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FUNCTIONAL EQUATIONS OF GENERALIZED EPSTEIN ZETA FUNCTIONS IN SEVERAL COMPLEX VARIABLES

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Let $S^{(n)}$ be the matrix of a positive definite quadratic form and $(\rho_1, \cdots, \rho_{r-1}) \in \boldsymbol{C}^{r-1}$. Define

(1) $$\zeta_{n_1,\cdots,n_r}(S, \rho_1, \cdots, \rho_{r-1}) = \sum \prod_{i=1}^{r-1} |S[U_i]|^{-\rho_i}.$$

Here the sum is over unimodular matrices $U^{(n)} = (U_i^{(n,N_i)} *)$ which lie in a complete set of representatives for the equivalence relation $U \sim V$ if $U = VP$, with P unimodular and having block form

$$P = \begin{pmatrix} P_1^{(n_1)} & & * \\ & \ddots & \\ 0 & & P_r^{(n_r)} \end{pmatrix}.$$

The following notation shall be used throughout:

$A^{(n,m)}$ for an n by m matrix A

$A^{(n)} = A^{(n,n)}$

$|A|$ = determinant of A

$S[A] = {}^tASA$, ${}^tA$ being the transpose of A

$N_i = \sum_{j=1}^{i} n_j, \ i = 1, 2, \cdots, r, \ N_r = n = \sum_{j=1}^{r} n_j.$

A unimodular matrix $U^{(n)}$ is one with integral entries and determinant ± 1.

The function (1) is clearly a generalization of Epstein's zeta function, as well as a generalization of functions defined by Koecher [1], Maass [3], and Selberg [4], [5], [6]. It can also be viewed as an Eisenstein series; for which, see Langlands [2].

If $n_i = 1$ for all $i = 1, 2, \cdots, r$ and $r = n$, denote the function defined by (1) as $\zeta_{(n)} = \zeta_{1,\cdots,1}$.

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We know from [7] that (1) converges for $\mathrm{Re}\,\rho_i > \frac{1}{2}n$, for $i = 1, 2, \cdots, r-1$. It was also shown in [7] that

(2) $$\zeta_{n_1,\cdots,n_r}(S, \rho_{N_1}, \cdots, \rho_{N_{r-1}}) = \zeta_{(n)}(S, \rho)\Big|_{\rho_i = 0, \text{ for } i \notin \{N_1, \cdots, N_r\}}.$$

Also from [7], every permutation σ of $\{1, 2, \cdots, n]$ induces a functional equation of $\zeta_{(n)}(S, \rho)$. More precisely, one introduces the new variables z_i, $i = 1, 2, \cdots, n$ with $\rho_i = z_{i+1} - z_i + \frac{1}{2}$. That means

$$\rho = \begin{pmatrix} \rho_1 \\ \vdots \\ \vdots \\ \rho_n \end{pmatrix} = \begin{pmatrix} -1+1 & & \\ & -1+1 & 0 \\ & & \ddots \\ 0 & & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ \vdots \\ z_n \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ \vdots \\ \vdots \\ \frac{1}{2} \end{pmatrix} = Az + \frac{1}{2}.$$

Define $\phi(x) = \Gamma(x)\zeta(2x)$. Here ζ denotes the ordinary Riemann zeta function. Now define

$$\Phi'(z) = |S|^{z_n - \frac{1}{2}} \pi^{-2\sum_{j=1}^{n} jz_j} \prod_{1 \le i < j \le n} \phi\Big(z_j - z_i + \frac{1}{2}\Big).$$

Then the functional equations of $\zeta_{(n)}(S, \rho) = \zeta'_{(n)}(S, z)$, for $\rho = Az + \frac{1}{2}$, are

$$\Phi'(z^\sigma)\zeta'_{(n)}(S, z^\sigma) = \Phi'(z)\zeta'_{(n)}(S, z),$$

where $(z^\sigma)_i = z_{\sigma(i)}$, that is,

$$z^\sigma = P^\sigma z, \text{ where } P^\sigma = (\delta_{\sigma(i)j}), \ \delta_{ij} \text{ being}$$

the usual Kronecker delta.

One can rewrite the functional equations in terms of the variables ρ by defining

$$\Phi(\rho) = \Phi'(z), \text{ for } \rho = Az + \frac{1}{2};$$

$$\rho^\sigma = AP^\sigma A^{-1}\Big(\rho - \frac{1}{2}\Big) + \frac{1}{2};$$

$$F(\rho, \sigma) = \frac{\Phi(\rho)}{\Phi(\rho^\sigma)}.$$

Then the functional equations become

(3) $$\zeta_{(n)}(S, \rho^\sigma) = F(\rho, \sigma)\zeta_{(n)}(S, \rho).$$

Note that we have defined the map $\rho \to \rho^\sigma$ to be the map induced on the

ρ coordinates by the map $z \to z^\sigma$. It is clear that $(\rho^\sigma)_i \in \boldsymbol{Q}[\rho_1, \cdots, \rho_{n-1}]$, for $i = 1, \cdots, n-1$.

Now equation (2) suggests that some of the equations (3) induce functional equations or relations on the functions $\zeta_{n_1,\cdots,n_r}$. The purpose now is to discover which r-tuples $(n_1, \cdots, n_r)$, $(m_1, \cdots, m_r)$ such that $\sum_{i=1}^{r} n_i = n = \sum_{i=1}^{r} m_i$ have the property that the corresponding zeta functions $\zeta_{n_1,\cdots,n_r}$ and $\zeta_{m_1,\cdots,m_r}$ are related by (3) for some σ. A result in this direction is proved in [7] for $r = 2$. Relations were obtained between $\zeta_{i,n-i}$ and $\zeta_{n-i,i}$ in that case for $i = 1, 2, \cdots, n-1$. This means that if $i = n-i$, a non-trivial functional equation is induced on $\zeta_{i,i}$ by some permutation of $\{1, 2, \cdots, 2i\}$, in (3).

In order to state the problem more precisely, define $F_{n_1,\cdots,n_r}(\rho,\sigma) = F(\rho,\sigma)\Big|_{\rho_j = 0,\ \text{for}\ j \notin \{N_1, \cdots, N_r\}}$.

We say that for $(n_1, \cdots, n_r)$ and $(m_1, \cdots, m_r)$ such that $\sum_{i=1}^{r} n_i = n = \sum_{i=1}^{r} m_i$,

$$\zeta_{n_1,\cdots,n_r} \sim \zeta_{m_1,\cdots,m_r}$$

if and only if there exists a permutation σ of $\{1, 2, \cdots, n\}$ such that setting $\rho_j = 0$, for $j \notin \{N_1, \cdots, N_r\}$ in (3), one obtains:

$$F_{n_1,\cdots,n_r}(\rho,\sigma)\zeta_{n_1,\cdots,n_r}(S, \rho_{N_1},\cdots,\rho_{N_{r-1}}) = \zeta_{m_1,\cdots,m_r}(S, \rho'_{M_1}, \cdots, \rho'_{M_{r-1}}).$$

Here $N_i = \sum_{j=1}^{i} n_j$ and $M_i = \sum_{j=1}^{i} m_j$. And $\rho'_{M_i} = (\rho^\sigma)_{M_i}\Big|_{\rho_j = 0,\ j \notin \{N_1, \cdots, N_r\}}$;
So $\rho'_{M_i} \in \boldsymbol{Q}[\rho_{N_1}, \cdots, \rho_{N_{r-1}}]$.

We shall soon see that "$\sim$" is indeed an equivalence relation.

Clearly $\zeta_{n_1,\cdots,n_r} \sim \zeta_{m_1,\cdots,m_r}$ under σ is equivalent to the condition that the ρ's which are set equal to zero to produce $\zeta_{n_1,\cdots,n_r}$ from $\zeta_{(n)}$ must be sent by the map $\rho \to \rho^\sigma$ to the ρ's which are set equal to zero to produce $\zeta_{m_1,\cdots,m_r}$ from $\zeta_{(n)}$.

Define $\mathscr{H}_{n_1,\cdots,n_r}$ to be the set of $n-r$ hyperplanes defined by $z_{j+1} - z_j + \frac{1}{2} = 0$, for $j \notin \{N_1, \cdots, N_r\}$. Recall that $N_i = \sum_{j=1}^{i} n_j$, $i = 1, 2, \cdots, r$. It is obvious that $\zeta_{n_1,\cdots,n_r} \sim \zeta_{m_1,\cdots,m_r}$ under σ is equivalent to requiring that the map $z \to z^\sigma$ take the set $\mathscr{H}_{n_1,\cdots,n_r}$ one-to-one, onto the set $\mathscr{H}_{m_1,\cdots,m_r}$.

THEOREM 1. *For every* $(n_1, \cdots, n_r)$ *and* $(m_1, \cdots, m_r)$ *such that* $\sum_{i=1}^{r} n_i = n = \sum_{i=1}^{r} m_i$.

$$\zeta_{n_1,\cdots,n_r} \sim \zeta_{m_1,\cdots,m_r}$$

if and only if there exists a permutation μ *of* $\{1, 2, \cdots, r\}$ *such that* $n_i = m_{\mu(i)}$.

Proof. Suppose for some permutation μ of $\{1, 2, \cdots, r\}$, $n_i = m_{\mu(i)}$. Define a permutation σ_μ of $\{1, 2, \cdots, n\}$ as follows. Set $N_i = \sum_{j=1}^{i} n_j$ and $M_k = \sum_{j=1}^{k} m_j$. Then define

$$\sigma_\mu = \begin{pmatrix} \cdots N_i - (n_i - 1) \cdots N_i - j \cdots N_i \cdots \\ \cdots M_{\mu(i)} - (n_i - 1) \cdots M_{\mu(i)} - j \cdots M_{\mu(i)} \cdots \end{pmatrix}.$$

Then for $1 \leq j < n_i = m_{\mu(i)}$,

$$z_{\sigma_\mu(N_i - j + 1)} - z_{\sigma_\mu(N_i - j)} + \frac{1}{2}$$
$$= z_{M_{\mu(i)} - j + 1} - z_{M_{\mu(i)} - j} + \frac{1}{2}.$$

This means that $z \to z^{\sigma_\mu}$ takes $\mathcal{H}_{n_1,\cdots,n_r}$ one-to-one, onto $\mathcal{H}_{m_1,\cdots,m_r}$.

Suppose next that $\zeta_{n_1,\cdots,n_r} \sim \zeta_{m_1,\cdots,m_r}$ under σ. Then $z \to z^\sigma$ takes $\mathcal{H}_{n_1,\cdots,n_r}$ one-to-one, onto $\mathcal{H}_{m_1,\cdots,m_r}$. The first claim is that there exists a permutation μ of $\{1, 2, \cdots, r\}$ such that if

$$N_i = \sum_{j=1}^{i} n_j, \quad M_i = \sum_{j=1}^{i} m_j,$$

then $\sigma(N_i) = M_{\mu(i)}$.

To see this, note that

$$\left(z_{N_i} - z_{N_i - 1} + \frac{1}{2} = 0\right) \in \mathcal{H}_{n_1,\cdots,n_r}, \text{ but } \left(z_{N_i + 1} - z_{N_i} + \frac{1}{2} = 0\right) \notin \mathcal{H}_{n_1,\cdots,n_r}.$$

And for all $k \in \{1, 2, \cdots, n\} - \{N_1, N_2, \cdots, N_r\}$,

$$\left(z_k - z_{k-1} + \frac{1}{2} = 0\right) \in \mathcal{H}_{n_1,\cdots,n_r}, \text{ and } \left(z_{k+1} - z_k + \frac{1}{2} = 0\right) \in \mathcal{H}_{n_1,\cdots,n_r}.$$

Thus for σ to send $\mathcal{H}_{n_1,\cdots,n_r}$ one-to-one, onto $\mathcal{H}_{m_1,\cdots,m_r}$, it is necessary that $\sigma(N_i) = M_{\mu(i)}$, for some permutation μ of $\{1, 2, \cdots, r\}$.

The next claim is that if $n_i > 1$, then $m_{\mu(i)} > 1$, and $\sigma(N_i - 1) = M_{\mu(i)} - 1$. If this is not the case, then $\left(0 = z_{N_i} - z_{N_i - 1} + \frac{1}{2}\right) \in \mathcal{H}_{n_1,\cdots,n_r}$ is mapped by

σ onto $\left(0 = z_{M_{\mu(i)}} - z_{\sigma(N_i-1)} + \frac{1}{2}\right) \notin \mathscr{H}_{m_1,\cdots,m_r}$. This is a contradiction.

Thus, by induction, we obtain, for $1 \leq j < n_i$, that $\sigma(N_i - j) = M_{\mu(i)} - j$, $m_{\mu(i)} \geq n_i$. It must be shown that $m_{\mu(i)} = n_i$. If this were not so, then $n_i < m_{\mu(i)}$ and $0 = z_{M_{\mu(i)}-(n_i-1)} - z_{M_{\mu(i)}-n_i} + \frac{1}{2} \in \mathscr{H}_{m_1,\cdots,m_r}$. Let $M_{\mu(i)} - n_i = \sigma(x)$. We know that $M_{\mu(i)} - (n_i - 1) = \sigma(N_i - (n_i - 1))$. Now where can x lie in the interval between 1 and n? It cannot lie between N_{k-1} and N_k for any k. If it did, then $\sigma(x)$ would be between $M_{\mu(k)}$ and $M_{\mu(k)} - m_{\mu(k)}$. This means x cannot exist, a contradiction.

Note that Theorem 1 implies that "$\sim$" is an equivalence relation.

Now define $[n_1, \cdots, n_r]$ to be the equivalence class of all $\zeta_{m_1,\cdots,m_r}$ which are related to $\zeta_{n_1,\cdots,n_r}$ by $\sim$.

Define $C(n_1, \cdots, n_r)$ to be the number of elements of $[n_1, \cdots, n_r]$. And define $N(n, r)$ to be the number of distinct classes $[n_1, \cdots, n_r]$ for fixed n and r.

COROLLARY. *$N(n, r)$ is the number of partitions of n into r parts (disregarding order).*

Thus $N(n, r)$ is also equal to the number of partitions of n having largest summand r.

Now define $E(n_1, \cdots, n_r)$ as the number of functional equations of $\zeta_{n_1,\cdots,n_r}$ induced by functional equations $\zeta_{(n)}(\rho) \sim \zeta_{(n)}(\rho^\sigma)$, for permutations σ of $\{1, 2, \cdots, n\}$.

And lastly define $G(n, r)$ to be the number of functions $\zeta_{n_1,\cdots,n_r}$ for fixed n and r. Clearly $G(n, r) = \binom{n-1}{r-1}$.

Given $(n_1, \cdots, n_r)$, let τ permute the indices $\{1, 2, \cdots, r\}$ so that

$$n_{\tau(1)} = n_{\tau(2)} = \cdots = n_{\tau(r_1)} < n_{\tau(r_1+1)} = n_{\tau(r_1+2)} = \cdots = n_{\tau(r_1+r_2)} < \cdots$$
$$\cdots = n_{\tau(r_1+r_2+\cdots+r_{s-1})} < n_{\tau(r_1+\cdots+r_{s-1}+1)} = \cdots$$
$$\cdots = n_{\tau(r_1+\cdots+r_s)}.$$

That is, using our usual notation for $(r_1, r_2, \cdots, r_s)$, namely $R_j = \sum_1^j r_i$, we have

$$n_{\tau(R_j+1)} = n_{\tau(R_j+2)} = \cdots = n_{\tau(R_{j+1})} < n_{\tau(R_{j+1}+1)},$$

for $j = 0, 1, \cdots, s$. Here $R_0 = 0$.

THEOREM 2. $E(n_1, \cdots, n_r) = (r_1)!\,(r_2)! \cdots (r_s)!$.

Proof. The problem is to count the number of permutations σ of $\{1, 2, \cdots, n\}$ such that $z \to z^\sigma$ fixes $\mathscr{H}_{n_1, \cdots, n_r}$. By the proof of Theorem 1, such a permutation σ has the following properties: $\sigma(N_i) = N_{\mu(i)}$, for some permutation μ of $\{1, 2, \cdots, r\}$. And $\sigma(N_i - j) = N_{\mu(i)} - j$, for $1 \leq j < n_i$. Finally $n_{\mu(i)} = n_i$.

Therefore, if $i = 1, \cdots, s$,

$$\mu(\tau(R_i + j)) \in \{\tau(R_i + j) \mid j = 1, 2, \cdots, r_{i+1}\}, \quad j = 1, 2, \cdots, r_{i+1}.$$

There are exactly $(r_1)!(r_2)! \cdots (r_s)!$ such μ, and therefore the same number of σ.

COROLLARY. $C(n_1, \cdots, n_r) = \dfrac{r!}{E(n_1, \cdots, n_r)}$.

Proof. Clear.

We shall consider some examples:

1. $r = 2, \ n > 1$.

   $N(n, 2) = \left[\frac{n}{2}\right]$

   $E(n_1, n_2) = 1 + \delta_{n_1 n_2}$

   $G(n, 2) = n - 1$

2. $r = n - 1, \ n > 1$.

   $N(n, n-1) = 1$

   $E(1, \cdots, 1, 2) = (n-1)!$

   $G(n, n-1) = n - 1$

3. $n = 7$

   a. $r = 5$

   $N(7, 5) = 2$

   $E(2, 2, 1, 1, 1) = 3!\,2!$ $\qquad C(2, 2, 1, 1, 1) = 10$

   $E(3, 1, 1, 1, 1) = 4!$ $\qquad C(3, 1, 1, 1, 1) = 5$

   $G(7, 5) = 15$

   b. $r = 4$

   $N(7, 4) = 3$

   $E(4, 1, 1, 1) = 3!$ $\qquad C(4, 1, 1, 1) = 4$

   $E(3, 2, 1, 1) = 2!$ $\qquad C(3, 2, 1, 1) = 12$

   $E(2, 2, 2, 1) = 3!$ $\qquad C(2, 2, 2, 1) = 4$

   $G(7, 4) = 20$

c. $\gamma = 3$

$N(7,3) = 4$	
$E(5,1,1) = 2$	$C(5,1,1) = 3$
$E(4,2,1) = 1$	$C(4,2,1) = 3!$
$E(3,3,1) = 2$	$C(3,3,1) = 3$
$E(3,2,2) = 2$	$C(3,2,2) = 3$
$G(7,3) = 15$	

Added in proof (October 18, 1971):
For another way of obtaining the results in this paper, see
Maass, H., *Siegel's Modular Forms and Dirichlet Series*, Lecture Notes in Mathematics, Vol. 216, New York, Springer Verlag, 1971.

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