcal{O}$ is a topology on E. It is called the fine topology on E. For terminologies relative to the fine topology we add the adverb "$\mathcal{O}$-finely". Suppose that x is in A and A^c is thin at x. Then there exists a compact set K such that $x \in K \subset A$ and K^c is thin at x. (See Blumenthal-Getoor [1], p. 85.) Hence the first half of the following remark is proved.

Remark 1. Let $\mathcal{O}_i$, $i = 1, 2$, be fine topologies induced by Markov processes X_i, $i = 1, 2$, on E respectively. Then

a) $A_{X_1}^r \subset A_{X_2}^r$ *for every open subset* A

implies

b) $\mathcal{O}_1$ *is stronger than* $\mathcal{O}_2$.

Conversely, if X_1 has a reference measure, *b*) implies *a*).

For the proof of the latter half we note that A is finely closed if and only if $A_X^r \subset A$ and the fine closure of A is $A \cup A_X^r$ for a nearly Borel set A (see. (4.9), p. 87, [1]). Further if X has a reference measure, the above statement is also valid for any subset A. (See Prop. 1.8, p. 199, [1].) Let B be open. Since $B_{X_2}^r = B \cup B_{X_2}^r$, $B_{X_2}^r$ is $\mathcal{O}_2$-finely closed , and accordingly $B_{X_2}^r$ is $\mathcal{O}_1$-finely closed by b). Hence it follows that $B_{X_1}^r \subset (B_{X_2}^r)_{X_1}^r \subset B_{X_2}^r$.

Remark 2. If there exists a compact set $B \subset E$ such that $B_{X_1}^r \neq \phi$ and $B_{X_2}^r = \phi$, then $\mathcal{O}_1$ is not equivalent to $\mathcal{O}_2$ provided that $P_X^i(\sigma_{\{y\}} < +\infty) = 0$, $i = 1, 2$, for each $x, y \in E$.

Indeed, if we set $K = (E - B) \cup \{x_0\}$ for some fixed $x_0 \in B_{X_1}^r$, we have

$$(E - K)_{X_2}^{ir} = (B - \{x_0\})_{X_2}^{ir} = E \supset K,$$

and

$$(E - K)_{X_1}^{ir} = (B - x_0)_{X_1}^{ir} = B_{X_1}^{ir} \not\ni x_0.$$

Hence K is $\mathcal{O}_2$-finely open but not $\mathcal{O}_1$-finely open.

Now we will list up some conditions which will be assumed on Markov processes on E in theorems of **§3** and **§4**. Let $\{G_\alpha\}_{\alpha>0}$ be a resolvent on E

and $\{T_t\}$ be a semi-group of X.

M 1) *G_α maps $C_K(E)$ into $C(E)$*[3] *for each $\alpha > 0$*

M 2) *$\int_0^{+\infty} T_t f dt$ is bounded on E for $f \in C_K(E)$.*

M 3) *For each points $x_1, x_2 \in E$, $P_{x_1}(\sigma_{\{x_2\}} < +\infty) = 0$.*

Let us consider the following condition.

R 1) *For every compact set K and a sequence $\{O_n\}_{n=1,2,\dots}$ of open sets such that $\bigcap_n O_n = K$, it holds that*

$$\lim_{n\to+\infty} P_x(\sigma_{O_n} < +\infty) = P_x(\sigma_K < +\infty),\ x \notin K.$$

Then we have

LEMMA 1 *Let X be a Markov process on E with the properties M 1) and M 2). Then X satisfies R 1).*

Proof Since K is compact, it is sufficient to show that for each fixed $x_0 \notin K$ we can choose a sequence $\{O_n\}_{n=1,2,\dots}$ of open sets such that $O_n \supset K$ and $P_{x_0}(\sigma_{O_n} < +\infty) \downarrow P_{x_0}(\sigma_K < +\infty)$. Let $\{O_n\}$ be a sequence of open sets such that $O_n \downarrow K$ and $P_{x_0}(\sigma_{O_n} \uparrow \sigma_K) = 1$ (for the existence see (11.3) in [1]). Let A be a compact set containing O_n for all n. Then it follows from M 1) and M 2) that

$$P_{x_0}(+\infty > {}^{\exists}\delta_A(\omega) > 0,\ t > \delta_A(\omega),\ x_t \notin A) = 1.$$

(See for example, (4.24), p. 89, [1].) Noting

$$\Big(\bigcap_n \{\sigma_{O_n}(\omega) < +\infty\}\Big) \cap \{\sigma_K(\omega) = +\infty\} \subset \{{}^{\exists}n_0(\omega);\ {}^{\forall}n > n_0(\omega),$$
$$+\infty > \sigma_{O_n}(\omega) > \delta_A(\omega)\};\ P_{x_0} - a.e.,$$

we have

$$P_{x_0}\Big(\sigma_K(\omega) = +\infty,\ \bigcap_n \{\sigma_{O_n}(\omega) < +\infty\}\Big) = 0.$$

Hence it holds that

$$P_{x_0}(\sigma_K < +\infty) = P_{x_0}\Big(\bigcap_n \{\sigma_{O_n} < +\infty\}\Big).$$

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3) $C(E)$, $C_0(E)$ and $C_K(E)$ denote the space of continuous functions on E which are bounded, vanishing at infinity and of compact support respectively.

We say that $G(x, y)$ is a *kernel* on E if it is a universally measurable function[4] on $E \times E$. We will sometimes discuss a kernel $G(x, y)$ on E with the following properties:

GB) *$G(x, y)$ is bounded outside each neighborhood of the diagonal set of $E \times E$;*

GC) *$G(x, y)$ is continuous except at the diagonal set of $E \times E$ and lower semi-continuous on $E \times E$;*

GS) *for each $z \in E$ and a sequence $\{O_n\}_{n=1,2,\dots}$ of open sets in E such that $\bigcap_n O_n = \{z\}$, it holds that*

$$\lim_{n \to +\infty} \inf_{x, y \in O_n} G(x, y) = +\infty.$$

In this article we will adopt the next definition of Green functions.

Definition 1. A nonnegative kernel $G(x, y)$ is called a *Green function of a Markov process X on E* if it satisfies:

*G*i) *$G(x, y)$ is an excessive function of x relative to X for each fixed $y \in E$;*

*G*ii) *there exists a σ-finite measure dy on E such that*

$$\int_E G(x, y) f(y) dy = \int_0^{+\infty} T_t f(x) dt < +\infty$$

for every $f \in C_K(E)$.

For a Green function $G(x, y)$ we write $Gf(x)$ instead of $\int_E G(x, y) f(y) dy$ for simplicity.

The next condition on Markov process X plays essential roles in later discussions on regular points.

R 2) *There exists a kernel $G(x, y)$ on E satisfying:*

i) *$G(x, y)$ is an excessive function of x relative to X for each fixed $y \in E$;*

ii) *For each compact set $K \subset E$ there exists a finite measure $\mu_K(dy)$ concentrated on K such that*

$$P_x(\sigma_K < +\infty) = \int_E G(x, y) \mu_K(dy), \quad x \in E.$$

For convenience we call $G(x, y)$ in *R* 2) the *potential kernel of X* and

4) In this paper a function on a set S may attain the value $+\infty$ on S.

$\mu_K(dy)$ the *capacitary measure on* K for (X, G). If we can choose Green function $G(x, y)$ of X as a potential kernel of X, we call it a *Green function with the property* $R\,2)$. Note that $M\,3)$ holds provided X has a Green function $G(x, y)$ with $R\,2)$ and $GS)$. We will close this section with the remark that Hunt's condition $F)$ and $G)$ is sufficient for $R\,2)$ (see G.A. Hunt [**10**]) and another sufficient condition on $R\,2)$ is given in [**14**], [**15**].

§ 3. Comparison theorems (I)

In this section we will show certain results on the comparison of regular points and hitting probabilities. First we introduce some notations which are convenient to state out results. As in § **2** X is a Markov process on E.

DEFINITION 2 Let Q be an open set in E containing x_0 and C_k, $k = 1$, 2 be constants such that $+\infty > C_2 \geqq 1 \geqq C_1 > 0$. A universally measurable function f on E is called *C_1-subharmonic* (*C_2-superharmonic*) at (x_0, Q) relative to X provided that for each open set S such that $x_0 \in S \subset \bar{S} \subset Q$ one has

$$E_{x_0} f(x_{\tau_S}) \geqq C_1 f(x_0) \quad (\text{resp. } E_{x_0} f(x_{\tau_S}) \leqq C_2 f(x_0)).$$

DEFINITION 3 Let D be a subset of E. We say that two kernels $G_k(x, y)$, $k = 1, 2$, *have the same local singularity on* D provided that for each point of D there exists a neighborhood $V \subset E$ and constants $C_1 \geqq C_2 > 0$[5] such that

$$C_2 G_2(x, y) \leqq G_1(x, y) \leqq C_1 G_2(x, y), \quad x, y \in D \cap V. \tag{1}$$

It is clear that the above inequality implies

$$1/C_1 G_1(x, y) \leqq G_2(x, y) \leqq 1/C_2 G_1(x, y), \quad x, y \in D \cap V. \tag{2}$$

Sometimes we will write

$$G_1(x, y) \approx G_2(x, y) \text{ on } D,$$

if $G_k(x, y)$, $k = 1, 2$, have the same local singularity on D.

In the sequel we use following symbols for a kernel $G(x, y)$ on $A \times A$: i) $G^y(x) = G(x, y)$; ii) $G^y|_A(x) = G(x, y)$ if $x \in A$ and $G^y|_A(x) = 0$ if $x \notin A$.

Remark 3 Let $G_2(x, y)$ be a kernel on E which is an excessive function of x relative to X_2 and $G_1(x, y)$ be a kernel satisfying (1) on an open set $V \subset E$. Then $G_1^y|_V$ is C_1/C_2-superharmonic at (x_0, V) relative to X_2 for each fixed x_0, $y \in V$.

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5) C_k, $k = 1, 2$, may depend on V.

Indeed we have

$$E^2_{x_0}G^y_1|_V(x^2_{\tau_s}) \leqq C_1 E^2_{x_0}G^y_2|_V(x^2_{\tau_s}) \leqq C_1 G_2(x_0, y), \quad x_0, y \in V,$$

where S is an open set such that $x_0 \in S \subset \bar{S} \subset V$.

Definition 4. Let X_k, $k = 1, 2$, be Markov processes on E and $x_0 \in D \subset E$. We say that *hitting probabilities of X_1 are C_1-dominated by those of X_2 at (x_0, D)* provided that

$$P^2_{x_0}(\sigma_B < +\infty) \geqq C_1 P^1_{x_0}(\sigma_B < +\infty)^{6)}$$

holds for each compact set B in D. We say that hitting probabilities of X_k, $k = 1, 2$, are (C_1, C_2) dominated each other at (x_0, D), if in addition hitting probabilities of X_2 are C_2-dominated by those of X_1 at (x_0, D). Here C_k, $k = 1, 2$ denote positive constants.

Now we prepare the following preliminary but essential Lemmas in discussing regular points.

Lemma 2. *Let X be a Markov process on E with the properties $R1)$ and $M3)$. Then, for each nearly Borel set B,*

i) $x \in B^r_X \iff \forall n,\ P_x(\sigma_{B\cap O_n} < +\infty) = 1;$

ii) $x \in B^{ir}_X \iff \lim_{n\to+\infty} P_x(\sigma_{B\cap O_n} < +\infty) = 0;$

where $\{O_n\}_{n=1,2,\dots}$ is a sequence of open sets in E such that $\bar{O}_{n+1} \subset O_n$ and $\bigcap_n O_n = \{x\}$.

Proof Let us fix n_0 and denote O_{n_0} by O'. Then

$$P_x(\sigma_{B\cap O_k} < +\infty) = E_x(P_{x_{\tau_{O'}}}(\sigma_{B\cap O_k} < +\infty);\ \sigma_B > \tau_{O'}) + P_x(\sigma_B \leqq \tau_{O'},\ \sigma_{B\cap O_k} < +\infty).$$

Combining $M3)$ with $R1)$ we have

$$\lim_{k\to+\infty} P_x(\sigma_{B\cap O_k} < +\infty) = \lim_{k\to+\infty} P_x(\sigma_B \leqq \tau_{O'},\ \sigma_{B\cap O_k} < +\infty). \tag{3}$$

On the other hand if $x \in B^{ir}_X$, it holds

$$1 = P_x(\sigma_B > 0) = P_x(\cup\ (0 < \forall t < \tau_{O_n},\ x_t \notin B)).$$

6) Precisely $P^2_{x_0}(\sigma^2_B < +\infty) \geqq C_1 P^1_{x_0}(\sigma^1_B < +\infty)$. We will remove the suffix of the hitting time in the sequel without confusions.

because $P_x(\lim_{n\to+\infty}\tau_{O_n}=0)=1$. Hence, for each $\varepsilon>0$, we can choose n_0 so that $P_x(0<{}^{\forall}t<\tau_{O'},\ x_t\notin B)>1-\varepsilon$. Accordingly, by (3), we have

$$\lim_{k\to+\infty} P_x(\sigma_{B\cap O_k}<+\infty)\leqq\varepsilon$$

for every $\varepsilon>0$ provided $x\in B_X^{ir}$. It is clear that $P_x(\sigma_{B\cap O_n}<+\infty)=1$ for every n provided $x\in B_X^r$. Consequently we can finish the proof of i) and ii) if only we note $E=B_X^r\cup B_X^{ir}$.

In the next Lemma 3 X_k, $k=1,2$, denote Markov processes on E with properties $M3$) and $R1$) without referring. Choose a point $x_0\in E$ and an open set $Q\subset E$ containing x_0 and fix them.

LEMMA 3a *Suppose further that X_1 has a potential kernel $G_1(x,y)$ satisfying $R2$). If, for each fixed $y\in Q$, $G_1^y(x)$ $(G_1^y|_Q(x))$ is C_1-subharmonic at $(x_0, Q-\{y\})$ (resp. C_2-superharmonic at $(x_0. Q)$) relative to X_2, where C_1 (resp. C_2) is independent of y, then $x_0\in B_{X_1}^r$ implies $x_0\in B_{X_2}^r$ (resp. $x_0\in B_{X_2}^r$ implies $x_0\in B_{X_1}^r$) for each compact or open set B in Q.*

LEMMA 3b *In addition to the assumption in Lemma 3a, suppose that X_2 has a potential kernel $G_2(x,y)$ satisfying $R2$) and both $G_k(x,y)$ $k=1,2$, have properties GS) and GB). Then hitting probabilities of X_1 are $C_1/2$-dominated by those of X_2 (resp. hitting probabilities of X_2 are $1/2C_2$-dominated by those of X_1) at $(x_0,\tilde{Q})$ for a certain open set $\tilde{Q}$ such that $x_0\in\tilde{Q}\subset Q$.*

Proof of Lemma 3a We will divide the proof into two steps. Let us fix an open set Q' in E such that $x_0\in Q'\subset\bar{Q}'\subset Q$.

step 1. We will show that

(4) $$P_{x_0}^2(\sigma_M<+\infty)\geqq C_1P_{x_0}^1(\sigma_M<+\infty)-\int_{Q'^c}P_z^1(\sigma_M<+\infty)P_{x_0}^2(x_{\tau_{Q'}}^2\in dz)$$

(resp. (4') $P_{x_0}^2(\sigma_M<+\infty)\leqq C_2P_{x_0}^1(\sigma_M<+\infty)+\int_{Q'^c}P_z^2(\sigma_M<+\infty)P_{x_0}^2(x_{\tau_{Q'}}^2\in dz)$)

for each compact or open set M in E such that $\bar{M}\subset Q'$. We prove (4) at first by breaking up the proof into three cases.

Case (I): M is compact in Q' and $M\not\ni x_0$. Choose an open set S in E such that $M\subset S\subset\bar{S}\subset Q'$ and $\bar{S}\not\ni x_0$. Then we have

(5) $$P_{x_0}^2(\sigma_S<+\infty)\geqq E_{x_0}^2(P_{x_{\sigma_S}^2}^1(\sigma_M<\infty),\ \sigma_S<\tau_{Q'})$$

$$= \int_{\bar{S}} P_z^1(\sigma_M < +\infty) P_{x_0}^2(x^2_{\tau_{Q'-\bar{S}}} \in dz).$$

Combining (5) and (R 2) we get

$$P_{x_0}^2(\sigma_S < +\infty) \geqq \int_{\bar{S}}\int_M G_1(z, y)\mu_M^1(dy) P_{x_0}^2(x^2_{\tau_{Q'-\bar{S}}} \in dz). \tag{6}$$

$$\geqq \int_M \left\{\int_E G_1(z, y) P_{x_0}^2(x^2_{\tau_{Q'-\bar{S}}} \in dz)\right\} \mu_M^1(dy) - \int_{Q'^c} P_z^1(\sigma_M < +\infty) P_{x_0}^2(x^2_{\tau_{Q'-\bar{S}}} \in dz).$$

Using C_1-subharmonicity of G_1^y at $(x_0, Q-\{y\})$ and the fact that $P_{x_0}^2(x^2_{\tau_{Q'-\bar{S}}} \in A) \leqq P_{x_0}^2(x^2_{\tau_{Q'}} \in A)$ for $A \subset Q'^c$, we have, by (6),

$$P_{x_0}^2(\sigma_S < +\infty) \geqq C_1 P_{x_0}^1(\sigma_M < +\infty) - \int_{Q'^c} P_z^1(\sigma_M < +\infty) P_{x_0}^2(x^2_{\tau_{Q'}} \in dz). \tag{7}$$

Since M is compact, we can get the inequality (4) for M by (R 1).

Case (II): M is compact in Q' and $M \ni x_0$. Choose a sequence $\{O_k\}_{k=1,2,\dots}$ of open sets such that $O_{k+1} \subset \bar{O}_k$ and $\bigcap_k O_k = \{x_0\}$ and set $M_k = M \cap O_k^c$. Then the inequality (4) holds for every M_k, $k = 1, 2, \dots$ Since $\{\sigma_{M_k} < +\infty\}$ is monotone increasing as $k \to +\infty$, (4) also holds for $M - \{x_0\}$. Noting that $\sigma_M = \inf\{\sigma_{M-\{x_0\}}, \sigma_{\{x_0\}}\}$ and M 3), we see that (4) is valid for M.

Case (III): M is open in Q. Choose an increasing sequence of $\{M_k\}$, $k = 1, 2, \dots$ of compact sets such that $\bigcup_k M_k = M$. Then it is clear that $P_{x_0}^2(\sigma_{M_k} < +\infty) \uparrow P_{x_0}^2(\sigma_M < +\infty)$. Accordingly (4) holds.

Secondly we prove (4′). Let M be the set of the case (I). Then we have

$$P_{x_0}^2(\sigma_M < \tau_{Q'}) = \int_M P_z^1(\sigma_{\bar{S}} < +\infty) P_{x_0}^2(x^2_{\tau_{Q'-M}} \in dz), \tag{8}$$

where S is an open set in Q' such that $M \subset S \subset \bar{S} \subset Q'$, $\bar{S} \not\ni x_0$. Using (R 2) and C_2-superharmonicity of $G_1^y|_Q$ at (x_0, Q), we have, by (8),

$$P_{x_0}^2(\sigma_M < \tau_{Q'}) \leqq \int_M \int_{\bar{S}} G_1(z, y)\mu_{\bar{S}}^1(dy) P_{x_0}^2(x^2_{\tau_{Q'-M}} \in dz) \leqq C_2 P_{x_0}^1(\sigma_{\bar{S}} < +\infty). \tag{9}$$

Since S is arbitrary, we get, by (9) and (R 1),

$$P_{x_0}^2(\sigma_M < \tau_{Q'}) \leqq C_2 P_{x_0}^1(\sigma_M < +\infty). \tag{10}$$

Noting that $P_{x_0}^2(\sigma_M < \tau_{Q'}) = P_{x_0}^2(\sigma_M < +\infty) - E_{x_0}^2(P^2_{x^2_{\tau_{Q'}}}(\sigma_M < +\infty), \tau_{Q'} < +\infty)$, the inequality (4′) holds for M in the case (I). The proof of (4′) in other cases is similar to that of (4).

step 2. Suppose $x_0 \in B^r_{X_1}$. Then, by Lemma 2, $P^1_{x_0}(\sigma_{B\cap O_n} < +\infty) = 1$ for all n, where $\{O_n\}$ is a sequence of open sets such that $\bigcap_n O_n = \{x_0\}$. On the other hand $\lim_{n\to+\infty} P^1_z(\sigma_{B\cap O_n} < +\infty) = 0$ for $z \in Q'^c$ by $(R1)$ and $M3)$. Combining this fact with the inequality (4), we get

$$\varliminf_{n\to+\infty} P^2_{x_0}(\sigma_{B\cap O_n} < +\infty) \geqq C_1,$$

which implies $x_0 \in B^r_{X_2}$ by Lemma 2. On the same way we can prove that $x_0 \in B^r_{X_2}$ implies $x_0 \in B^r_{x_1}$ by using (4′) provided $G^y_1|_Q$ is C_2-superharmonic at (x_0, Q). The proof is complete.

Proof of Lemma 3b Using GB) and GS) for $G_k(x, y)$, $k = 1, 2$, we can choose an open set $\tilde{Q}$ such that $x_0 \in \tilde{Q} \subset Q'$ and

$$\inf_{y\in\tilde{Q}} G_k(x_0, y) \geqq 2/C_1 \sup_{\substack{z\in Q'^c \\ y\in\tilde{Q}}} G_k(z, y),\ k = 1, 2.$$

Then, for each compact set $M \subset \tilde{Q}$, it holds that

$$(11)\quad \int_{Q'^c} P^k_z(\sigma_M < +\infty) P^2_{x_0}(x^2_{\tau_{Q'}} \in dz) \leqq \sup_{\substack{z\in Q'^c \\ y\in\tilde{Q}}} G_k(z, y)\mu^k_M(M) \leqq (C_1/2) P^k_{x_0}(\sigma_M < +\infty),$$

$k = 1, 2$. Combining (11) with (4) ((4′)), we get

$$P^2_{x_0}(\sigma_M < +\infty) \leqq (C_1/2) P^1_{x_0}(\sigma_M < +\infty)\ \text{(resp. } (1 - C_1/2) P^1_{x_0}(\sigma_M < +\infty) \leqq C_2 P^2_{x_0}(\sigma_M < +\infty))$$

for every compact set $M \subset \tilde{Q}$. The proof is complete.

Remark 4 Further suppose that $G_1(x, y)$ in Lemma 3a satisfies GC). Then the following conditions are equivalent.

i) For each fixed $y \in Q$, $G^y_1|_Q(x)$ is C_2-superharmonic at (x_0, Q) relative to X_2.

ii) For each fixed $y \in Q$, $G^y_1|_Q(x)$ is C_2-superharmonic at $(x_0, Q - \{y\})$ relative to X_2.

We will prove that ii) implies i). Let S be an open set such that $x_0, y \in S \subset \bar{S} \subset Q$, and let $\{Q_n\}_{n=1,2\dots}$ be a sequence of open sets converging to y. Then, setting $S_n = S - \bar{Q}_n$, it holds for every n that

$$(12)\qquad E^2_{x_0} G^y_1|_Q(x^2_{\tau_S}) \leqq E^2_{x_0} G^y_1|_Q(x^2_{\tau_{S_n}}) + E^2_{x_0}(P^2_{x_{\sigma_{Q_n}}}(G^y_1|_Q(x^2_{\tau_S})),\ \sigma_{Q_n} < +\infty)$$

$$\leqq C_2 G_1^y(x_0) + \sup_{z \in S^c} G_1^y|_Q(z) \cdot P_{x_0}^2(\sigma_{Q_n} < +\infty).$$

Since $\sup_{z \in S^c} G_1^y|_Q(z) < +\infty$ by GC) and $\lim_{n \to +\infty} P_{x_0}^2(\sigma_{Q_n} < +\infty) = 0$ by $M3$) and $R1$), it follows from (11) that

$$E_{x_0}^2 G_1^y|_Q(x_{\tau_s}^2) \leqq C_2 G_1^y(x_0). \tag{13}$$

Next let us consider the case that $x_0 \in S \subset \bar{S} \subset Q$ but $y \notin S$, $y \in \bar{S}$. Let $\{y_n\}_{n=1,2,\dots}$ be a sequence converging to y such that $y_n \notin \bar{S}$ for evrey n. Then it follows from the assumption (ii) that

$$E_{x_0}^2 G_1^{y_n}|_Q(x_{\tau_s}^2) \leqq C_2 G_1^{y_n}(x_0). \tag{14}$$

Combinig (14) with GC), we have (13) for the above case. Thus we have proved (i). This remark will be used in § 5.

Now we are ready to state our theorem. Let

$\Phi_0^m = \{\varphi;\ \varphi$ is nonnegative, monotone decreasing function on $[0, +\infty]$ such that $\varphi(0) = +\infty$ and $\varphi(+\infty) = 0\}$.

Then we have

THEOREM 1 *Let X_k, $k = 1, 2$ be Markov processes on E with $M1)\sim M3)$ which have Green functions $G_k(x, y)$, $k = 1, 2$, with $R2$). Suppose that there exists an open set Q and a finite nonnegative kernel $\rho(x, y)$ on Q such that*

$$G_k(x, y) \approx \varphi_k(\rho(x, y)) \text{ on } Q,\ k = 1, 2, \tag{15}$$

where $\varphi_k(r) \in \phi_0^m$. If

$$\varphi_2(r)/\varphi_1(r) \text{ is monotone decreasing on } (0, +\infty), \tag{16}$$

then it follows that

$$K_{X_2}^r \subset K_{X_1}^r \tag{17}$$

for each compact or open set $K \subset Q$.

Proof Fix an arbitrary $x_0 \in Q$. Let us choose a neighborhood V of x_0 such that $V \subset Q$ and

$$C_{2,k}\varphi_k(\rho(x, y)) \leqq G_k(x, y) \leqq C_{1,k}\varphi_k(\rho(x, y)),\quad x, y \in V, \tag{18}$$

where $C_{l,k} > 0$, $l, k = 1, 2$. For a fixed $y \in V$ we set $V_1 = V \cap \{z;\ \rho(x_0, y) \leqq \rho(z, y)\}$ and $V_2 = V \cap \{z;\ \rho(x_0, y) > \rho(z, y)\}$. Then, for each open set S such that $x_0 \in S \subset \bar{S} \subset V$, it follows from (18) and (16) that

$$(19)\qquad E^2_{x_0}G^y_1|_V(x^2_{\tau_s}) \leqq C_{11}\int_V \varphi_1(\rho(z,\ y))P^2_{x_0}(x^2_{\tau_s} \in dz)$$
$$\leqq C_{11}\varphi_1(\rho(x_0,\ y)) \text{ if } \rho(x_0.\ y) = 0$$
$$\leqq C_{11}\varphi_1(\rho(x_0,\ y) + \frac{\varphi_1(\rho(x_0,\ y))}{\varphi_2(\rho(x_0,\ y))}\int_{V_2}\varphi_2(\rho(z,\ y))P^2_{x_0}(x^2_{\tau_s} \in dz),$$
$$\text{if } \rho(x_0,\ y) > 0.$$

Combining (19) with (18) we have

$$(20)\quad E^2_{x_0}G^y_1|_V(x^2_{\tau_s}) \leqq \begin{cases} \dfrac{C_{11}}{C_{21}}G_1(x_0,\ y), \text{ if } \rho(x_0,\ y) = 0 \\ \dfrac{C_{11}}{C_{21}}\cdot G_1(x_0, y) + \dfrac{C_{11}C_{12}}{C_{21}C_{22}}\cdot\dfrac{G_1(x_0,\ y)}{G_2(x_0,\ y)}\displaystyle\int_E G_2(z,\ y)P^2_{x_0}(x^2_{\tau_s} \in dz), \\ \text{if } \rho(x_0,\ y) > 0. \end{cases}$$

Since $G_2(x,\ y)$ is an excessive function of x relative to X_2, we have

$$(21)\qquad E^2_{x_0}G^y_1|_V(x^2_{\tau_s}) \leqq \Big(\frac{C_{11}}{C_{21}} + \frac{C_{11}C_{12}}{C_{21}C_{22}}\Big)G_1(x_0,\ y).$$

In other words $G^y_1|_V$ is $(C_{11}/C_{21} + C_{11}C_{21}/C_{12}C_{22})$-superharmonic at $(x_0,\ V)$ relative to X_2 for each $y \in V$. Noting that X_k, $k = 1,\ 2$, satisfy $R1)$ by Lemma 1, the conclusion follows from Lemma 3a immediately. The proof has been finished.

Corollary 1 *If $G_k(x,\ y)$, $k = 1,\ 2$, have the same local singularity on Q, then*

$$K^r_{X_1} = K^r_{X_2}$$

holds for each compact or open set $K \subset Q$.

Indeed it suffices to choose $\rho(x,\ y) = 1/G_2(x,\ y)$ and $\varphi_k(r) = 1/r$, $k = 1,\ 2$, in Theorem 1. We note that Corollary 1 also follows immediately from Remark 3 and Lemma 3a.

Corollary 2 *Let $\varphi \in \Phi^m_0$ be such that $r\varphi(r)$ is monotone increasing on* $(0,\ +\infty)$. *If*

$$G_1(x,\ y) \approx \varphi(1/G_2(x,\ y)) \text{ on } Q,\ G_2(x,\ y) > 0 \text{ on } Q,$$

then it follows that

$$K^r_{X_1} \supset K^r_{X_2}$$

for each compact or open set $K \subset Q$.

Indeed it is sufficient to choose $\rho(x, y) = 1/G_2(x, y)$, $\varphi_1(r) = \varphi(r)$ and $\varphi_2(r) = 1/r$ in Theorem 1.

Next we will refine the above Theorem 1. *A* subset D of E is said to have the property $D)_X$ provided

$D)_X$ *D is closed and* $D_X^r = D$.

THEOREM 1′. *If the assumption* (15) *holds for a subset D with* $D)_{X_k}$, $k = 1, 2$, *instead of Q, then the conclusion* (17) *follows from* (16) *for each compact or relatively open set* $K \subset D$.

After this theorem is established, the refinement of Corollary 1, 2 will be clear. We denote them by Corollary 1′, 2′ respectively.

To prove Theorem 1′ we will study the time changed process by a local time on D introduced by M. Motoo [22]. Let us consider Markov process X on E with a reference measure and a subset D with $D)_X$. For each $\alpha > 0$ fixed there exists a unique additive functional $\Phi_\alpha(t, \omega)$ defined by

$$E_x\Big(\int_0^{+\infty} e^{-\alpha t} d\Phi_\alpha(t)\Big) = E_x\Big(\int_{\sigma_D}^{\infty} e^{-\alpha t} d(t \wedge \zeta)\Big), \quad x \in E,$$

where ζ is the killing time of X. It is called the α-th order sweeping-out on D of $\inf\{t, \zeta\}$, or the local time on D for X. Let τ be the inverse of Φ_α. Then, choosing an adequate set Ω^D such that $P_x(\Omega - \Omega^D) = 0$ for every $x \in E$, we can construct a Markov process $X^D = (\Omega^D, \mathscr{M}^D, \mathscr{M}_t^D, x_t^D, \theta_t^D, P_x^D)$ on D, where $x_t^D(\omega) = x_{\tau(t)}(\omega)$ if $t < +\infty$, $x_t^D(\omega) = \Delta$ if $t = +\infty$ and P_x^D is the restriction of P_x on Ω^D. (M. Motoo [22].) Moreover we have

LEMMA 4 i) (*M. Motoo* [22]) $P_x(\sigma_B < +\infty) = P_x^D(\sigma_B^D < +\infty)$ *for each Borel set B in D and* $x \in D$. ii) $B_X^r = B_{X^D}^r$ *for each Borel set B in D.*

Proof The statement i) is Lemma 6.13 of [22] itself[7]. For the proof of ii) we note that $\tau(t)$ is right continuous and strictly increasing. (See [22]). Now it is clear that $x \in B_{X^D}^r$ implies $x \in B_X^r$ by the definition of X^D. Suppose $x \in B_X^r$. Then, for almost all ω there is a sequence $t_n \downarrow 0$ such that $x_{t_n} \in B$. Since $x_{\tau(t_n)}^D = x_{t_n}$ and $\lim_{n \to +\infty} \tau(t_n) = 0$, it follows that $x \in B_{X^D}^r$. The proof is complete.

[7] In Lemma 6.13 in [22] the statement is asserted for a closed set B. But it is valid for a Borel set B. (See for example [1] p. 233, (4.13), ii).)

Proof of theorem 1′ First we note that, if f is an excessive function of X, where $X=X_1$ or X_2, then it is an excessive function of X^D. This is proved as follows. Since $f(x_t)$ is right continuous on $[0,\infty)$ almost surely P_x (Theorem 5.7, iii), [1]), it holds that

$$\varliminf_{n\to+\infty} E_x^D f(x^D_{\tau(t_n)}) = E_x(\lim_{n\to+\infty} f(x_{\tau(t_n)})) = f(x) \tag{22}$$

for each monotone decreasing sequence $\{t_n\}_{n=1,2,\dots}$ converging to 0. On the other hand we have

$$E_x^D f(x_t^D) = E_x f(x_{\tau(t)}) \leqq f(x). \tag{23}$$

Combining (22) with (23), we see that f is an excessive function of X^D. Since from the above result $G_k(x,y)$, $k=1,2$, are excessive functions of x relative to X_k^D, $k=1,2$, we can choose them as potential kernels of X_k^D, $k=1.2$ by using Lemma 4 i). It will be clear by Lemma 4 i) that X_k^D satisfy $R1$), $k=1,2$, because X_k satisfy $R1$), $k=1,2$. Now let us note that Theorem 1 is also valid even if we replace the conditions $M1$) and $M2$) by $R1$). Then, applying it to X_k^D, $k=1,2$, we see that $B^r_{X_2^D}\subset B^r_{X_1^D}$. Therefore $B^r_{x_2}\subset B^r_{X_1}$ holds by Lemma 4 ii). The proof is complete.

Even if the state spaces E_k, $k=1, 2$, of X_k, $k=1, 2$, are different, Theorem 1′ is also valid provided that both E_k are subspaces of E and $D\subset E_1\cap E_2$ satisfying $D)_{X_{k1}}$, $k=1,2$.

In the following we will discuss the converse of the above results. Let X_k, $k=1,2$, be Markov processes on E with $M1)\sim M3)$. Suppose that X_k, $k=1,2$ have Green functions $G_k(x,y)$, $k=1,2$ with $R2$).

Let us consider the next three conditions.

i) For each point there exists a neighborhood V and positive constants C_k, $k=1,2$ so that hitting probabilities of X_k, $k=1,2$ are (C_1,C_2)-dominated each other at (x,V) for every $x\in V$.

ii) For each point there exists a neighborhood V and positive constants $L_k>0$, $k=1,2$ such that $G_y^k(x)$, $k=1,2$ is L_k-superharmonic at (x,V) for each x, $y\in V$ relative to X_l, $l=1,2$, $l\neq k$.

iii) $G_k(x,y)$, $k=1,2$, have the same local singularity on E.

Then we have the following

THEOREM 2 *Suppose that $G_k(x, y)$, $k = 1, 2$, have the properties GB), GC) and GS). Then i), ii) and iii) are equivalent each other.*

Proof. The fact that iii) $\Longrightarrow$ ii) follows from Proposition 2. We can prove that ii) $\Longrightarrow$ i) on the same way as in the proof of Lemma 3b. The proof of i) $\Longrightarrow$ iii) is as follows. Let us choose an open set V of E such that $\inf_{x, y \in V} G_k(x, y) > 0$ for $k = 1, 2$ and

$$C_1 P_x^1(\sigma_M < +\infty) \leqq P_x^2(\sigma_M < +\infty) \leqq 1/C_2\, P_x^1(\sigma_M < +\infty) \tag{24}$$

for every compact set $M \subset V$. Let V_k, $k = 1, 2$, be open sets such that $V_1 \subset \bar{V}_1 \subset V_2 \subset \bar{V}_2 \subset V$ and set $M_1^k = \sup G_k(x, y)$ and $M_2^k = \inf G_k(x, y)$, where the supremum and the infimum are taken over the set $(V - \bar{V}_2) \times \bar{V}_1$. Then it follows from (24) that

$$C_2 M_2^2 / M_1^1 \cdot \mu_M^2(M) \leq \mu_M^1(M) \leq M_1^2 / C_1 M_2^1 \cdot \mu_M^2(M) \tag{25}$$

for every compact set $M \subset V_1$, where $\mu_M^k(dy)$, $k = 1, 2$, denote the capacitary measures on M for (X_k, G_k), $k = 1, 2$. Now let us fix arbitrary x, $y \in V_1$, $x \neq y$, and choose a neighborhood U of y such that $\bar{U} \subset V_1$ and $\sup_{z \in \bar{U}} G_k(x, z) \leq 2 \inf_{z \in \bar{U}} G_k(x, z)$, $k = 1, 2$. Then, substituting $\bar{U}$ in (24) and (25) instead of M, we get

$$\frac{C_1 C_2 M_2^1}{4 M_1^2} G_2(x, y) \leq G_1(x, y) \leq \frac{4 M_1^1}{C_1 C_2 M_2^2} G_2(x, y).$$

Consequently $G_k(x, y)$, $k = 1, 2$ have the same local singularity on E. The proof is complete.

Naturally i) implies that $K_{X_1}^r = K_{X_2}^r$ for each compact or open set $K \subset E$ by Lemma 2. But it is open whether the converse is valid. We will give a certain converse to Theorem 1 concerning regular points within a restricted class of Markov processes on R^n in the next section.

We close this section with the remark that the conclusions of Theorem 1 and its Corollaries are expressed in the strength and the weakness of the fine topology by using Remark 1 of § 2.

§ 4. Comparison theorems (II)

Throughout this section we will consider Markov processes in R^n $(n \geqq 3)$. We always assume that Markov processes satisfy $M1) \sim M3)$ and have Green functions with $R2)$ without referring.

Set

$\Phi = \{\varphi$; φ is positive, continuous and monotone decreasing function on $(0, \delta)$ for some $\delta > 0$ such that $\int_0^\delta r^{n-1}\varphi(r)dr < +\infty$ and $\lim_{r\to 0}\varphi(r) = +\infty\Big\}$;

$\Phi_p = \{\varphi$; $\varphi\in\Phi$ and $r^p\varphi(r)$ is monotone on $(0, \delta)\}$.

We say that a kernel $G(x, y)$ on R^n has an *isotropic singularity* $\varphi\in\Phi(\Phi_p)$ provided that

$$G(x, y) \approx \varphi(|x-y|) \quad \text{on} \quad R^n.$$

Let X_0 be a Markov process on R^n which has a Green function $G_0(x, y)$ with isotropic singularity $\varphi_0\in\Phi_{n-\alpha}$ for some $2\geqq\alpha>0$. Moreover let us assume that X_0 satisfies Hunt's condition (H)[8]. In other words, $K^r_{X_0}\neq\phi$ for a compact set K provided that $P^0_x(\sigma_K<+\infty)>0$ for some $x\in R^n$.

Our aim is to show the following

THEOREM 3[9] *Let X be a Markov process which has a Green function $G(x, y)$ with isotropic singularity $\varphi\in\Phi$. Suppose that X satisfies Hunt's condition (H) and*

$$K^r_X = K^r_{X_0} \tag{1}$$

for every compact set $K\subset R^n$. Then it follows that

$$\varphi(r) \asymp \varphi_0(r), \quad r\to 0\text{[10]} \tag{2}$$

For the proof of Theorem 2 we will prepare two lemmas.

LEMMA 5 *Let X_k, $k=1, 2$ be Markov processes which have Green functions $G_k(x, y)$, $k=1, 2$ with isotropic singularities $\varphi_k\in\Phi$, $k=1, 2$, respectively. If we suppose that*

i) *$\varphi_1\in\Phi_n$ and there exists a positive constant λ such that $\frac{1}{\lambda}r^{-n}\int_0^r s^{n-1}\varphi_1(s)ds\leqq\varphi_1(r)$ for $0<s<\delta$;*

ii) $$\varliminf_{r\to 0}\varphi_1(r)/\varphi_2(r) = 0,$$

then there exists a compact set K such that

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[8] The condition (H) holds for a fairly large class of Markov processes. See Remark 6.

[9] In case $2\geqq\alpha>1$ this theorem has been established in [15].

[10] We write $\varphi_1(r)\asymp\varphi_2(r)$, $r\to a$, if

$$0<\varliminf_{r\to a}\varphi_2(r)/\varphi_1(r)\leqq\varlimsup_{r\to a}\varphi_2(r)/\varphi_1(r)<+\infty.$$

(3) $$\mu_K^1(K) > 0 \quad \textit{and} \quad \mu_K^2(K) = 0,$$

where $\mu_K^k(dy)$, $k = 1, 2$, *denote the capacitary measures on* K *for* X_k, $k = 1, 2$.

This lemma follows immediately from Theorem 4 and Remark in S.J. Taylor [29]. Indeed, if we choose $n+1$ as k in [29], $\varphi_1(t)$ satisfies (12) and $t^{-k+1}\int_0^t s^{k-2}\varphi_1(s)ds \leqq \lambda\varphi_1(t)$ in Theorem 4 and Remark [29] respectively by the condition (i). Hence, using Theorem 4 in [29], we see that there exists a compact set $K \subset R^{k-1} = R^n$ such that $C^{\varphi_1}(K) > 0$[11] and $h_2\text{-}m(K) < +\infty$[11], where $h_2(t) = 1/\varphi_2(t)$ under the condition (i) and (ii). Now (4) follows from the fact that $C^{\varphi_k}(K) > 0$ is equivalent to $\mu_K^k(K) > 0$, $k = 1, 2$ and $h_2\text{-}m(K) < +\infty$ implies $C^{\varphi_2}(K) = 0$.

LEMMA 6. *Let* X *be a Markov process which has a Green function* $G(x, y)$ *with isotropic singularity* $\varphi \in \Phi$ *and* B_α *be an isotropic stable process of index* α, $0 < \alpha \leqq 2$. *Suppose*

(4) $$K_X^r \supset K_{B_\alpha}^r$$

for every compact set $K \subset R^n$. *Then it holds*

(5) $$1/\varphi(r) \asymp \mu_{Q_r}(Q_r) \asymp 1/\tilde{\varphi}(r), \quad r \to 0,$$

where $Q_r = \{x \,;\, |x| \leq r\}$ *and*

(6) $$\tilde{\varphi}(r) = r^{-n}\int_0^r \varphi(s)s^{n-1}ds.$$

Proof. Set $Q(x_0, r) = \{x \,;\, |x - x_0| \leq r\}$ and $\tilde{Q}_r = \{x \,;\, r/2 \leq |x| \leq r\}$. Let us fix a constant C such that $0 < C < 1/2$ and choose a sequence $\{r_k\}_{k=1,2,\dots}$ decreasing to zero. Let $\{x_k\}_{k=1,2,\dots}$ be a sequence of points such that $|x_k| = r_k(1-C)$. We define

$$\tilde{Q} = \bigcup_k \tilde{Q}_{r_k} \cup \{0\}, \quad Q = \bigcup_k Q(x_k, Cr_k) \cup \{0\}.$$

In the following discussions we will denote the total mass of finite measure $\mu(dy)$ by $\bar{\mu}$ and denote various positive absolute constants by M_k, $k = 1, 2, 3, \cdots$. Let $\mu_K^\alpha(dy)$ be a capacitary measure on K relative to B_α. Since $\overline{\mu_{Q(x_0,r)}^\alpha} = M_1 r^{n-\alpha}$ (for example see [21], p. 204), we have

11) $C^\varphi(K)$ denotes the φ-capacity of K and $h\text{-}m(K)$ denote the h-measure of K in the sense of [29].

(7) $$\overline{\mu^{\alpha}_{\tilde{Q}_r}} \geqq \overline{\mu^{\alpha}_{Q_r}} - \overline{\mu^{\alpha}_{Q_{r/2}}} \geqq M_2 r^{n-\alpha}.$$

Hence it holds

(8) $$P^{\alpha}_0(\sigma_{\tilde{Q}_{r_n}} < +\infty) \geqq M_3 \text{ and } P^{\alpha}_0(\sigma_{Q(x_k, Cr_k)} < +\infty) \geqq M^{c}_4.$$

Combining Lemma 4.2 in [13] with (8), we have

(9) $$0 \in \tilde{Q}^{r}_{B_\alpha} \cap Q^{r}_{B_\alpha}.$$

Next we will prove that

(10) $$\varliminf_{r \to 0} P_0(\sigma_{\tilde{Q}_r} < +\infty) = M_5 > 0,$$

(11) $$\varliminf_{r \to 0} P_0(\sigma_{Q(x_r, Cr)} < +\infty) = M^{c}_6 > 0,$$

where $|x_r| = (1-C)r$. If (10) ((11)) did not hold, we can choose a sequence $\{r_k\}_{k=1,2,\dots}$ decreasing to zero such that

(12) $$\sum_{k=1}^{+\infty} P_0(\sigma_{\tilde{Q}_{r_k}} < +\infty) < +\infty \text{ (resp } \sum_{k=1}^{+\infty} P_0(\sigma_{Q(x_{r_k}, Cr_k)} < +\infty) < +\infty).$$

Using the Borel-Cantelli lemma, it follows from (12) that

(13) $$0 \notin Q^{r}_{X} \cap \tilde{Q}^{r}_{X}.$$

Since (13) contradicts to (9) and (4), both (10) and (11) must hold. From (10), we get

(14) $$1/\varphi(r) \geqq \overline{\mu_{\tilde{Q}_r}} \geqq M_5/2 \cdot 1/\varphi(r/2).$$

From (11) we get

(15) $$\varphi((1-2C)r)\overline{\mu_{Q_{Cr}}} \geqq M^{c}_7.\text{[12)]}$$

Since $\overline{\mu_{Q_{Cr}}} \leqq M_8/\varphi(Cr)$, it follows from (15) that

(16) $$\frac{\varphi((1-2C)r)}{\varphi(Cr)} \geqq M^{c}_9.$$

Combining (14) with (16), we have

(17) $$1/\varphi(r) \asymp \overline{\mu_{\tilde{Q}_r}}, \quad r \to 0.$$

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12) Note that $\overline{\mu_{Q_{Cr}}} \asymp \overline{\mu_{Q(x_r, Cr)}}$, $r \to 0$.

Now, noting that

$$\overline{\mu_{Q_{r/2}}} \leqq M_{10}\, 1/\varphi(r/2) \leqq M_{11}\, \overline{\mu_{\tilde{Q}_r}} \leqq M_{11} \overline{\mu_{Q_r}}, \tag{18}$$

it follows from (17) that

$$1/\varphi(r) \asymp \overline{\mu_{Q_r}}, \quad r \to 0. \tag{19}$$

Next we will prove

$$\frac{1}{\overline{\mu_{Q_r}}} \asymp \tilde{\varphi}(r), \quad r \to 0. \tag{20}$$

Since we have

$$\sup_{y \in Q_r} \int_{Q_r} \varphi(|z-y|)\,dz = \int_{Q_r} \varphi(|z|)\,dz \leq 2^n \inf_{y \in Q_r} \int_{Q_r} \varphi(|z-y|)\,dz,$$

it holds that

$$M_{12}\tilde{\varphi}(r) \cdot \overline{\mu_{Q_r}} \leqq \frac{1}{\bar{Q}_r} \int_{Q_r} P_z(\sigma_{Q_r} < +\infty)\,dz \leqq M_{13}\,\tilde{\varphi}(r) \cdot \overline{\mu_{Q_r}},$$

where $\bar{Q}_r$ is the volume of Q_r. Hence (20) has been proved[13]. Combining (19) with (20), we have (5). The proof is complete.

Remark 5. If we assume that (4) holds for each open set K instead of each compact set, then (5) is also valid. For the proof we only need a slight modification of the above.

Proof of theorem 3. Since $\varphi_0 \in \Phi_{n-\alpha}$, it follows from Theorem 1 that the condition (4) holds for X_0 and X. Hence $\varphi_0(r) \asymp \tilde{\varphi}_0(r)$, $r \to 0$ and $\varphi(r) \asymp \tilde{\varphi}(r)$, $r \to 0$. Note that $\tilde{\varphi}_0(r)$ and $\tilde{\varphi}(r)$ satisfy i) in Lemma 5. Since X and X_0 satisfy Hunt's condition (H), it follows from (1) that $\tilde{\varphi}_0(r) \asymp \tilde{\varphi}(r)$, $r \to 0$ by using Lemma 5. Thus we have proved (2).

Using Remark 2 of §2 and Remark 5, we can prove

THEOREM 3′. *Let $\mathcal{O}_{X_0}$ and $\mathcal{O}_X$ be fine topologies of X_0 and X respectively. If $\mathcal{O}_X$ is equivalent to $\mathcal{O}_{X_0}$, then $\varphi(r) \asymp \varphi_0(r)$, $r \to 0$.*

Finally we note

Remark 6. Let X be a Markov process having a Green function $G(x,y)$ such that

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13) Note that the second term of the above inequality equals to 1.

$$G(x, y) \approx g(x-y) \text{ on } R^n,$$

where $g(x-y)$ is a Green function with GS) of some symmetric Lévy process $\tilde{X}$. Then X satisfies the condition (H).

This is proved as follows. Note that it follows immediately from Proposition (4.10) in [1], p. 289 that $\tilde{X}$ satisfies (H). Combining Lemma 3b with Remark 3, we can choose a neighborhood V for each fixed point and constants $C_1 \geqq C_2 > 0$ such that hitting probabilities of X and $\tilde{X}$ are (C_1, C_2)-dominated each other at V. On the other hand it holds by Corollary 1 of Theorem 1 that $K_X^r = K_{X_0}^r$ for each compact or open set K. Summing up the above results, we can show that X satisfies (H).

§ 5. Regular points for diffusion processes with continuous coefficients

Throughout this section we let $(a_{jk}(x))$ be a symmetric matrix such that

$$\lambda_2|\xi|^2 \geqq \sum_{j,k=1}^n a_{jk}(x)\xi_j\xi_k \geqq \lambda_1|\xi|^2, \quad |\xi| \neq 0, \quad \xi \in R^n, \tag{1}$$

where $+\infty > \lambda_2 \geqq \lambda_1 > 0$ and the entries $a_{jk}(x)$ are bounded, continuous on R^n. For a differential operator A defined by

$$Au(x) = \sum_{j,k=1}^n a_{jk}(x) \frac{\partial^2}{\partial x_j \partial x_k} u(x), \tag{2}$$

there exists a minimal diffusion process X_A on a bounded domain D with a smooth boundary ∂D. X_A satisfies

X_A i) *the strong infinitesimal operator* $\mathfrak{A}$ *of* $\{T_t\}$ *coincides with* A *on* $C^2(\bar{D})$;

X_A ii) $\{T_t\}$ *is strongly continuous on* $C_0(D)$;

X_A iii) X_A *is of strongly Feller type*;

X_A iv) X_A *satisfies* $M2$).

(See [17], [27] and [28].) Hereafter we shall always deal with the above process X_A. The property $M3$) does not always hold. But in case $n \geqq 3$ we can prove $M3$) by using the next Lemma obtained by Girbarg-Serrin [7].

LEMMA 7. *Suppose* $n \geqq 3$. *Let* $u(x)$ *be a non-constant function which is*

14) We say that u is subharmonic (harmonic) in an open set Q if it holds that $E_x u(x_{\tau_S}) \geqq u(x)$ (resp. $E_x u(x_{\tau_S}) = u(x)$) for every open set S such that $x \in S \subset \bar{S} \subset Q$.

subharmonic[14] *in the punctured ball* S_0 $(=\{x\,;0<|x|<r_0\})$ *and continuous on* $\{x\,;0<|x|\leqq r_0\}$. *We set* $M=\max\limits_{|x|=r_0} u(x)$, *Then, if*

$$u(x)=0(|x|^{2-n+\delta}) \quad as \quad |x|\to 0$$

for some $\delta>0$, *it follows that* $u<M$ *in* S_0, *and furthermore* $\limsup\limits_{x\to 0} u(x)<M$.

The above Lemma is proved for $u\in C^2(S_0)$ such that $Au\geqq 0$ in S_0 in [7], but without any change of the proof the assertion is valid for the function u in the above Lemma 7.

Now, set $u(x)=p_x^A(\sigma_{\{x_0\}}<+\infty)$. Then u is harmonic[14] in $D-\{x_0\}$. By X_A iii) $u(x)$ is continuous on $D-\{x_0\}$. Further $\lim\limits_{x\to\partial D} u(x)=0$ by X_A i) and X_A ii). Applying Lemma 5 to u, we have $\lim\limits_{x\to x_0} u(x)=0$. Since u is excessive relative to X_A, it follows that $u\equiv 0$. Consequently we get

X_A v) *X_A satisfies M3) provided* $n\geqq 3$.

In order to state our result we will prepare some notations. We let the matrix $(A_{jk}(x))$ be the inverse of the coefficients matrix $(a_{jk}(x))$. Set

$$L=\sup_{j,k,x\in D_{\delta_0}} |A_{jk}(x)| \qquad M=L^2n^3/\lambda_1,$$

where $D_{\delta_0}=\{x\,;\text{distance }(x,D)<\delta_0/\lambda_1\}$. We define

$$a(r)=\sup_{\substack{j,k\\ x\in D_{\delta_0}}}\ \sup_{|h|<r/\lambda_1} |a_{jk}(x+h)-a_{jk}(x)| \tag{3}$$

$$\rho_y(x)=\{\sum_{j,k=1}^n A_{jk}(y)(x_j-y_j)(x_k-y_k)\}^{1/2}. \tag{4}$$

We will denote $\bar{F}_A(\underline{F}_A)$ the collection of positive continuous functions f on $(0,s_0)$ for some $0<s_0<\delta_0$ which saitsfies

$$f(\rho)\geqq\frac{(n-1)+n^2(M+L)a(\rho)}{\rho(1-n^2Ma(\rho))},\quad 0<\rho<s_0, \tag{5}$$

(resp. (5′) $$f(\rho)\leqq\frac{(n-1)-n^2(M+L)a(\rho)}{\rho(1+n^2Ma(\rho))},\quad 0<\rho<s_0.)$$

For a positive continuous function f on $(0,s_0)$ for some $0<s_0<\delta_0$, we define

$$\tilde{f}(r)=\int_r^{s_0}\exp\left(\int_t^{s_0}f(\rho)d\rho\right)dt. \tag{6}$$

Let us consider the following conditions on a function φ on $(0,s_0)$:

($\phi 1$) *for every fixed* $0<t<1$ *it holds* $\varphi(tr)/\varphi(r)\leq C_t<+\infty$ *for every* $0<r<s_0$:

$$(2\phi)\quad((\phi 2'))\qquad \varphi(r)\asymp \tilde{f}(r),\ r\to 0,\ \textit{for some}\ f\in\bar{F}_A\ (\textit{resp.}\ f\in\underline{F}_A).$$

Set $\bar{\phi}_A(\underline{\phi}_A)=\{\varphi\ ;\ \varphi$ is a positive function on $(0,s_0)$ for some $0<s_0<\delta_0$ which satisfies $\phi 1)$ and $\phi 2)$ (resp $\phi 2')$)$\}$. The sets $\bar{\phi}_A$ and $\underline{\phi}_A$ depend on the degree of the continuity of the coefficients of A.

LEMMA 8. *Suppose* $n\geqq 3$. i) *For each* $0<\alpha<2$, $r^{\alpha-n}\in\bar{\phi}_A$ ii) *If the coefficients of* A *are uniformly Dini continuous, that is,*

$$\int_0^{\delta_0}\frac{a(\rho)}{\rho}<+\infty, \tag{7}$$

then r^{2-n} *belongs to both* $\bar{\phi}_A$ *and* $\underline{\phi}_A$.

Proof. i) For $0<\alpha<2$, if we choose sufficientely small $s_0>0$, we see that $F(\rho)=(n+1-\alpha)/\rho\in\bar{F}_A$. It is clear that $\tilde{f}(r)\asymp r^{\alpha-n}$, $r\to 0$. ii) If s_0 is sufficiently small, we can choose constants $M_k>0$, $k=1,2$, so that $(n-1)/\rho+M_1a(\rho)/\rho\in\bar{F}_A$ and $(n-1)/\rho-M_2a(\rho)/\rho\in\underline{F}_A$. Hence, using (7), r^{2-n} belongs to both $\bar{\phi}_A$ and $\underline{\phi}_A$.

Now we are ready to state our theorem.

THEOREM 4. *Suppose* $n\geqq 3$. *Let* X *be a Markov process on* R^n *with the properties* $M1)\sim M3)$ *which has a Green function* $G(x,y)$ *with* GC) *and* R2). *If* $G(x,y)$ *has an isotropic singularity* $\varphi\in\bar{\phi}_A(\underline{\phi}_A)$, *then*

$$K_X^r\subset K_{X_A}^r\quad (\textit{resp.}\ (8')\quad K_{X_A}^r\subset K_X^r) \tag{8}$$

for each compact or open set $K\subset D$.

Combining Theorem 4 with Lemma 8, we get the followings.[15]

COROLLARY 1. *For an isotropic stable process* B_α *of index* α, $0<\alpha<2$, *it follows that*

$$K_{B_\alpha}^r\subset K_{X_A}^r$$

COROLLARY 2. *Suppose that the coefficients of* A *are uniformly Dini continuous. Then*

$$K_{X_A}^r=K_B^r,$$

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[15] In the sequel we assume that $n\geqq 3$ and K is a compact or an open set in D.

where B denotes the n-dimensional Brownian motion.

Proof of theorem 4. We define

$$A_y(x)=\Big(\sum_{j,k=1}^{n}a_{jk}(x)\frac{\partial\rho_y}{\partial x_j}(x)\frac{\partial\rho_y}{\partial x_k}(x)\Big)^{-1}A\rho_y(x),$$

$$B_{jk}^y(x)=\sum_{l,m=1}^{n}A_{jl}(y)(x_l-y_l)A_{km}(y)(x_m-y_m).$$

Then we have

$$A_y(x)= \tag{9}$$

$$\frac{\{(n-1)+\sum_{j,k=1}^{n}(a_{jk}(x)-a_{jk}(y))A_{jk}(y)\}\rho_y(x)-(1/\rho_y(x))\sum_{j,k=1}^{n}(a_{jk}(x)-a_{jk}(y))B_{jk}^y(x)}{\rho_y(x)^2+\sum_{j,k=1}^{n}(a_{jk}(x)-a_{jk}(y))B_{jk}^y(x)}$$

and

$$\sum_{j,k=1}^{n}|B_{jk}^y(x)|\leqq M\rho_y(x)^2 \tag{10}$$

Combining (3), (9) and (10), we get

$$\frac{(n-1)-n^2(L+M)a(\rho_y(x))}{\rho_y(x)+n^2Ma(\rho_y(x))\rho_y(x)}\leqq A_y(x)\leqq\frac{(n-1)+n^2(L+M)a(\rho_y(x))}{\rho_y(x)-n^2Ma(\rho_y(x))\rho_y(x)}, \tag{11}$$

Let us choose $f\in\bar{F}_A(\underline{F}_A)$ and set $F_y(x)=\tilde{f}(\rho_y(x))$ for each fixed y, where $\tilde{f}$ is defined by (6). Since we have

$$AF_y(x)=\sum_{j,k}a_{jk}(x)\frac{\partial\rho_y}{\partial x_j}(x)\frac{\partial\rho_y}{\partial x_k}(x)\exp\Big(\int_{\rho_y(x)}^{s_0}f(\rho)d\rho\{-A_y(x)+f(\rho_y(x))\}\tag{12}$$

for $x\in Q_y-\{y\}$, where $Q_y=\{x\,;\,\rho_y(x)<s_0\}$, it follows from (12) and (5) (resp. (5′)) that

$$AF_y(x)\geqq 0\quad(\text{resp. } AF_y(x)\leqq 0)$$

for $x\in Q_y-\{y\}$. Accordingly $F_y(\cdot)$ is 1-subharmonic (resp. 1-superharmonic) at $(x,Q_y-\{y\})$ relative to X_A. On the other hand, since $\varphi\in\bar{\Phi}_A$ (resp. $\varphi\in\underline{\Phi}_A$), there exists constants $C_k>0$, $k=1,2,3,4$ and $\delta>0$ such that

$$C_4\tilde{f}(\rho_y(x))\leqq C_3\varphi(\rho_y(x))\leqq\varphi(|x-y|)\leqq C_2\varphi(\rho_y(x))\leqq C_1\tilde{f}(\rho_y(x)),\ 0<|x-y|<\delta.$$

Hence, setting $Q=\Big\{x\,;\,|x-x_0|<\frac{\delta}{2\lambda_2}\Big\}$ for a fixed $x_0\in D$, $\varphi(|\cdot-y|)$ is C_4/C_1-subharmonic (resp. C_1/C_4-superharmonic) at $(x_0,Q-\{y\})$ for every $y\in Q$.

Since X_A satisfies $M1)$, $M2)$ by X_A ii), X_A iv) respectively, $R1)$ holds for X_A by Lemma 1. Now let us apply Lemma 3a (resp. Lemma 3a and Remark 4) to X_A and X. Then (8) (resp (8′)) follows immediately. The proof is complete.

§ 6. Regular points for Markov processes subordinate to the diffusion process with uniformly Hölder continuous coefficients.

Our object of this section is the class of Markov processes subordinate to the diffusion processes with uniformly Hölder continuous coefficients. Singularities of Green functions of Markov processes of such a class are fairly abound in the variety, though they are isotropic.

Let $\mathscr{X}$ be the class of diffusion processes X[16] on R^n whose generator is a uniformly elliptic partial differential operator A of second order with bounded, uniformly Hölder continuous coefficients. For convenience we will denote by $(B(t), P^b)$ the n-dimensional Brownian motion. A process $(z(t), P)$ is called a subordinator provided that it is one-sided Lévy process on $[0, +\infty)$ starting at the origin which has increasing paths. It is known that for such a process $E\{e^{-sz(t)}\} = e^{-t\phi(s)}$ for all $t \geqq 0$ and $s \geqq 0$, where

$$\phi(s) = bs + \int_0^{+\infty} (1 - e^{-su})\nu(du). \tag{1}$$

In (1), b is a nonnegative constant and ν is a Borel measure on $(0, +\infty)$ satisfying $\int_0^{+\infty} u(1+u)^{-1}\nu(du) < +\infty$. The function ϕ is called the exponent of $z(t)$ and ν is called the Lévy measure of $z(t)$. We let $\mathscr{Z}$ be a collection of the subordiantors. If we set

$$\begin{aligned} P_z(t, x, dy) &= \int_0^{+\infty} P(s, x, dy) P(z(t) \in ds) \\ (P_z^b(t, x, dy) &= \int_0^{+\infty} P^b(s, x, dy) P(z(t) \in ds))^{17)}, \end{aligned} \tag{2}$$

then there exists a Markov process on R^n whose transition probability is $P_z(t, x, dy)$ (resp. $P_z^b(t, x, dy)$) and the semi-group of such a process is strongly

16) X is of strongly Feller type and its semi-group is strongly continuous on $C_0(R^n)$. (For example see [**11**].)

17) $P(s, x, dy)$ $(P^b(s, x, dy))$ denotes the transition probability of X (resp. B). It is known that $p(s, x, dy)$ $(P^b(s, x, dy))$ has a density $p(s, x, y)$ (resp. $p^b(s, x, y)$) with respect to the Lebesque measure dy such that

$pi)$ *$p(s, x, y)$ is positive, continuous on* $(0, +\infty) \times R^n \times R^n$.

continuous on $C_0(R^n)$. (See, for example, N. Ikeda-S. Watanabe [11].) We will denote it by X_z (resp. B_z) in the sequel. Set

(3) $$U(t)=\int_0^{+\infty}P(z(s)\leq t)ds$$

for $z(t)\in\mathscr{Z}$ with an exponent $\psi(u)$. Then

(4) $$1/\psi(u)=\int_0^{+\infty}e^{-ut}dU(t)^{18)}.$$

First note that

LEMMA 9. *Suppose the Lévy measure $\nu(du)$ of $z(t)\in\mathscr{Z}$ is non-trival. Then, for $U(t)$ of the form* (3), *we have*

(5) $$\int_\delta^\infty\frac{1}{t^{1+a}}dU(t)<+\infty$$

for every fixed $a>0$ and $\delta>0$.

Proof. Since ν is non-trivial, we can choose constants $K>0$ and $u_0>0$ so that for every $0<s\leq u_0$

(6) $$\psi(s)\geqq Ks.$$

Combining (6) with (4) it follows that

$$1/K\cdot\int_0^{u_0}s^{a-1}ds\geqq\int_0^{u_0}s^a ds\int_0^{+\infty}e^{-st}dU(t)\geqq\int_\delta^{+\infty}\frac{1}{t^{1+a}}dU(t)\Big(\int_0^{\delta u_0}v^a e^{-v}dv\Big)$$

for each fixed $\delta>0$, which implies (5). The proof is complete.

For the transition probability density $p(t,x,y)$ $(p^b(t,x,y))$ of $X\in\mathscr{X}$ (resp. B) we define

$$G_z(x,y)=\int_0^{+\infty}p(t,x,y)dU(t),\quad(\text{resp. } g_z(|x-y|)=\int_0^{+\infty}p^b(t,x,y)dU(t)).$$

Combining (5) with the following estimate:

p ii) $M_0t^{-n/2}\exp(-\alpha_0|y-x|^2/t)\geqq p(t,x,y)\geqq$

$$M_1t^{-n/2}\exp(-\alpha_1|y-x|^2/t)-M_2t^{-n/2+\lambda}\exp(-\alpha_2|y-x|^2/t),$$

where M_0, M_1, M_2, α_0, α_1, α_2 and λ are positive constants [12]; we get

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18) In the following the integral sign $\int_a^b\cdot dU(t)$ means that the Lebesgue-Stieltjes integral on $(a,b]$ in case a,b are finite. $\int_a^\infty\cdot dU(t)$ is defined as usual.

LEMMA 10. *If* $n \geqq 3$, $G_z(x, y)$ *is a Green function of* X_z *which has the properties GB) and GC).*

Indeed, noting that for each fixed $\delta > 0$

$$\sup_{|x-y|>\delta} p(t, x, y) \leq M_0 t^{-n/2} \exp(-\alpha_0 \delta^2/t),$$

$GB)$ follows immediately and $G_z(x, y)$ is continuous on $|x-y| > \delta$ by Lebesgue convergence theorem.

Hereafter we shall always assume that $n \geqq 3$.

Let $\mathscr{L}$ be the class of continuous positive functions L on $(0, +\infty)$ which vary slowly at zero, that is, $\lim_{t\to 0} L(tx)/L(t) = 1$ for each fixed $x > 0$. The following relation is essential in our theorem.:

$$1/\psi(u) \sim u^{-\alpha} L(1/u), \quad u \to +\infty \text{[19]} \tag{7}$$

is equivalent to

$$U(t) \sim \frac{1}{\Gamma(1+\alpha)} t^\alpha L(t), \quad t \to 0, \tag{8}$$

where $L \in \mathscr{L}$ and $U(t)$, $\psi(t)$ are the ones of (3) and (4) respectively. (See, for example, W. Feller [6], Th. 3, p. 422.) Set $\mathscr{L}_i = \{L(t) \in \mathscr{L}$ which is monotone increasing on $(0, \delta)$ for some $\delta > 0\}$ and $\mathscr{L}_d = \{L(t) \in \mathscr{L}$ which is monotone decreasing on $(0, \delta)$ for some $\delta > 0\}$. We define

$$\mathscr{Z}^i_\alpha \, (\mathscr{Z}^d_\alpha) = \{z(t) \in \mathscr{Z} \text{ whose exponent } \psi \text{ satisfies (7) for } L \in \mathscr{L}_i \text{ (resp. } L \in \mathscr{L}_d)\}$$

Remark 7. $L(t) \in \mathscr{L}$ has the following representation:

$$L(t) = b(t) \exp\left\{ -\int_t^\delta \frac{a(u)}{u} du \right\}, \tag{9}$$

where $a(u)$, $b(u)$ are continuous such that $\lim_{u\to 0} a(u) = 0$ and $\lim_{u\to 0} b(u) > 0$. From (9) it is easily proved that $t^\gamma L(t)/b(t)$ $(t^{-\gamma} L(t)/b(t))$, $\gamma > 0$, is monotone increasing (resp. monotone decreasing) on some interval $(0, \delta_0)$.

Now we are ready to state our results.

THEOREM 5. *Let* $X \in \mathscr{X}$ *and* $z(t) \in \mathscr{Z}^i_\alpha (\mathscr{Z}^d_\alpha)$ *for some* $0 < \alpha \leqq 1$. *Then it follows that, for every* α' *such that* $\alpha < \alpha' \leqq 1$[20] *(resp.* $0 < \alpha' < \alpha$*),*

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19) We will write $f(x) \sim g(x)$, $x \to a$ provided $\lim_{x\to a} f(x)/g(x) = 1$

20) In this case we assume that $1 > \alpha > 0$.

(10) $$K^r_{B_{2\alpha}} \subset K^r_{X_z} \subset K^r_{B_{2\alpha'}} \quad (resp.\ K^r_{B_{2\alpha'}} \subset K^r_{X_z} \subset K^r_{B_{2\alpha}})$$

holds for each compact or open set $K \subset R^n$, *where* $B_{2\alpha}$, $B_{2\alpha'}$ *are isotropic stable processes of index* 2α, $2\alpha'$ *respectively. Furthermore there exists compact set* K, $\tilde{K}$ *such that*

(11) $$K^r_{B_{2\alpha}} \subsetneqq K^r_{X_z},\ \tilde{K}^r_{X_z} \subsetneqq \tilde{K}^r_{B_{2\alpha'}} \quad (resp.\ K^r_{B_{2\alpha}} \subsetneqq K^r_{X_z},\ \tilde{K}^r_{X_z} \subsetneqq \tilde{K}^r_{B_{2\alpha}})$$ [21]

provided that $\lim_{t\to 0} L(t) = 0$ (*resp.* $\lim_{t\to 0} L(t) = +\infty$.)

THEOREM 6. *Let* $X^k \in \mathscr{X}$ *and* $z_k(t) \in \mathscr{Z}^i_\alpha$ $(\mathscr{Z}^d_\alpha)$, $k=1,2$, *for* $0<\alpha\leqq 1$. *Suppose that*

(12) $$\psi_1(s) \asymp \psi_2(s), \quad s \to +\infty,$$

where $\psi_k(s)$, $k=1,2$, *are exponents of* $z_k(t)$, $k=1,2$, *respectively. Then*

(13) $$K^r_{X^1_{z_1}} = K^r_{X^2_{z_2}}$$

holds for each compact or open set $K \subset R^n$.

For the proof we will prepare two Lemmas.

LEMMA 11.

(14) $$g_z(r) \asymp r^{2\alpha-n} L(r^2), \quad r \to 0,$$

provided $z(t) \in \mathscr{Z}^i_\alpha$ *or* $\mathscr{Z}^d_\alpha$ *for* $0<\alpha\leqq 1$.

Proof. For simplicity we assume that $L(t)$ is monotone on $(0,2]$. Let us set

$$I_1(x) = \int_{|x|^2}^{2|x|^2} p^b(t,0,x)\,dU(t).$$

Then, by the formula of the integral by part and (8), we get

(15) $$\lim_{x\to 0} \frac{I_1(x)}{|x^{2\alpha-n} L(|x|^2)|} = (2\pi)^{-\frac{n}{2}} \frac{1}{\Gamma(1+\alpha)} \left\{ 2^{-\frac{n}{2}+\alpha} e^{-\frac{1}{4}} - e^{-\frac{1}{2}} + K_1 - K_2 \right\} = K_3,$$

where

$$K_1 = \frac{n}{2} \int_1^2 u^{-\frac{n}{2}-1+\alpha} e^{-\frac{1}{2u}} du, \quad K_2 = \frac{1}{2} \int_1^2 u^{-\frac{n}{2}-2+\alpha} e^{-\frac{1}{2u}} du.$$

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[21] As we see from the proof below, we can choose K and $\tilde{K}$ such that $\tilde{K}^r_{B_{2\alpha}} = K^r_{X_z} = \phi$ (resp. $K^r_{B_{2\alpha'}} = \tilde{K}^r_{X_z} = \phi$).

Now, replacing $U(t)$ with t^α in the integral $I_1(x)$, we have

(16) $$\lim_{x\to 0}\frac{I_1(x)}{|x|^{2\alpha-n}}=K_3.$$

On the other hand, in case $U(t)=t^\alpha$, it holds by changing the variable

(17) $$I_1(x)=|x|^{2\alpha-n}\alpha\int_1^2(2\pi u)^{-\frac{n}{2}}u^{\alpha-1}e^{-\frac{1}{2u}}du.$$

Combining (16) with (17), we see that $K_3>0$, which together with (15) implies

(18) $$I_1(x)\asymp|x|^{2\alpha-n}L(|x|^2),\qquad |x|\to 0.$$

If we set

$$I_2(x)=\int_0^{|x|^2}p^b(t,0,x)dU(t),$$

then we have

(19) $$I_2(x)\leqq K_4|x|^{2\alpha-n}L(|x|^2),$$

for sufficiently small x and some constant K_4, because $I_2(x)\leqq p^b(|x|^2,0,x)\times U(|x|^2)+n/2\cdot(2\pi)^{-\frac{n}{2}}|x|^{-n}\int_0^1 u^{-n/2-1}e^{-1/2u}U(u|x|^2)du$. Choose $\varepsilon>0$ so that $U(t)\leqq 2t^\alpha L(t)/\Gamma(1+\alpha)$ for $0<t\leqq\varepsilon$ and define

$$I_3(x)=\int_{|x|^2}^{\varepsilon}p^b(t,0,x)dU(t).$$

Then it holds that, for some constant K_5,

(20) $$I_3(x)\leqq K_5+n(2\pi)^{-n}/\Gamma(1+\alpha)\cdot|x|^{-n+2}\alpha\int_1^{\varepsilon/|x|^2}u^{-\frac{n}{2}-1+\alpha}e^{-\frac{1}{2u}}L(u|x|^2)du.$$

If $L\in\mathscr{L}_d$, we have from (20)

(21) $$I_3(x)\leqq K_5+K_6|x|^{-n+2\alpha}L(|x|^2).$$

In case $L\in\mathscr{L}_i$ we will use the representation (9). Choose γ_0 such that $1<\gamma_0<n/2$. Then, by Remark 7, $u^{-n/2+\gamma_0}L(u)/b(u)$ is monotone decreasing. Hence, from (20) we have (21) for $L\in\mathscr{L}_i$. Since $I_2(x)+I_3(x)+\int_\varepsilon^{+\infty}p^b(t,0,x)\times dU(t)\geqq g_z(|x|)\geqq I_1(x)$, we have (14) by combining (5), (19), (21) and (18). The proof is complete.

Remark 8. $g_z(r)$ for $z(t)\in\mathscr{Z}_\alpha^i$ or $\mathscr{Z}_\alpha^d$ satisfies

(22) $\forall C>0,\ \exists K_C>0,\ \exists\delta_C>0,\ g_z(Cr)/g_z(r)\leqq K_C \quad \text{for} \quad 0<r<\delta_C.$

LEMMA 12. *For,* $z(t)\in\mathscr{X}^i_\alpha$ *or* $\mathscr{X}^d_\alpha$ *it holds that*

(23) $$G_z(x,y)\approx g_z(|x-y|) \quad on \quad R^n.$$

Proof. For each fixed $\delta>0$ we have, by P ii),

(24) $$G_z(x,y)\geqq\int_0^\delta p(t,x,y)dU(t)\geqq I_1(x,y)-M_2\delta^\lambda(2\pi)^{n/2}g_z(\sqrt{2\alpha_2}\,|x-y|),$$

$$I_1(x,y)=M_1\int_0^\delta t^{-n/2}\exp(-\alpha_1|y-x|^2/t)dU(t).$$

On the other hand

(25) $$I_1(x,y)\geqq M_1(2\pi)^{n/2}g_z(\sqrt{2\alpha_1}\,|x-y|)-M_1I(\delta),$$

where $I(\delta)=\int_\delta^{+\infty}t^{-n/2}dU(t)$. Let us choose $\delta_1>0$, $K(\alpha_1,\alpha_2)>0$ such that

(26) $$g_z(\sqrt{2\alpha_2}\,|x-y|)\leqq K(\alpha_1,\alpha_2)g_z(\sqrt{2\alpha_1}\,|x-y|)$$

for $|x-y|<\delta_1$. This is possible by (22). Set

$$\delta_0=\left(\frac{M_1}{4M_2}\frac{1}{K(\alpha_1,\alpha_2)}\right)^{1/\lambda}.$$

Since $I(\delta_0)<+\infty$ by (5) and $\lim_{x\to y}g_z(|x-y|)=+\infty$ by (14), we can choose $\delta_2>0$ so that

(27) $$M_1I(\delta_0)\leqq M_1/2\cdot(2\pi)^{n/2}g_z(\sqrt{2\alpha_1}\,|y-x|)$$

for $|y-x|<\delta_2$. Combining (24) with (27), we get

(28) $$G_z(x,y)\geqq\frac{M_1}{4}((2\pi)^{n/2}g_z(\sqrt{2\alpha_1}\,|x-y|)$$

for $0<|x-y|<\min(\delta_1,\delta_2)$. Since it is clear that

(29) $$G_z(x,y)\leqq M(2\pi)^{n/2}g_z(\sqrt{2\alpha_0}\,|x-y|)$$

by P i), the proof of (23) is complete by using (22).

Proof of theorem 5 *and theorem* 6. As mentioned before, the semi-group of X_z is strongly continuous on $C_0(R^n)$. Furthermore, $G_z(x,y)$ satisfies GB) and GC) by Lemma 8 and has an isotropic singularity $g_z(r)$ by Lemma 9. Therefore it follows immediately from Lemma 1 in **[15]** that $R2$) holds for X_z. Using Lemma 1 in **§2**, $R1$) follows from $M1$) and $M2$). $M3$) is clear from $R2$) and from the fact that $\lim_{x\to y} G_z(x, y) = +\infty$. Consequently we can apply the result in § 3 to the above process X_z. If we assume (12), then $g_{z_1}(r) \asymp g_{z_2}(r)$, $r \to 0$ by (14). Hence $G^1_{z_1}(x, y) \approx G^2_{z_2}(x, y)$, $|x-y| \to 0$ by (23), where $G^1_{z_1}(x, y)$ and $G^2_{z_2}(x, y)$ are Green functions of $X^1_{z_1}$ and $X^2_{z_1}$ respecitvely. This implies (13) by the Corollary 1 of Theorem 1. Thus Theorem 5 has been proved. If $z(t) \in \mathscr{Z}^i_\alpha$ ($\mathscr{Z}^d_\alpha$), it follows immediately from Corollary 3 of Theorem 1 that $K^r_{B_{2\alpha}} \subset K^r_{B_z}$ (resp. $K^r_{B_{2\alpha}} \supset K^r_{B_z}$). Using the representation (9), we have

$$g_z(r) \asymp r^{2\alpha-n} \exp\left\{ -\int_{r^2}^{\delta} \frac{a(u)}{u} du \right\}, \quad r \to 0,$$

and $r^{2(\alpha-\alpha')} \exp\left\{ -\int_{r^2}^{\delta} \frac{a(u)}{u} du \right\}$ is monotone increasing (monotone decreasing) provided that $\alpha > \alpha'$ (resp. $\alpha' < \alpha$). Accordingly $K^r_{B_z} \subset K^r_{B_{2\alpha'}}$ (resp. $K^r_{B_{2\alpha'}} \subset K^r_{B_z}$) holds for $z \in \mathscr{Z}^i_\alpha$ (resp. $\mathscr{Z}^d_\alpha$) provided $\alpha < \alpha'$ (resp. $\alpha > \alpha'$). Since $K^r_{B_z} = K^r_{X_z}$ by Lemma 12 and Corollary 1 of Theorem 1, we have proved (10). Noting that X_z, $B_{2\alpha}$ and $B_{2\alpha'}$ satisfy the condition (H) by Remark 6, (11) follows from Theorem 3.

Using Remark 1 and Theorem 5, we get

THEOREM 5′. *Let $\mathcal{O}, \mathcal{O}_{2\alpha}$ and $\mathcal{O}_{2\alpha'}$ be fine topologies induced by X_z, $B_{2\alpha}$ and $B_{2\alpha'}$ respectively. Then*

$$\mathcal{O}_{2\alpha} \prec \mathcal{O} \prec \mathcal{O}_{2\alpha'} \quad (\textit{resp. } \mathcal{O}_{2\alpha'} \prec \mathcal{O} \prec \mathcal{O}_{2\alpha})^{22)}$$

Furthermore

$$\mathcal{O}_{2\alpha} \underset{\neq}{\prec} \mathcal{O} \underset{\neq}{\prec} \mathcal{O}_{2\alpha'}, \quad (\textit{resp. } \mathcal{O}_{2\alpha'} \underset{\neq}{\prec} \mathcal{O} \underset{\neq}{\prec} \mathcal{O}_{2\alpha})$$

provided that $\lim_{t\to 0} L(t) = 0$ (*resp.* $\lim_{t\to 0} L(t) = +\infty$).

Finally we will give simple examples. Consider

$$\phi(s) = \int_{\alpha'}^{\alpha} s^\beta d\beta, \quad 1 \geqq \alpha > \alpha' \geqq 0. \tag{30}$$

Since ϕ has a completely monotone derivative and $\phi(0) = 0$, it is an exponent of some $z(t) \in \mathscr{Z}$ (for example, see W. Feller [6], Theorem 1, p. 425). By a computation

22) $\mathcal{O}_1 \prec \mathcal{O}_2$ ($\mathcal{O}_1 \underset{\neq}{\prec} \mathcal{O}_2$) implies that $\mathcal{O}_1$ is stronger than $\mathcal{O}_2$ (resp. $\mathcal{O}_1$ is stronger than $\mathcal{O}_2$ and $\mathcal{O}_1$ is not equivalent to $\mathcal{O}_2$).

$$1/\psi(s) \sim s^{-\alpha} \log s, \quad s \to +\infty.$$

Hence

$$g_z(r) \asymp r^{2\alpha-n} \log 1/r, \quad r \to 0.$$

If we set

$$\psi_1(s) = \int_{\alpha'}^{\alpha} [\psi(s)]^{\beta} d\beta, \quad 1 \geqq \alpha > \alpha' > 0,$$

where ψ is of the form (30), then it is also an exponent of some $z_1(t) \in \mathscr{Z}$ and

$$1/\psi_1(s) \sim s^{-\alpha^2} (\log s)^{1+\alpha}, \quad s \to +\infty.$$

Hence

$$g_{z_1}(r) \asymp r^{2\alpha^2-n} (\log 1/r)^{1+\alpha}, \quad r \to 0.$$

§7. Green functions and regular points for a certain class of Markov processes with homogeneity (I).

In this section we will study Lévy processes with homogeneity. Let $\mathscr{S}$, $\mathscr{S}'$, $\mathscr{D}$, $\mathscr{D}'$, $\mathscr{B}$, $\mathscr{B}'$, $\mathscr{D}_{L^1}$, etc. be the space of distributions or functions in Schwartz' sense [24]. For $f \in \mathscr{S}$, $\mathscr{L}^1(R^n)$ or $\mathscr{L}^2(R^n)$ we denote the Fourier transform. (the Fourier inverse transform.) by

$$\hat{f}(x) = \int_{R^n} e^{-i<x,\xi>} f(\xi) d\xi \text{ (resp. } \check{f}(\xi) = \int_{R^n} e^{i<x,\xi>} f(x) \tilde{d}x, \ \tilde{d}x = (2\pi)^{-n} dx),$$

and denote the extension of $\wedge$ (resp. $\vee$) to $\mathscr{S}'$ by $\mathscr{F}$ (resp. $\mathscr{F}^{-1}$) as usual.

Now we will summarize some elementary facts about Lévy processes on R^n. Let X be a Lévy process on R^n such that

$$E_0(e^{-i<\xi, x_t>}) = e^{-t\psi(\xi)}, \quad \xi \in R^n \tag{1}$$

$\psi(\xi)$ is called the exponent of X. It is known that $\psi(\xi)$ is a negative difinite function on R^n. Suppose that $\mathscr{F}^{-1}(e^{-t\psi})(x)$ is a bounded continuous function for each fixed $t > 0$. Then, setting $p(t, x) = \mathscr{F}^{-1}(e^{-t\psi})(x)$, $p(t, x-y)$ is a transition probability density of X. If in addition it holds that

$$1/\psi(\xi) \in \mathscr{L}^1_{\text{loc}}(R^n), \tag{2}$$

X has the Green function $g(x-y)$ given by $g(x-y) = \int_0^{+\infty} p(t, x-y) dt$. Moreover $g(x-y)$ satisfies $GS)$ provided $n \geqq 3$ and symmetric. Indeed, since Re

$\psi(\xi) \leqq C|\xi|^2$ for large $|\xi|$ and some constant $C > 0$, we have

$$\text{(3)} \qquad \varliminf_{x\to 0} g(x) \geqq \int_0^{+\infty} \varliminf_{x\to 0} \mathscr{F}^{-1}(e^{-t\psi})(x)\,dt = \int_{R^n} 1/Re\psi(\xi)d\xi = +\infty.$$

In the above case X satisfies $R1$) by Lemma 1 and also satisfies $R2$), because Hunt's conditions F) and G) hold for X (see G.A. Hunt [**10**]).

Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, \cdots, \beta_n)$ be real vectors. We write as $\boldsymbol{\alpha} \geqq \boldsymbol{\beta}$ provided $\alpha_k \geqq \beta_k$ for all k. If $\alpha_k = \alpha$ for all k, we write simply as α instead of $\boldsymbol{\alpha}$. A function f on R^n is called a homogeneous function of degree $\boldsymbol{\alpha}$ provided $f(t^{1/\alpha_1}\xi_1, \cdots, t^{1/\alpha_n}\xi_n) = tf(\xi)$ for $t > 0$. If in addtion $f \in C^\infty(R^n - \{0\})$, we say that f is *a C^∞-homogeneous function of degree* $\boldsymbol{\alpha}$. Define

$\mathscr{A}_{\boldsymbol{\alpha}}(\mathscr{A}_{\boldsymbol{\alpha}}^\infty) = \{\psi(\xi)$; *a homogeneous* (*resp.* C^∞-*homogeneous*) *function of degree* $\boldsymbol{\alpha}\}$;

$\mathscr{A}_{\boldsymbol{\alpha}}^+(\mathscr{A}_{\boldsymbol{\alpha}}^{\infty,+}) = \mathscr{A}_{\boldsymbol{\alpha}}$ (*resp.* $\mathscr{A}_{\boldsymbol{\alpha}}^\infty) \cap \{\psi(\xi)$; *a negative definite function*$\}$;

$\mathscr{A}_{\boldsymbol{\alpha}}^{++}(\mathscr{A}_{\boldsymbol{\alpha}}^{\infty,++}) = \mathscr{A}_{\boldsymbol{\alpha}}^+$ (*resp.* $\mathscr{A}_{\boldsymbol{\alpha}}^{\infty,+}) \cap \{\psi(\xi)$; $Re\,\psi(\xi) > 0$ *for* $|\xi| \neq 0\}$.

In this section we consider the following two types of Lévy processes in R^n $(n \geqq 3)$[23]. Let $2 > \boldsymbol{\alpha} > 0$:

$(I)_{\boldsymbol{\alpha}}$ *Lévy process whose exponents belong to* $\mathscr{A}_{\boldsymbol{\alpha}}^{\infty,++}$[24] *and symmetric,*

$(II)_{\boldsymbol{\alpha}}$ *Lévy processes obtained by assuming that the coordinate processes are independent symmetric stable processes of index* α_k, $k = 1, \cdots, n$ *in* R^1.

Note that the exponent $\psi(\xi)$ of a Lévy process of type $(II)_{\boldsymbol{\alpha}}$ has the form

$$\text{(4)} \qquad \psi(\xi) = \sum_{k=1}^n C_k|\xi_k|^{\alpha_k}, \quad \text{where } C_k > 0.$$

Hence $\psi(\xi) \in \mathscr{A}_{\boldsymbol{\alpha}}^{++}$ but $\psi(\xi) \notin \mathscr{A}_{\boldsymbol{\alpha}}^{\infty,++}$.

Define

$$\text{(5)} \qquad r_{\boldsymbol{\alpha}}(y) = (\sum_{j=1}^n y_j^{2\alpha_j})^{1/2} \text{ for real vector } \boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_n).$$

LEMMA 13. *Let* $2 > \boldsymbol{\alpha} > 0$. *Suppose that* X *is a Lévy process of type* $(I)_{\boldsymbol{\alpha}}$ *or* $(II)_{\boldsymbol{\alpha}}$. *Then* X *has Green function* $g(x-y)$ *with* GS) *and* $R2$). *Furthermore*

i) *if* X *is of type* (I),

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[23] We always assume $n \geqq 3$ in the sequel without referring.

[24] We do not discuss about the existence of such Lévy processes here. For the existence of such process for certain $\boldsymbol{\alpha}$, see Proposition 1.

(6) $$g(x-y) \approx r_{\alpha}(x-y)^{1-\sum_{j=1}^{n} 1/\alpha_j} \quad on\ R^n\ ;$$

ii) *if* X *is of type* $(II)_{\alpha}$ *the following cases occur*:

ii. 1) *in case* $n \geqq 5$, $g(x)$ *is infinite on each coordinate axis*;

ii. 2) *in case* $n \geqq 3$ *and* $1-\sum_{j \neq k}^{n} 1/\alpha_j \leqq -1$ *for some* k, $g(x)$ *is infinite on the* x_k*-axis*;

ii. 3) *in case* $n=3$ *or* 4 *and* $1-\sum_{j \neq k}^{n} 1/\alpha_j > -1$ *for every* k, *it follows that* (6) *holds*.

Remark 9. If $n=3$ and $2>\boldsymbol{\alpha}>1$, then $1-\sum_{j \neq k}^{n} 1/\alpha_j > -1$ for all k.

For the proof of Lemma 13 we will prepare some facts. Suppose $\boldsymbol{\beta}=(\beta_1, \beta_2, \cdots, \beta_n)>0$ or $\boldsymbol{\beta}<0$. Let $\rho_{\beta}(y)$ be a positive C^{∞}-function on R^n-$\{0\}$ uniquely defined by

(7) $$\sum_{j=1}^{n} \frac{y_j^2}{\rho_{\beta}(y)^{2/\beta_j}}=1.$$

Then we can easily prove that

(8) $$C_2 r_{\beta}(y) \leq \rho_{\beta}(y) \leq C_1 r_{\beta}(y), \quad y \neq 0$$

provided $\boldsymbol{\beta}>0$, and

(9) $$C_2 r_{-\beta}(y) \leq \rho_{\beta}(y)^{-1} \leq C_1 r_{-\beta}(y), \quad y \neq 0$$

provided $\boldsymbol{\beta}<0$, where $C_1 \geqq C_2>0$ are absolute constants and r_{β}, $r_{-\beta}$ are functions defined by (5). Let us note that for a C^{∞}-homogeneous function f of degree $\boldsymbol{\beta}$ it holds

(10) $$\left|\sum_{j=1}^{n}\left(\frac{\partial}{\partial \xi_j}\right)^{2k} f(\xi)\right| \leq M_{2k} \sum_{j=1}^{n}\left(1/\rho_{\beta}(\xi)\right)^{\frac{2k}{\beta_j}-1}, \quad \xi \neq 0,$$

where k is a positive integer and M_{2k} is a positive absolute constant.

Proof of Lemma 13. Let ψ be the exponent of X. Then $p(t,x)=\mathscr{F}^{-1}(e^{-t\psi})(x)$ is a bounded continuous function of x for each fixed $t>0$. Since ψ satisfies (2) because $n \geqq 3$, there exists a Green function $g(x-y)=\int_0^{+\infty} p(t, x-y) dt$ with GS) and R) as mentioned before. We will first prove (i). Note that

(11) $$g(x) = \mathscr{F}^{-1}(1/\psi)(x).$$

Combining (10) with (9), we have

(12) $$\left| \sum_{j=1}^{n} \left(\frac{\partial}{\partial \xi_j}\right)^{2k} 1/\psi(\xi) \right| \leq M'_{2k} \sum_{j=1}^{n} r_\alpha(\xi)^{-\frac{2k}{\alpha_j}-1}, \quad \xi \neq 0,$$

Since $n + r_\alpha(\xi)^2 \geqq \sum_{j=1}^{n} |\xi_j|^{2\alpha_0} \geqq (1/n)^{\alpha_0}|\xi|^{2\alpha_0}$ for $\alpha_0 = \min_{1<j\leq n} \alpha_j$, it follows from (12) that

(13) $$\left| \sum_{j=1}^{n} \left(\frac{\partial}{\partial \xi_j}\right)^{2k} 1/\psi(\xi) \right| \leq M''_{2k} |\xi|^{-k\alpha_0/2}$$

for large $|\xi|$. Combining (11) with (13), we can show that $g(x) \in C^\infty(R^n\text{-}\{0\})$ by the standard method. If we set

$$x' = (x_1/\rho_\alpha(x)^{1/\alpha_1}, \cdots, x_n/\rho_\alpha(x)^{1/\alpha_n}) \text{ for } x \neq 0,$$

it holds that

(14) $$g(x) = (\rho_\alpha(x))^{1-\sum_{j=1}^{n} 1/\alpha_j} \mathscr{F}^{-1}(1/\psi)(x')$$

by changing the variable of the coordinates in (11). Combining (8), GS) and the fact that $g(x) \in C^\infty(R^n\text{-}\{0\})$, it follows from (14) that (6) holds. Next we will prove ii). For the estimate of $g(x)$, we note the following : Let $p(t, x-y)$ be transition probability density of X of type $(II)_\alpha$. Then

(15) $$p(t, x-y) = \prod_{k=1}^{n} p_k(t, x_k - y_k), \quad x = (x_1, \cdots, x_n), \quad y = (y_1, \cdots, y_n),$$

where $p_k(t, x_k - y_k)$ denotes the transition probability density of a symmetric stable process $x_k(t)$ of index α_k on R^1. We use the estimate **[25]** :

$$C_1 \leq p_k(1, x) \leq C_2 \quad \text{for} \quad |x| \leqq 1$$
$$C_3 \leqq |x|^{1+\alpha_k} p_k(1, x) \leq C_4 \text{ for } |x| \geqq 1$$

and

$$p_k(rt, r^{1/\alpha_k}x) r^{1/\alpha_k} = p_k(t, x) \quad \text{for each } r > 0,$$

where $x \in R^1$ and $C_j, j = 1, 2, 3, 4$, are positive constants. Let us fix $x = (0, \cdots, 0, x_k, 0, \cdots, 0)$ where $|x_k| > \delta > 0$. Then we have[25)]

[25)] In the following M_l, $l = 1, 2, \cdots$ denote positive absolute constants.

$$g(x) \geqq M_1 \int_0^\delta t^{1-\sum_{j\neq k}^n 1/\alpha_j} dt.$$

Hence ii. 2) follows immediately. Since $1-\sum_{j\neq k}^n 1/\alpha_j \leqq 1-\frac{n-1}{2}$ ii. 1) also holds. Now we will estimate $g(x)$ on $\{|x|=1\}$. If x is on the x_k-axis, we have

$$\begin{aligned} g(x) &\leqq M_2 \int_0^{+\infty} p_k(1, t^{-1/\alpha_k} x_k) t^{-\sum_{j=1}^n 1/\alpha_j} dt. \\ &\leqq M_3 \int_0^1 t^{1-\sum_{j\neq k}^n 1/\alpha_j} dt + M_4 \int_\delta^\infty t^{-\sum_{j=1}^n 1/\alpha_j} dt. \end{aligned} \tag{16}$$

Let $x=(x_1, x_2, \cdots, x_n)$ be a point on $\{|x|=1\}$ such that $x_k \neq 0$, $k=1, \cdots, l$, $x_{l+1}=\cdots=x_n=0$ and $|x_1|^{\alpha_1} \leqq |x_2|^{\alpha_2} \leqq \cdots \leqq |x_l|^{\alpha_l}$ where $l \geqq 2$. We define

$$I_j = \int_{|x_{j-1}|^{\alpha_{j-1}}}^{|x_j|^{\alpha_j}} p(1, t^{-/\alpha_1} x_1) \cdots p(1, t^{-1/\alpha_l} x_l) t^{-\sum_{m=1}^n 1/\alpha_m} dt, \quad j=1, \cdots, l,$$

$$I = \int_{|x_l|^{\alpha_l}}^{+\infty} t^{-\sum_{m=1}^n 1/\alpha_m} dt, \quad x_0 = 0.$$

Since

$$|x_l|^{\alpha_l} \geqq (1/n)^{(1+1/\alpha_0)/2\alpha} \equiv C, \quad \alpha_0 = \min_{1<j\leq n} \alpha_i, \quad \alpha = \max_{1\leq j\leq n} \alpha_j, \tag{17}$$

we have

$$I \leqq M_5. \tag{18}$$

Combining (17) with

$$I_j < M_6 \prod_{m=j}^l |x_m|^{-1-\alpha_m} \int_0^{|xj|^{\alpha_j}} t^{-\sum_{m=1}^n 1/\alpha_m} \prod_{m=j}^l t^{1+1/\alpha_m} dt, \quad j=1, \cdots l$$

we get

$$\begin{aligned} I_j &\leqq M_7 \prod_{m=j}^l |x_m|^{-1-\alpha_m} \int_0^{|x_j|^{\alpha_j}} t^{1-\sum_{m\neq l}^n 1/\alpha_m} dt\, |x_j|^{\alpha_j(l-j)+\sum_{m=j}^{l-1} \alpha_j/\alpha_m} \\ &\leqq M_7 C^{-1-1/\alpha_l} \prod_{m=j}^{l-1} |x_m|^{-1-\alpha_m} |x_j|^{\alpha_j(l-j)+\sum_{m=j}^{l-1} \alpha_j/\alpha_m} \int_0^{|x_j|^{\alpha_j}} t^{1-\sum_{m\neq l}^n 1/\alpha_m} dt \\ &\leqq M_8 \int_0^1 t^{1-\sum_{m\neq l}^n 1/\alpha_m} dt \end{aligned} \tag{19}$$

Combining (16) (18) and (19), it follows that

$$g(x) \leq M_9 \text{ on } |x| = 1,$$

provided that $1 - \sum_{j \neq k}^{n} 1/\alpha_j > -1$ for all k. Noting that $g(x-y)$ satisfies GS), we can prove (6) by using the representation (14). The proof is complete.

THEOREM 7. *Let $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ be real vectors such that $2 > \boldsymbol{\alpha} = C\boldsymbol{\beta} >$ for some $C \geqq 1$ and*

$$1 - \sum_{j \neq k}^{n} 1/\alpha_j > -1 \quad \textit{for every } k. \tag{20}$$

Let $X_1(X_2)$ be a Lévy process of type $(I)_\alpha$ or type $(II)_\alpha$ (resp. type $(I)_\beta$ or type $(II)_\beta$). Then

$$K^r_{X_2} \subset K^r_{X_1} \tag{21}$$

holds for every compact set $K \subset R^n$. If both X_1 and X_2 are of type (I), then (21) *holds without the assumption* (20).

Proof. By Lemma 13 $X_1(X_2)$ has Green function $g_1(x-y)$ (resp. $g_2(x-y)$) such that

$$g_1(x-y) \approx r_\alpha(x-y)^{1-\sum_{j=1}^{n} 1/\alpha_j} \quad (\text{resp. } g_2(x-y) \approx r_\beta(x-y)^{1-\sum_{j=1}^{n} 1/\beta_j}) \tag{22}$$
$$\text{on } R^n.$$

On the other hand it holds

$$n^C r_\alpha(x)^2 \geqq r_\beta(x)^{2C} \geqq \frac{1}{n} r_\alpha(x)^2. \tag{23}$$

Hence, setting $\rho(x, y) = r_\beta(x-y)$, $\varphi_1(r) = r^{C-\sum_{j=1}^{n} 1/\beta_j}$ and $\varphi_2(r) = r^{1-\sum_{j=1}^{n} 1/\beta_j}$, it follows from (22) and (23)

$$g_k(x-y) \approx \varphi_k(\rho(x,y)) \text{ on } R^n, \ k = 1, 2.$$

Using Theorem 1, we can prove (21). The proof is complete.

Next we will construct Lévy processes of type (I) for a certain class of $\boldsymbol{\alpha}$. Let X be a Lévy process on R^n and A be the generator of X. We say that $n(dy)$ is Lévy measure of X if for each $f \in \mathscr{D}$ vanishing on a neighborhood of the origin it holds

$$\int_{R^n} f(y) n(dy) = Af(x). \tag{24}$$

For convenience we introduce

$N_\beta^\infty = \{n(y)\ ;\ a\ C^\infty$ homogeneous function of degree $-(n+\beta)$ such that $n(y)>0$ for $y\neq 0\}$.

We define $\mathscr{A}\in\mathscr{D}'_{L^1}$ by

$$(25)\qquad (\mathscr{A},u)=\int_{R^n}\Big\{u(y)-u(0)-\sum_{j=1}^{n}\frac{\partial u}{\partial y_j}(0)y_j\Big\}n(y)dy,\quad u\in\mathscr{B}$$

provided $n(y)\in N_\beta^\infty$ for $2>\beta>1$ and

$$(26)\qquad (\mathscr{A},u)=\int_{R^n}\{u(y)-u(0)\}n(y)dy,\quad u\in\mathscr{B}$$

provided $n(y)\in N_\beta^\infty$ for $1>\beta>0$. Set

$$(27)\qquad Au(x)=\mathscr{A}*u(x),\quad u\in\mathscr{B}$$

and

$$(28)\qquad \psi(\xi)=-\mathscr{F}(\mathscr{A})(\xi).$$

Then it is known that there exists a Lévy process whose generator coincides with A of (27) on $\mathscr{B}$ and exponent is $\psi(\xi)$ defined by (28). Furthermore, it we set

$$(29)\qquad \alpha=\Big(1-\sum_{j=1}^{n}\frac{1}{\beta_j+n}\Big)(\beta+n)^{26)},$$

it is a Lévy process of type $(I)_\alpha$ as is shown in the following proposition.

PROPOSITION 1. *Suppose* $2<\beta<1$ *or* $1>\beta>0$. *Then, for each* $n(y)\in N_\beta^\infty$ *the function* $\psi(\xi)$ *defined through* (25) *or* (26) *and* (28) *belongs to* $\mathscr{A}_\alpha^{\infty,++}$, *where* α *is defined by* (29).

Proof. Note that $\psi(\xi)=-(\mathscr{A},e^{-i\langle\cdot,\xi\rangle})$, because $\mathscr{A}\in\mathscr{D}'_{L^1}$. Changing the variable of the coordinates, we see that $\psi(\xi)$ is homogeneous of degree α. It is known that $\psi(\xi)$ is negative definite. Further

$$Re\psi(\xi)\leqq\min_{|y|=1}n(y)\int_{R^n}(1-\cos\langle\xi,y\rangle)\rho_{-(n+\beta)}(y)dy$$
$$\leqq M\int_{R^n}(1-\cos\langle\xi,y\rangle)\frac{1}{r_{n+\beta}(y)}dy$$

26) If $\beta=\alpha$, then $\alpha=\alpha$. If there exist $j\neq k$ such that $\beta_j\neq\beta_k$, then $\sup_{1\leq j\leq n}\beta_j>\alpha>\inf_{1\leq j\leq n}\beta_j$.

by using (9), where M is positive constant. Hence $\psi(\xi)\in\mathscr{A}_{\alpha}^{++}$. Next we will prove that $\psi(\xi)\in C^{\infty}(R^n-\{0\})$. Let $Q(y)\in C^{\infty}$ such that $Q(y)=0$ for $|y|\leqq 1/2$ and $Q(y)=1$ for $|y|\geqq 1$. Set

$$\psi_1(\xi)=\int_{R^n}(1-Q(y))\{1-e^{-i<y,\xi>}-\sum_{j=1}^{n}iy_j\xi_j\}n(y)dy,\quad \psi_2(\xi)=\int_{R^n}Q(y)n(y)dy$$

$$\psi_3(\xi)=-\int_{R^n}Q(y)e^{-i<y,\xi>}n(y)dy,\quad \psi_4(\xi)=-\int_{R^n}Q(y)\sum_{j=1}^{n}iy_j\xi_j n(y)dy,$$

in case $2>\boldsymbol{\beta}>1$ and

$$\psi_1(\xi)=\int_{R^n}(1-Q(y))\{1-e^{-i<y,\xi>}\}n(y)dy,\quad \psi_4(\xi)\equiv 0,$$

in case $1>\boldsymbol{\beta}>0$. Then it follows immediately that $\psi_1(\xi)$, $\psi_2(\xi)$ and $\psi_4(\xi)\in C^{\infty}(R^n-\{0\})$. On the other hand we have

$$\left|\sum_{j=1}^{n}\left(\frac{\partial}{\partial y_j}\right)^{2k}Q(y)n(y)\right|\leqq M|y|^{-\frac{nk}{n+2}}$$

for large y on the same way as in the proof of (13). Hence we can prove that $\psi_3(\xi)\in C^{\infty}(R^n-\{0\})$ by the standard method. Since $\psi(\xi)=\sum_{k=1}^{4}\psi_k(\xi)$, it follows that $\psi(\xi)\in C^{\infty}(R^n-\{0\})$. Consequently, $\psi(\xi)\in\mathscr{A}_{\alpha}^{\infty,++}$. The proof is complete.

We will close this section with the following Remarks.

Remark 10. Let $\boldsymbol{\alpha}$ be a vector defined by (29) for $2>\boldsymbol{\beta}>1$. Then there exists a Lévy process X_1 of class $(I)_{\boldsymbol{\alpha}}$ on R^3 by Proposition 2. Furthermore it follows from Theroem 7 that

$$K_{X_1}^{r}=K_{X_2}^{r}$$

for every compact or open set $K\subset R^3$, where X_2 is a Lévy process of type $(II)_{\boldsymbol{\alpha}}$ on R^3. On the other hand the Lévy measures of X_1 and X_2 are

$$n(y)dy \text{ and } M_1\frac{dy_1}{|y_1|^{1+\alpha_1}}\times\delta(dy_2dy_3)+M_2\frac{dy_2}{|y_2|^{1+\alpha_2}}\times\delta(dy_1dy_3)+M_3\frac{1}{|y_3|^{1+\alpha_3}}\times\delta(dy_2dy_1),$$

respectively, where $n(y)\in N_{\boldsymbol{\beta}}^{\infty}$ and $\delta(dy_jdy_k)$ denote the Dirac measure at the origin on $y_j\times y_k$-space.

Remark 11. Using Corollary 2′ of Theorem 1′, we can show the following. Let $X_1(X_2)$ be a Lévy process of type $(I)_{\boldsymbol{\alpha}}$ (resp. type $(I)_{\boldsymbol{\beta}}$) on R^n, where $2>\boldsymbol{\alpha},\boldsymbol{\beta}>1$. Suppose that $(\alpha_1,\cdots,\alpha_{n-1})=(\beta_1,\cdots,\beta_{n-1})$ and $\alpha_n\geqq\beta_n$.

Then

$$K^{r}_{X_1} \supset K^{r}_{X_2}$$

holds for every compact or relatively open set K in $(x_1, \cdots, x_{n-1})$-space.

§8. Green functions and regular points for Markov processes with homogeneity (II).

Let us consider a function $n(x, y)$ which satisfies:

$n1)$ *$n(x, \cdot) \in \mathscr{A}^{\infty}_{-(n+\alpha)}$ for each fixed $x \in R^n$;*

$n2)$ *for each multi-indices β, γ, $(D_x)^{\beta}(D_y)^{\gamma} n(x, y)$ is bounded on $R^n \times \{|y| = 1\}$;*

$n3)$ *for some constants $C_1 \geqq C_2 > 0$, $C_2 < n(x, y) < C_1$ on $R^n \times \{|y| = 1\}$;*

$n4)$ *there exists $L > 0$ and $n(\infty, y) \in \mathscr{A}^{-(n+\alpha)}_{\infty}$ such that $n(x, y) = n(\infty, y)$ for $|x| \geqq L$.*

For the above $n(x, y)$ we define a distribution $\mathscr{A}_x$ by

$$(\mathscr{A}_x, u) = \int_{R^n} \left\{ u(y) - u(0) - \sum_{j=1}^{n} \frac{\partial u}{\partial y_j}(0) y_j \right\} n(x, y) dy \tag{1}$$

provided $1 < \alpha < 2$ and

$$(\mathscr{A}_x, u) = \int_{R^n} \{ u(y) - u(0) \} n(x, y) dy \tag{2}$$

provided $0 < \alpha < 1$. We let A be a operator on $\mathscr{B}$[27] defined by

$$Au(x) = \mathscr{A}_x * u(x). \tag{3}$$

We call $n(x, y)dy$ the Lévy measure of A as usual. Our result is the following

THEOREM 8. *Suppose that $n \geqq 3$ and $2 > \alpha > 1$ or $1 > \alpha > 0$. Then there exists a Markov process X on R^n[28] which has a Green function $G(x, y)$ with GB), GC) and R2) such that*

$$\begin{aligned} &G(x, y) \approx |x - y|^{\alpha - n}, \quad \text{on } R^n, \\ &AGf = -f, \quad f \in \mathscr{D}. \end{aligned} \tag{4}$$

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27) $\mathscr{B}$ $(\dot{\mathscr{B}})$ denotes the space of C^∞-functions whose derivatives of any order are bounded (resp. vanishing at infinity). The topology in $\mathscr{B}$ $(\dot{\mathscr{B}})$ is that introduced by L. Schwartz [**24**].

28) It is known that there exists a Markov process on R^n whose generator is A [**26**]. Our aim is to construct the kernel $G(x, y)$ satisfying (4). But in our proof the existence of a Markov process also follows in this connection.

Furthermore $\{T_t\}$ *of* X *is strongly continuous on* $C_0(R^n)$.

Combining (4) with $R2$), $M3$) holds for the above X. Hence, using Corollary 1 of Theorem 1, we have

COROLLARY. *Let* B_α *be an isotropic stable process in* $R^n (n \geqq 3)$. *Then*

$$K_X^r = K_{B_\alpha}^r$$

holds for every compact or open set $K \subset R^n$.

We will break up the proof of Theorem 6 into several Lemmas. Set

$$a(x, \xi) = \mathscr{F}(\mathscr{A}_x)(\xi). \tag{5}$$

Then we can prove the following on the similar way as in Lemma 10.

LEMMA 14. $a(x, \xi)$ *of* (5) *satisfies:*

$a1$) $-a(x, \cdot) \in \mathscr{A}_\alpha^{\infty,+}$;

$a2$) *for each multi-index* β, γ, $(D_x)^\beta (D_\xi)^\gamma a(x, \xi)$ *is bounded on* $R^n \times \{|\xi| = 1\}$;

$a3$) *for some constants* $M_1 \geqq M_2 > 0$, $M_2 \leqq -Re\, a(x, \xi) \leqq M_1$ *on* $R^n \times \{|\xi| = 1\}$;

$a4$) $a(x, \xi)$ *is independent of* x *for* $|x| \geqq L$.

we set $a^\infty(\xi) \equiv a(x, \xi)$ *for* $|x| \geqq L$.

Suppose $u \in \mathscr{S}$. Then, since $\mathscr{F}(\mathscr{A}_x * u) = \mathscr{F}(\mathscr{A}_x)\hat{u}$, it holds

$$Au(x) = \int_{R^n} e^{i<x, \xi>} a(x, \xi) \hat{u}(\xi) \tilde{d}\xi, \quad u \in \mathscr{S}. \tag{6}$$

We call $a(x, \xi)$ the symbol of A. Let $u \in \dot{\mathscr{B}}$ and let $\{u_n\}$ be a sequence of functions belonging to $\mathscr{S}$ such that $u_n \to u$ in $\dot{\mathscr{B}}$. Then, since $\mathscr{F}(u_n) \to \mathscr{F}(u)$ in $\mathscr{S}'$ and $\mathscr{A}_x * u_n \to \mathscr{A}_x * u$ in $\mathscr{S}'$, it follows that $\mathscr{F}(\mathscr{A}_x * u) = \lim_{n \to +\infty} \mathscr{F}(\mathscr{A}_x * u_n) = a(x, \xi)\mathscr{F}(u)$ in $\mathscr{S}'$. Therefore we have

$$Au(x) = \mathscr{F}^{-1}(a(x, \cdot)\mathscr{F}(u))(x), \quad u \in \dot{\mathscr{B}}. \tag{7}$$

Next for our later use we will prepare some notations. For any real number s we define the norm $\|u\|_s$:

$$\|u\|_s^2 = \int_{R^n} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi, \quad u \in \mathscr{S},$$

and denote by H_s the Hilbert space obtained by the completion of $\mathscr{S}$ in this norm. We let H_∞ ($H_{-\infty}$) be $\bigcap_s H_s$ (resp. $\bigcup_s H_s$). Then $H_\infty \subset \dot{\mathscr{B}}$. A linear operator $L : \mathscr{S} \to \mathscr{S}$ is said to have order r, or to be of order r, if for each real s there exists a constant C_s such that

$$\|Lu\|_s < C_s\|u\|_{s+r} \quad \text{for all } u\in\mathcal{S}.$$

Let $a(x,\xi)\in\mathcal{A}_\alpha^\infty$, $\alpha\geqq 0$, be the one with $a2)$ and $a4)$ in Lemma 14 and let $f(\xi)$ be a bounded measurable function. We define $A(a,f)$ as follows.

$$A(a,f)u(x) = \int_{R^n} e^{i<x,\xi>}a(x,\xi)f(\xi)\hat{u}(\xi)\bar{d}\xi, \quad u\in\mathcal{S}, \tag{8}$$

If $f\equiv 1$ we will simply write $A(a)$ instead of $A(a,1)$. Especially, for $a_0(x,\xi)\in\mathcal{A}_0^\infty$ with $a2)$ and $a4)$, $A(a_0)$ has order zero. (See Kohn-Nirenberg [**16**], Theorem 1 and Lemma 3.1). Hence $A(a_0)$ can be extended to the operator mapping H_s continuously into H_s for every s. We use the same symbol $A(a_0)$ for such an extended operator. Further suppose that $a_0(x,\xi)$ satisfies $a3)$. Then $A(a_0)$ is a Fredholm operator on $\mathcal{L}^2(R^n)$. This is proved as follows. Set $b_0(x,\xi) = 1/a_0(x,\xi)$. Then $A(a_0)A(b_0)-I$ and $A(b_0)A(a_0)-I$ have order -1 where I denotes the identity operator. (See, [**16**], Lemma 5.1 and Lemma 3.1.) Let Φ be a bounded set in $\mathcal{L}^2(R^n)$. Then the set $\psi = (A(a_0)A(b_0)-I)\Phi$ or $(A(b_0)A(a_0)-I)\Phi$ satisfies that for each fixed $R>0$ the collection of the Fourier transform of the elements of ψ are uniformly equicontinuous on $\mathcal{L}^2(|\xi|<R)$. This can be proved on the same way as in the proof of Theorem 7 in [**16**]. Hence ψ is relatively compact in $\mathcal{L}^2(R^n)$ by Lemma 8 in [**16**]. In other words $A(a_0)A(b_0)-I$ and $A(b_0)A(a_0)-I$ are compact operators on $\mathcal{L}^2(R^n)$, which implies that $A(a_0)$ is a Fredholm operator on $\mathcal{L}^2(R^n)$ by the definition. Next we define the quantity

$$\tilde{K}_t(\xi,\eta) = \exp(-|\xi-\eta|^2[(1/t)-1]), \quad 0<t\leqq 1,$$

$$\tilde{K}_t = \int_{S_{n-1}}\tilde{K}_t(\xi,\eta)d\sigma(\xi)^{29)},\ S_{n-1}\ ;\ \textit{the surface of a unit ball} \subset R^n, \eta\in S_{n-1},$$

$$K_t(\xi,\eta) = 1/\tilde{K}_t\cdot\tilde{K}_t(\xi,\eta),$$

$$a_0^t(x,\xi) = \int_{S_{n-1}}K_t(\xi/|\xi|,\eta)a_0(x,\eta)d\sigma(\eta),$$

$$a_0^{\infty,t}(\xi) = \int_{S_{n-1}}K_t(\xi/|\xi|,\eta)a_0(\infty,\xi)d\sigma(\eta),\quad a_0(\infty,\xi) = a_0^\infty(\xi).$$

Set $a_0'^t(x,\xi) = a_0^t(x,\xi) - a_0^{\infty,t}(\xi)$. Then, using the following estimate

$$\|A(a_0'^t)u\|_0 \leqq M\|u\|_0 \sup_{\xi\in S_{n-1}}\int_{R^n}\left|\left[1-\sum_{j=1}^n\left(\frac{\partial}{\partial x_j}\right)^2\right]^p a_0'^t(x,\xi)\right|dx,$$

29) $d\sigma(\eta)$ is the Lebesgue measure on S_{n-1}.

where p is the integer such that $p > n/2$ (Palais et al [23], Th. 4), we can prove that $\{A(a_0^t)\}_{0\leq t\leq 1}$[30] is a strongly continuous family of operators on $\mathscr{L}^2(R^n)$ by a computation. From this the index of $A(a_0)$ equals to that of $A(a_0^1)$ (see [23], Th. 4). Since the index of $A(a_0^1)$ equals to zero, it follows that the index of $A(a_0)$ is zero. Summing up the above results, we have

LEMMA 15. *Suppose that $a_0(x,\xi)\in\mathscr{A}_0^\infty$ for each fixed x and satisfies $a2)\sim a4)$ in Lemma 14. Then the operator $A(a_0)$ is a Fredholm operator on $\mathscr{L}^2(R^n)$ whose index is zero*

A function $\theta\in C^\infty$ is called a "patch function" if θ vanishes in a neighborhood of zero and $1-\theta$ vanishes in a neighborhood of ∞. The next two properties[31] will be used in the proof of Lemma 16.

A1) Let $a(x,\xi)\in\mathscr{A}_\alpha^\infty$[32], α ; real, be the one with a2) and a3). Suppose that, $A(a,\theta)u=f$ for $f,u\in H_{-\infty}$ and $f\in C^\infty(U)$, where U is an open set. Then $u\in C^\infty(U)$. (See Hörmander [8].)

A2) Let $a(x,\xi)\in\mathscr{A}_\alpha^\infty$, α ; real, be the one with a2) and a4). Suppose that $u\in H_{-\infty}\cap C^\infty(U)$ for some open set U. Then $A(a,\theta)u\in C^\infty(U)$. (See Kohn-Nirenberg [16], Corollary 9.2.)

LEMMA 16. *Let $a_0(x,\xi)\in\mathscr{A}_0^\infty$ be the one with $a2)\sim a4)$. Suppose that $u\in\mathscr{L}^2(R^n)$ and $A(a_0)u\in\mathscr{S}(H_\infty)$. Then u can be represented in the form*

$$u(\xi)=\mathscr{F}^{-1}\left(\frac{g(\cdot)}{a_0^\infty(\cdot)}\right)(\xi), \tag{9}$$

where $g\in\mathscr{S}$ (resp. H_∞).

Proof. Set $f=A(a_0)u$. Since we can easily show that $A(a_0,1-\theta)u\in C^\infty(R^n)\cap\mathscr{L}^2(R^n)$, $A(a_0,\theta)u=f-A(a_0,1-\theta)u\in C^\infty(R^n)\cap\mathscr{L}^2$. Hence $u\in C^\infty)(R^n)$ by $A1$). Set $a_0'(x,\xi)=a_0(x,\xi)-a_0(\infty,\xi)$ and $a_0^\infty(\xi)=a_0(\infty,\xi)$. Then $A(a_0')u\in\mathscr{D}$. Indeed $A(a_0',\theta)u\in C^\infty(R^n)$ by $A2$ and $A(a_0',1-\theta)\in C^\infty(R^n)$. Consequently $A(a_0^\infty)u=f-A(a_0')u\in\mathscr{S}$ (resp. H_∞). Setting $g=f-A(a_0')u$, we get (9). The proof is complete.

Remark 12. The above u belongs to H_∞ by (9).

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30) Here $a_0^0(x,\xi)=a_0(x,\xi)$.

31) In the following we will always denote a patch function by θ.

32) Precisely $a(x,\xi)\in\mathscr{A}_\alpha^\infty$ for each fixed x. In the sequel we will simply write as $a(x,\xi)\in\mathscr{A}_\alpha^\infty$.

Now we will study the operator A defined by (3). First we give

Remark 13. (Maximum principle) Suppose that $u(x_0) = \sup_{x\in R^n} u(x)$ $(u(x_0) = \inf_{x\in R^n} u(x))$ for real function $u\in\dot{\mathscr{B}}$. Then $u\equiv 0$ or $Au(x_0) < 0$ (resp. $Au(x_0) > 0$). From this we see that $u\equiv 0$ provided $Au\equiv 0$ for $u\in\dot{\mathscr{B}}$.

LEMMA 17. *Let A be a operator defined by* (3) *with a Lévy measure $n(x, y)dy$ satisfying $n1)\sim n4)$ for $0<\alpha<1$ or $1<\alpha<2$. Then there exists a unique solution $v\in\dot{\mathscr{B}}$ of the equation*

$$Av = f \tag{10}$$

for every $f\in H_\infty$ provided $n\geqq 3$.

Proof. Let $a(x, \xi)$ be the one defined by (5). Then $a(x, \xi)$ satisfies $a1)\sim a4)$ by Lemma 14. Hence, setting $a_0(x, \xi) = a(x, \xi)/|\xi|^\alpha$, $a_0(x, \xi)\in\mathscr{A}_0^\infty$ and satisfies $a2)\sim a4)$. We will prove this lemma dividing into steps.

step 1. Let u be a function of the form $u(\xi)=\mathscr{F}^{-1}(g(\cdot)/a_0^\infty(\cdot))(\xi)$, where $g\in H_\infty$. Define

$$v(x) = \Gamma\Big(\frac{n-\alpha}{2}\Big)\Big/\Gamma\Big(\frac{\alpha}{2}\Big)\int_{R^n}\frac{1}{|x-y|^{n-\alpha}}u(y)dy. \tag{11}$$

Then $v\in\dot{\mathscr{B}}$. Indeed, since $\hat{v}(\xi) = \hat{u}(\xi)/|\xi|^\alpha$, we have $v(x) = \mathscr{F}^{-1}(Q(\xi)\hat{g}(\xi)/a^\infty(\xi))(x) + \mathscr{F}^{-1}((1-Q(\xi))\hat{g}(\xi)/a^\infty(\xi))(x)$, where $Q(\xi)\in\mathscr{D}$ such that $Q(\xi)=1$ on some neighborhood of the origin. The first term belongs to $\dot{\mathscr{B}}$ using Riemann-Lebesgue lemma repeatedly. The second term also belongs to $H_\infty(\subset\dot{\mathscr{B}})$, because $g\in H_\infty$. Hence $v\in\dot{\mathscr{B}}$.

step 2. For every given $f\in\mathscr{L}^2(R^n)$ there exists a unique solution $u\in\mathscr{L}^2(R^n)$ of the equation $A(a_0)u = f$. This is proved as follows. Since the index of $A(a_0)$ equals to zero by Lemma 15, we have only to prove $\ker A(a_0) = \{0\}$. If $A(a_0)u = 0$, then $u(\xi) = \mathscr{F}^{-1}(\hat{g}(\cdot)/a_0^\infty(\cdot))(\xi)$ for some $g\in\mathscr{S}$ by Lemma 16. Let v be a function defined by (11) for the above u. Since $v\in\dot{\mathscr{B}}$ by the result of step 1, we have

$$Av(x) = \mathscr{F}^{-1}(a(x, \cdot)\mathscr{F}(v))(x) \tag{12}$$

by (7). Noting that $\mathscr{F}(v) = \hat{u}(\xi)/|\xi|^\alpha$, it follows from (12) that $Av(x) = \mathscr{F}^{-1}(a_0(x, \xi)\hat{u}(\xi)) = A(a_0)u(x) = 0$. Therefore, using the remark 13, we have $v\equiv 0$, which implies $u\equiv 0$. Hence $\ker A(a_0) = \{0\}$.

step 3. For a given $f \in H_\infty$ we let $u \in \mathscr{L}^2(R^n)$ be a solution of $A(a_0)u = f$ in the step 2. Then the function $v(x)$ defined by (11) for u belongs to $\dot{\mathscr{B}}$ and satisfies $Av = f$, as is shown in step 2. Thus the proof of Lemma 15 is complete.

Remark 14. Let G be a operator: $H_\infty \to \dot{\mathscr{B}}$ defined by

$$G : f \in H_\infty \to v \in \dot{\mathscr{B}}, \tag{13}$$

where $Av = -f$. Then G maps H_∞ into $\dot{\mathscr{B}}$ continuously.

By the closed graph theorem we have only to show that G is closed. Let $\{f_n\}$ and $\{v_n\}$ be sequences such that $f_n \to f$ in H_∞ and $v_n \to v$ in $\dot{\mathscr{B}}$ respectively. Since we see that $\lim_{n \to +\infty} Av_n(x) = Av(x)$ for every $x \in R^n$, it follows that $-f(x) = -\lim_{n \to +\infty} f_n(x) = Av(x)$, which implies $Gf = v$.

Next we will give a kernel representation of the above operator G. For the symbol $a(x, \xi)$ of A we choose a sequence $\{e_j(x, \xi)\}_{j=0,1,2,\dots}$ such that

$$e_0(x, \xi)a(x, \xi) = 1$$

$$\sum_{j,\gamma} (1/\gamma!)\partial_\xi^\gamma e_j(x, \xi) D_x^\gamma a(x, \xi) = 1, \qquad \gamma = (\gamma_1, \cdots, \gamma_n).$$ [33]

Then $e_j(x, \xi) \in \mathscr{A}_{-a-j}$ for each fixed x and satisfy $a2)$. Let us fix a function $\varphi(\xi) \in C^\infty(R^n)$ such that $\varphi(\xi) = 0$ for $|\xi| < 1/2$ and $\varphi(\xi) = 1$ for $|\xi| > 1$. We choose a sequence $1 = t_0 < t_1 < t_2 < \cdots \to +\infty$ such that

$$|D_x^\gamma D_\xi^\beta(\varphi(\xi/t_j)e_j(x, \xi))| \leq (1/2^j)|\xi|^{-a-j-|\beta|} \tag{14}$$

for $|\xi| \geqq 2t_j$, $|\gamma| + |\beta| \leq j$. Define

$$E^k(x, \xi) = \sum_{j=0}^{k} \varphi(\xi/t_j)e_j(x, \xi), \quad E_k(x, \xi) = \sum_{j=k+1}^{+\infty} \varphi(\xi/t_j)e_j(x, \xi)$$

$$E(x, \xi) = E^k(x, \xi) + E_k(x, \xi).$$

Then, for a fixed patch function θ, $A(a, \theta)A(E^k) - I$ and $A(E^k)A(a, \theta) - I$ have order $-k - \alpha$, $k = 0, 1, 2, \cdots, +\infty$. (See Hörmander [8].) Let us set

$$L_j(x, z) = \int_{R^n} e^{i\langle z, \xi\rangle}\varphi(\xi/t_j)e_j(x, \xi)d\xi. \tag{15}$$

If we fix $P(z) \in \mathscr{D}$ such that $P(z) = 1$ on some neighborhood of the origin, then we have

33) $\partial_\xi^\gamma f(\xi)$ implies $\dfrac{\partial f^{|\gamma|}}{\partial \xi_1^{\gamma_1} \cdots \partial \xi_n^{\gamma_n}} f(\xi)$

(16) $$L_j(x,z) = l_j^1(x,z)P(z) + l_j^2(x,z),$$

where $l_j^1(x,z)\in\mathscr{A}^{\infty}_{\alpha+j-n}$ for each fixed x and $l_j^1(x,z)\in C^{\infty}(R^n\times R^n-\{0\})$, $l_j^2(x,z)\in C^{\infty}(R^n\times R^n)$. Moreover, for each multi-indices γ, β, $(D_x)^{\gamma}(D_z)^{\beta}l_j^2(x,z)$ is bounded on $R^n\times R^n$. If we set

$$K_1^{(k)}(x,z) = \sum_{j=0}^{k} L_j(x,z),$$

we have

(17) $$A(E^k)f(x) = \int_{R^n} K_1^{(k)}(x, x-y)f(y)dy, \quad f\in\mathscr{S}$$

On the other hand, if we set

(18) $$K_2^{(k)}(x,z) = \int_{R^n} e^{i<z,\xi>}E_k(x,\xi)d\xi, \quad k\geqq n,$$

we can prove that $K_2^{(k)}(x,z)\in C^{k-n}(R^n\times R^n)$ and bounded on $R^n\times R^n$, because for each multi-indices β, γ there exists a constants $C(k,\beta,\gamma)$ such that $D_x^{\beta}D_{\xi}^{\gamma}E_k(x,\xi)\leqq C(k,\beta,\gamma)|\xi|^{-\alpha-k}$ for large $|\xi|$. Moreover we have

(19) $$A(E_k)f(x) = \int_{R^n} K_2^{(k)}(x, x-y)f(y)dy, \quad f\in\mathscr{S}.$$

Next we will prove that $A(a, 1-\theta)$ has order $-\infty$[34]. For $u\in\mathscr{S}$, set $v_1 = A(a', 1-\theta)u$ and $v_2 = A(a^{\infty}, 1-\theta)u$, where $a'(x,\xi)-a^{\infty}(\xi)$ and $a^{\infty}(\xi)=a(\infty\ \xi)$. Since $\hat{v}_2(z) = a^{\infty}(z)(1-\theta(z))\hat{u}(z)$, it is clear that $A(a^{\infty}, 1-\theta)$ has order $-\infty$. Set $a_0'(x,\xi) = a'(x,\xi)/|\xi|^{\alpha}$ and let $\hat{a}_0'(x,\xi)$ be the Fourier transform of $a_0'(x,\xi)$ with respect to x. Then it holds, for each fixed real s, s'

(20) $$(1+|\eta|^2)^{s/2}\hat{v}_1(\eta) = \int_{R^n}\left(\frac{1+|\eta|^2}{1+|\xi|^2}\right)^{s/2}\hat{a}_0'(\eta-\xi,\xi)\frac{|\xi|^{\alpha}(1-\theta(\xi))}{(1+|\xi|^2)^{\frac{s'-s}{2}}}(1+|\xi|^2)^{s'/2}\hat{u}(\xi)d\xi.$$

Using Peetre's inequality, we have

(21) $$\left(\frac{1+|\eta|^2}{1+|\xi|^2}\right)^{s/2} \leqq 2^{|s/2|}(1+|\xi-\eta|^2)^{|s/2|}.$$

Because of the fact that $\hat{a}_0'(x,\xi)$ belongs to $\mathscr{S}$ uniformly in ξ we see that for any power p

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34) Since $a(x,\xi)$ becomes irregular at the origin with respect to ξ, we cannot refer to the result of pseudo-differential operators directly.

(22) $$|\hat{a}'_0(\eta-\xi,\xi)| \leq \frac{M}{(1+|\eta-\xi|^2)^p}$$

where M is a constant which is independent of η, ξ. Combining (20), (21) and (22) with p large, we get

$$\|v_1\|_s \leq M'\|u\|_{s'},$$

which implies that $A(a', 1-\theta)$ has order $-\infty$.

Now, if we set $L^\infty = A(a,\theta)A(E)-I$, L^∞ has order $-\infty$ as mentioned before. Since G maps $H_{-\infty}$ into $\dot{\mathscr{B}}$ continuously by Remark 14, $GA(a1-\theta,)A(E)$ and GL^∞ maps H_∞ into $\dot{\mathscr{B}}$ continuously. Hence, using Schwartz kernel theorem, we see that there exists a bounded kernel $K_3(x,y)\in C^\infty(R^n\times R^n)$ such that

(23) $$(GL^\infty + GA(a,\ 1-\theta)A(E))f(x) = \int_{R^n} K_3(x,y)f(y)dy, \quad f\in\mathscr{D}.$$

Since $A(a)A(E) = I + L^\infty + A(a,\ 1-\theta)A(E)$, we have

(24) $$-Gf = (A(E) + GL^\infty + GA(a,\ 1-\theta)A(E))f, \quad f\in\mathscr{D}.$$

Combining (17), (19) and (23), it follows from (24) that

(25) $$-Gf(x) = \int_{R^n} \{K_1^{(k)}(x, x-y) + K_2^{(k)}(x,\ x-y) + K_3(x,y)\}f(y)dy,$$

for every $k \geqq n$.

Consequently we have

Lemma 18[35]. *The operator G defined by* (13) *has a kernel representation*

(26) $$Gf(x) = \int_{R^n} G(x,y)f(y)dy, \qquad f\in\mathscr{D},$$

where $G(x,y)$ satisfies GB), GC). Furthermore $G(x,y)$ is C^∞ except at the diagonal set.

Next we are going to study the properties of the above $G(x,y)$.

Lemma 19. *$G(x,y)$ satisfies*

G1) if we set $Gf(x) = \int_{R^n} G(x,y)f(y)dy$, G maps $C_K(R^n)$ into $C_0(R^n)$;

G2) for every nonnegative $f\in C_K(R^n)$ such that $f \not\equiv 0$, $Gf > 0$;

35) We assume $n \geqq 3$.

G3) G satisfies the weak principle of the positive maximum; in other words, if m $(=\sup_{x\in R^n} Gf(x))$ *is positive for real* $f\in C_K(R^n)$, *m equals to* $\sup_{x\in S} Gf(x)$, *where* $S=\overline{\{x\,;\,f(x)>0\}}$;

G4) $G(x,y)\approx|x-y|^{\alpha-n}$ *on* R^n.

Proof. $G1$) follows immediately from GC) and the fact that $Gf\in\dot{\mathscr{B}}$ for $f\in\mathscr{D}$ by the definition. We prove $G2$). Let f be a nonnegative, nonconstant function belonging to $\mathscr{D}$. If $\inf_{x\in R^n} Gf(x)=Gf(x_0)$ for some $x_0\in R^n$, then $AGf(x_0)>0$. On the other hand $AGf(x_0)=-f(x_0)\leqq 0$. Hence Gf cannot attain the infimum in R^n, which implies $Gf>0$ everywhere. Here let us note that $G(x,y)\geqq 0$ by using the continuity of $G(x,y)$ except at the diagonal set. Next we prove $G3$). Suppose that $m>\sup_{x\in S} Gf(x)$ for $f\in\mathscr{D}$. Then, since $Gf\in C_0(R^n)$, there exists a point $x_0\in S^C$ such that $m=Gf(x_0)$. Then $AGf(x_0)<0$ by the maximum principle of A (see Remark 13), which contradicts to the fact that $AGf(x_0)=-f(x_0)\geqq 0$. Thus $G3$) holds for $f\in\mathscr{D}$. We can prove that $G3$) also holds for $f\in C_K(R^n)$, because there exists a sequence $\{f_n\}$ of functions in $\mathscr{D}$ such that $f_n\to f$ and $Gf_n\to Gf$ uniformly on R^n. Finally we will prove. $G4$). Set

$$g^x(z)=\mathscr{F}^{-1}\Big(\frac{-1}{a(x,\cdot)}\Big)(z).$$

Then, for each fixed x, $g^x(z-y)$ is a Green function of a Lévy process on R^n whose exponent $\psi(\xi)$ is $-a(x,\xi)$ and $g^x(z-y)\approx|z-y|^{\alpha-n}$, $|z-y|\to 0$ as in Lemma 13. Moreover we can prove $g^x(z)\in C^\infty(R^n\times(R^n-\{0\}))$ on the same way as in the proof of $a2$) in Lemma 14. Hence, using the homogeneity of $g^x(z)$ with respect to z, for each fixed compact set Q there exist constants $N_1\geqq N_2>0$ such that

$$N_2|z|^{\alpha-n}\leq g^x(z)\leq N_1|z|^{\alpha-n} \tag{27}$$

for every $x\in Q$ and $z\in R^n$. Since $-L_0(x,z)=g^x(z)+\mathscr{F}^{-1}((1-\varphi(\cdot))\times 1/a(x,\cdot))(z)$ and the second term belongs to $C^\infty(R^n\times R^n)$, we see that for some $\delta>0$

$$\frac{1}{2}N_2|z|^{\alpha-n}\leq -L_0(x,z)\leqq 2N_1|z|^{\alpha-n},\ \ x\in Q,\ \ |z|<\delta \tag{28}$$

by (27). Combining (28) with (25), we get

$$G(x,y)\approx|x-y|^{\alpha-n},\ \ |x-y|\to 0.$$

Thus the proof is complete.

Proof of Theorem 8. By the properties $G1)\sim G4)$ together with $GB)$, $GC)$ we can construct a Markov process X on R^n whose Green function is $G(x, y)$ in Lemma 17 by using Theorem 1.1 in **[13]**[36]. Further it has been proved in the above Theorem 1.1 that $\{T_t\}$ of X is strongly continuous on $C_0(R^n)$. From the properties $G1)$ $G2)$, $G4)$, $GB)$ and $GC)$ it follows that $R2)$ holds for X by Lemma 1 in **[15]**. The proof is complete.

Bibilography

[1] R.M. Blumenthal and R.K. Getoor, Markov processes and potential theory." Academic Press, New York and London, 1968.

[2] J.L. Doob, Semimartingales and subharmonic functions. Trans. Am. Math. Soc., **77,** 86–121 (1954).

[3] J.L. Doob, A probability approach to the heat equation. Trans. Am. Math. Soc., **80,** 216–280 (1955).

[4] E.B. Dynkin, Intrinsic topology and excessive functions connected with a Markov process. Dokl. Akad. Nauk SSSR **127** 17–19, (1959).

[5] E.B. Dynkin, Markov processes." Moscow, 1963. English translation (in two volumes): Springer, Berlin, 1965.

[6] W. Feller, An introduction to Probability Theory and its Applications," Vol. **11** Wiley, New York, 1966.

[7] D. Girbarg and J. Serrin, On isolated singularities of solutions of second order elliptic differential equations, J. Analyse Math. **4** 309–340 (1954–1955).

[8] L. Hörmander, Pseudo-differential operators, Comm. Pure Appl. Math. **18** 501–517 (1965).

[9] G.A. Hunt, Markoff processes and potentials I., Illinois J. Math. **1** 44–93 (1957).

[10] G.A. Hunt Markoff processes and potentials. III, Illinois J. Math. **2** 151–213 (1958).

[11] N. Ikeda and S. Watanabe, On some relations between the harmonic measure and the Lévy measure for a certain class of Markov processes. J. Math., Kyoto Univ., **1–1** 79–95 (1962).

[12] A.M. Il'in, A.S. Kalashnikov and O.A. Oleinik, Second order linear equations of parabolic type. Uspekhi Math. Nauk **17** 3–146 (1962).

[13] M. Kanda, Regular points and Green functions in Markov processes. J. Math. Soc. Japan **19** 46–69 (1967).

[14] M. Kanda, On the singularity of Green functions in Markov Processes. Nagoya Math. J., **33** 21–52 (1968).

[15] M. Kanda, On the singularity of Green functions in Markov processes II. Nagoya Math. J., **37** 209–217 (1970).

[16] J.J. Kohn and L. Nirenberg, An algebra of pseudo-differential operators. Comm. Pure Appl. Math., **18** 269–305 (1965).

[17] N.V. Krylov, On quasi-diffusion processes. Theory of Probability and its Applications **11** 424–443 (1966).

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36) Recall the footnote on p. 41.

[18] N.V. Krylov, On the green function for the Dirichlet problem. Uspekhi Math. Nauk **22** 116–118 (1967).

[19] N.V. Krylov, The first boundary value problem for elliptic equations of second order. Differential Equations **3** 315–325 (1967).

[20] E.M. Landis, S-capacity and its applications to the study of solutions of second order elliptic differential equations with discontinuous coefficients. Math. Sbornik **5** 177–204 (1968).

[21] N.S. Landkof, "Foundamentals of modern potential theory." Moscow, 1966.

[22] M. Motoo, The sweeping-out of additive functionals and processes on the boundary. Ann. Inst. Statist. Math., **16** 317–345 (1964).

[23] R.S. Palais, "Seminar on the Atiyah-Singer index theorem." Annals of Mathematics Studies **57** Princeton, New Jersey, 1965.

[24] L. Schwartz "Theorie des distributions." Hermann Sc Cie Paris, 1950.

[25] A.V. Skorokhad, Asymptotic formulas for stable distribution laws. Dokl. Akad. Nauk. SSSR **98** 731–734 (1954).

[26] A.V. Skorokhod, "Studies in the Theory of Random Processes." Kiev 1961. English translation Addison-Wesley, 1965.

[27] D.W. Stroock and S.R.S. Varadhan, Diffusion processes with continuous coefficients **I, II.** Comm. Pure Appl. Math. **21** and **22** 479–530 (1969).

[28] H. Tanaka, Existence of diffusions with continuous coefficients. Mem. Fac. Sci. Kyushu Univ. **18** 89–103 (1964).

[29] S.J. Taylor, On the connection between Hausdorff measures and generalized capacity. Proc. Cambridge Philos. Soc. **57** 524–531 (1961).

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