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RIEMANN DOMAINS WITH BOUNDARY OF CAPACITY ZERO

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§ 1. Introduction.

The well-known Thullen-Remmert-Stein's theorem ([9], [7]) asserts that, for a domain D in C^N and an n-dimensional irreducible analytic set S in D, a purely n-dimensional analytic set A in $D-S$ has an essential singularity at any point in S if A has at least one essential singularity in S. In [1], E. Bishop generalized this to the case that A has the boundary of capacity zero in his sense. Afterwards, in [8], W. Rothstein obtained more precise informations on the essential singularities of A under the assumption $\dim A = 1$. The main purpose in this paper is to generalize these Rothstein's results to the case of arbitrary dimensional analytic sets.

We consider a Riemann domain (X, π, M) with boundary of capacity zero, namely, a triple of a connected n-dimensional normal complex space X, a connected n-dimensional complex manifold M and a discrete holomorphic map $\pi : X \to M$ with the following properties:

For any $z_0 \in M$ there are a neighborhood U of z_0 and a plurisubharmonic function $u(x)$ on $\pi^{-1}(U)$ such that (i) $u(x) \leqq 0$, (ii) $u(x) \not\equiv -\infty$ on any connected component of $\pi^{-1}(U)$ and (iii) $\lim_{\nu\to\infty} u(x_\nu) = -\infty$ for any sequence $\{x_\nu\}$ without accumulation points in X if $\lim_{\nu\to\infty} \pi(x_\nu)$ exists in U.

The first main result is the following

THEOREM I. *If (X, π, M) is a Riemann domain with boundary of capacity zero, then $M - \pi(X)$ is of capacity zero* (c.f. Definition 2.5).

We define a direct boundary point of a Riemann domain on the analogy of a direct transcendental singularity in the theory of functions of one complex variable (c.f. Definition 4.3). As an application of Theorem I, we give the following theorem, which is a generalization of a result in [4].

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THEOREM II. *The projection image of the set of all direct boundary points of a Riemann domain with boundary of capacity zero is a set of capacity zero in the base space.*

Using Theorem II, we prove Rothstein's results in [8] without the assumption of dimension one. The following theorem is shown.

THEOREM III. *Let* $D = D_1 \times D_2$ *be a domain in* $\boldsymbol{C}^{n+k}(D_1 \subset \boldsymbol{C}^n,\ D_2 \Subset \boldsymbol{C}^k)$ *and* S *be a closed subset of* $\boldsymbol{C}^{n+k}(S \subset \bar{D})$. *If an irreducible n-dimensional analytic set* A *in* $D - S$ *satisfies the conditions* (i) *there is a plurisubharmonic function* $u(x)$ *on* A *such that* $0 \geqq u(x) \not\equiv -\infty$ *and* $\lim_{x \to S} u(x) = -\infty$, (ii) $\bar{A} \cap (D_1 \times \partial D_2) \subset S$ *and* (iii) *there is a set* P *of positive capacity in* D_1 *such that each* $\{z = c\} \cap A (c \in P)$ *is finite, then* $\bar{A} \cap (D_1 \times \boldsymbol{C}^k)$ *is analytic in* $D_1 \times \boldsymbol{C}^k$.

In this connection, we generalize the well-known Iversen's theorem to the case of Riemann domains with boundary of capacity zero and give some other applications.

§ 2. Preliminaries on plurisubharmonic functions.

In this paper a complex space is always assumed to be normal unless stated to the contrary. Moreover, we assume that all complex spaces and complex manifolds are σ-compact and connected.

Let X be a complex space and $u(x)$ be an extended real-valued function on X which permits the value $-\infty$ but not $+\infty$.

DEFINITION 2.1. $u(x)$ is said to be *plurisubharmonic* on X if (i) $u(x)$ is upper semi-continuous on X and (ii), for any open set W in the complex plane $\boldsymbol{C}$ and a holomorphic map $\psi : W \to C$, the composite $u \cdot \psi : W \to X$ is subharmonic on W in the usual sense or identically equal to $-\infty$.

As is easily seen, it holds that

(2.2) (i) *Let* ψ *be a holomorphic map of* X *into another complex space* Y. *If* $u(x)$ *is plurisubharmonic on* Y, *then* $u \cdot \psi$ *is also plurisubharmonic.*

(ii) *If* $u(x)$ *and* $v(x)$ *are plurisubharmonic on* X *and* c *is a positive real constant, then* cu, $u + v$ *and* $\max(u, v)$ *are also plurisubharmonic.*

The following assertion was proved by H. Grauert and R. Remmert in [2], Satz 3, p. 181.

(2.3) *Let S be a thin analytic set in a complex space X and $u(x)$ a plurisubharmonic function on $X-S$ which is bounded above. Then there exists exactly one plurisubharmonic function $\tilde{u}(x)$ on X such that $\tilde{u}(x)=u(x)$ on $X-S$ and it is given by* $\tilde{u}(x):=\overline{\lim}_{x'\to x,\ x'\in X-S} u(x')(x\in X)$.

The assertion (2.3) implies the following

(2.4) *If $u(x)$ is a plurisubharmonic function on X and S is a thin analytic subset of X, then we have* $u(x)=\overline{\lim}_{x'\to x, x\in X-S} u(x')$ *for any* $x\in X$.

Now, we give the definition of a subset of capacity zero in a complex space X (c.f. T. Nishino [6], p. 232).

Definition 2.5. We shall say a subset S of X to be of *capacity zero* in X and write it $\operatorname{cap}(S)=0$ if we can take a countable family $\{S_\nu\}$ of subsets of X such that $S=\cup_\nu S_\nu$ and, for each S_ν, there exists a plurisubharmonic function u_ν on a connected open set U_ν satisfying the conditions that (i) $u_\nu(x)\not\equiv-\infty$ on U_ν and (ii) $S_\nu\subset\{x\in U_\nu;\ u_\nu(x)=-\infty\}$. If S is not of capacity zero, it is said to be a set of *positive capacity* and denoted by $\operatorname{cap}(S)>0$.

Remark. A closed set in the complex plane $\boldsymbol{C}$ is of capacity zero in the sense of Definition 2.5 if and only if it is of logarithmic capacity zero in the usual sense.

Easily, we have

(2.6) (i) *If each $S_\nu(\nu=1,2,\cdots)$ is of capacity zero, so is the union $\cup_\nu S_\nu$.*

(ii) *Any subset of a set of capacity zero is of capacity zero.*

(iii) *If S is of positive capacity, the set*

$$S':=\{x\in S;\ \operatorname{cap}(S\cap U)>0 \text{ for any neighborhood } U \text{ of } x\}$$

is also of positive capacity.

§3. A generalization of the Riemann theorem on removable sigularities.

For later use, we shall prove

Proposition 3.1. *Let X be a complex space and S be a closed subset of capacity zero in X. If a holomorphic function f on $X-S$ is locally bounded on S, i.e., bounded on some neighborhood of each $x\in S$, then it has exactly one holomorphic continuation to X.*

Proof. Firstly, under the assumption that X is a complex manifold, we shall show that, for each $x_0 \in S$, f has a holomorphic continuation to a neighborhood of x_0. Take a sufficiently small neighborhood V of x_0 which can be written $V = \{|z_i| < 1,\ 1 \leqq i \leqq n\}$ with a system of local coordinates $z_1, \cdots, z_n$ defined on some neighborhood of $\bar{V}$. There is no harm in assuming that $S \subset V$ and $S = \cup_\nu S_\nu$, where each S_ν is included in $\{x \in U_\nu;\ u_\nu(x) = -\infty\}$ for a suitable plurisubharmonic function u_ν ($\not\equiv -\infty$) on a connected open subset U_ν of V.

Let $\{a^{(\kappa)};\ \kappa = 0, 1, 2, \cdots\}$ ($a^{(\kappa)} \neq 0$) and $\{\underline{b}^{(\mu)} := (b_2^{(\mu)}, \cdots, b_n^{(\mu)});\ \mu = 0, 1, 2, \cdots\}$ be countable dense subsets of the sets $\{|w_1| < 1/2\}$ in $\boldsymbol{C}$ and $\{|w_i| < 1/2,\ 2 \leqq i \leqq n\}$ in $\boldsymbol{C}^{n-1}$ respectively, where we let $\underline{b}^{(0)} := (0, \cdots, 0)$. By $\Phi_{\kappa\mu}$ we denote the non-singular linear transformation defined as follows;

$$
\Phi_{\kappa\mu}: \quad
\begin{aligned}
z_1 &= z_1' a^{(\kappa)} \\
z_2 &= z_2' a^{(\kappa)} + b_2^{(\mu)} \\
&\cdots\cdots \\
z_n &= z_n' a^{(\kappa)} + b_n^{(\mu)}.
\end{aligned}
$$

Then, each $v_{\kappa\mu\nu}(z') = u_\nu(\Phi_{\kappa\mu}(z'))$ ($\kappa, \mu, \nu = 0, 1, 2, \cdots$) is plurisubharmonic on $\Phi_{\kappa\mu}^{-1}(U_\nu)$, where $z' = (z_1', \cdots, z_n')$. Since any set $F_{\kappa\mu\nu} := \Phi_{\kappa\mu}^{-1}(U_\nu) \cap \{v_{\kappa\mu\nu}(z') = -\infty\}$ is of measure zero with respect to the coordinates z', we can find a point $(c, \underline{d}) := (c, d_2, \cdots, d_n)$ in $\boldsymbol{C} \times \boldsymbol{C}^{n-1}$ such that $0 < |c| < 1$, $|d_i| < 1$ ($2 \leqq i \leqq n$) and $(c, \underline{d}) \notin F_{\kappa\mu\nu}$ for any κ, μ, ν if $(c, \underline{d}) \in \Phi_{\kappa\mu}^{-1}(U_\nu)$. Consider the non-singular linear transformation

$$
\Psi: \quad
\begin{aligned}
z_1 &= c w_1 \\
z_2 &= d_2 w_1 + w_2 \\
&\cdots\cdots \\
z_n &= d_n w_1 + w_n.
\end{aligned}
$$

We have new local coordinates $w_1, w_2, \cdots, w_n$ which are well-defined on $W := \{|w_i| < 1/2,\ 1 \leqq i \leqq n\}$. Let $\underline{w} := (w_2, \cdots, w_n)$ and $w := (w_1, \underline{w}) = (w_1, \cdots, w_n)$. The functions $\tilde{u}_\nu(w_1, \underline{w}) = u_\nu(\Psi(w_1, \underline{w}))$ defined on $W \cap \Psi^{-1}(U_\nu)$ satisfy the condition $\tilde{u}_\nu(a^{(\kappa)}, \underline{b}^{(\mu)}) \neq -\infty$ if $(a^{(\kappa)}, \underline{b}^{(\mu)}) \in \Psi^{-1}(U_\nu)$. So, $u_\nu(w_1, \underline{b}^{(\mu)}) \not\equiv -\infty$ on any connected component of $W \cap \{\underline{w} = \underline{b}^{(\mu)}\} \cap \Psi^{-1}(U_\nu)$ because $\{a^{(\kappa)}\}$ is dense in $\{|w_1| < 1/2\}$. Since

$$W \cap S_\nu \cap \{\underline{w} = \underline{b}^{(\mu)}\} \subset \{u_\nu(w_1, \underline{b}^{(\mu)}) = -\infty\} \cap W \cap \Psi^{-1}(U_\nu)$$

for any μ and ν, it is considered as a set of capacity zero in the w_1-plane. In particular, $W \cap S \cap \{w_2 = \cdots = w_n = 0\}$ is of capacity zero in the w_1-plane. Then, as is well-known, there is an arbitrarily small real number s_1 such that

$$(\{|w| = s_1\} \times \{w_2 = \cdots = w_n = 0\}) \cap S = \phi$$

and so we can find real numbers s_1', s_1'', $s_2, \cdots, s_n$ such that $(\{s_1' \leqq |w_1| \leqq s_1''\} \times \{|w_2| < s_2, \cdots, |w_n| < s_n\}) \cap S = \phi$, where $0 < s_1' < s_1 < s_1'' < 1/2$ and $0 < s_i < 1/2$ $(2 \leqq i \leqq n)$.

Put $U = \{|w_i| < s_i \ (1 \leqq i \leqq n)\}$. We want to prove that f has a holomorphic continuation to U. It may be assumed that f is bounded. Consider the function

$$\tilde{f}(w_1, \underline{w}) = \frac{1}{2\pi i} \int_{|\zeta| = s_1''} \frac{f(\zeta, \underline{w})}{\zeta - w_1} d\zeta.$$

Obviouly, it is holomorphic on U. On the other hand, each $h_\mu(w_1) := f(w_1, \underline{b}^{(\mu)})$ is a bounded holomorphic function on $\{|w_1| < s_1\}$ except a closed set of capacity zero. It has exactly one holomorphic continuation to the whole $\{|w_1| < s_1\}$ (e.g. [8], p. 171), which ought to be equal to $\tilde{f}(w_1, \underline{b}^{(\mu)})$. It follows that $f(w_1, \underline{b}^{(\mu)}) = \tilde{f}(w_1, \underline{b}^{(\mu)})$ on $U - S$. Since $\{b^{(\mu)}\}$ is dense in $\{|w_i| < 1/2, \ 2 \leqq i \leqq n\}$, we conclude $f(w) = \tilde{f}(w)$ on the whole $U - S$. This shows that $\tilde{f}$ is a continuation of f to U.

Now, we set about the proof of Proposition 3.1 for an arbitrary complex space X. By X_{reg} we denote the set of all regular points of X. Then, it is considered as a complex manifold and $S \cap X_{\text{reg}}$ is a closed set of capacity zero in X_{reg}. By the above proof, f has a bounded holomorphic continuation to the whole X_{reg}. Since $X - X_{\text{reg}}$ is a thin analytic subset of X, Proposition 3.1 is an immediate consequence of the well-known Riemann theorem on removable singularities of holomorphic functions on a normal complex space.

COROLLARY 3.2. *If S is a closed set of capacity zero in a complex space, then $D - S$ is connected for any connected open subset D of X.*

Proof. Assume that $D - S$ is not connected, i.e. it can be written $D - S = D_1 \cup D_2$ with mutually disjoint non-empty open sets D_1 and D_2. The bounded holomorphic function $f(z)$ on $D - S$ defined as $f(z) = i$ on D_i

$(i=1,2)$ is not continuable to D, which is contrary to the assumption by Proposition 3.1.

§4. The boundary of a Riemann domain.

By definition, a Riemann domain (X, π, M) is a triple of complex spaces X and M with $\dim X = \dim M$ and a discrete holomorphic map $\pi : X \to M$. In this paper, the base space M is always assumed to be a complex manifold.

Let (X, π, M) be an arbitrary Riemann domain.

DEFINITION 4.1. We shall say a sequence $\{x_\nu\}$ in X to *converge to the boundary* of X (relative to M) if $\{x_\nu\}$ has no accumulation point in X and the sequence $\{\pi(x_\nu)\}$ has a limit z_0 in M. Such a point z_0, i.e. the limit of $\{\pi(x_\nu)\}$ for some $\{x_\nu\}$ converging to the boundary, is said a *boundary value* of (X, π, M).

DEFINITION 4.2. An *accessible boundary point* of (X, π, M) is defined as a filter $r = \{U_\iota\, ;\, \iota \in I\}$ satisfying the conditions (i) there exist a point z in M and a fundamental system $\{V_\iota\, ;\, \iota \in I\}$ of open connected neighborhoods of z such that each U_ι is a connected component of $\pi^{-1}(V_\iota)$ and (ii) $U_\iota \in r$ is not relatively compact in X.

We denote the set of all accessible boundary points of (X, π, M) by $\partial_a X$ and put $\check{X} = X \cup \partial_a X$. The set $\check{X}$ has a canonically defined locally connected Hausdorff topology such that a base for neighborhoods of each $r_0 = \{U_\iota\} \in \partial_a X$ is given by the system $\{\check{U}_\iota\}$, where

$$\check{U}_\iota := U_\iota \cup \{r \in \partial_a X\, ;\, \text{there exists some } V_\kappa \in r \text{ with } V_\kappa \subset U\}.$$

The projection map π is canonically extended to a continuous map $\pi : X \to M$. A sequence $\{x_\nu\}$ in X converges to some $r \in \partial_a X$ if and only if $\{x_\nu\}$ converges to the boundary of X and there is a continuous curve $\gamma(t)$ $(0 < t \leqq 1)$ in X such that $\gamma(1/\nu) = x_\nu$ for any ν and $\lim_{t \to 0} \pi(\gamma(t)) = \lim_{\nu \to \infty} \pi(x_\nu)$.

DEFINITION 4.3. An accessible boundary point $r \in \partial_a X$ is said to be a *direct boundary point* if there exists a neighborhood U of $\check{\pi}(r)$ such that the connected component U' of $\pi^{-1}(U)$ which gives a neighborhood $\check{U}$ of r satisfies the condition $U' \cap \pi^{-1}(\check{\pi}(r)) = \phi$.

By definition, we see easily

(4.4) *For a point $r \in \partial_a X$, if $\pi^{-1}(\check{\pi}(r))$ contains only finitely many points in X, r is a direct boundary point.*

DEFINITION 4.5. A Riemann domain (X, π, M) is said to *have the boundary of capacity zero* if each $z \in M$ has a neighborhood U such that there exists a plurisubharmonic function $u(x)$ on $\pi^{-1}(U)$ with the following properties;

(i) $u(x) \leqq 0$,

(ii) $u(x) \not\equiv -\infty$ on any connected component of $\pi^{-1}(U)$,

(iii) $\lim_{\nu \to \infty} u(x_\nu) = -\infty$ for any sequence $\{x_\nu\}$ in X which converges to the boundary.

EXAMPLE 4.6. (i) Let (X, π, M) be a Riemann domain with boundary of capacity zero and D be a connected open subset of M. Then, for any connected component D_ι of $\pi^{-1}(D)$, $(D_\iota, \pi | D_\iota, D)$ is also a Riemann domain with boundary of capacity zero.

(ii) If X is a Riemann surface of type O_G (e.g., see [10], p. 429) and $\phi(x)$ is a non-constant meromorphic function on X, a Riemann domain $(X, \phi, \boldsymbol{P})$ has the boundary of capacity zero, where $\boldsymbol{P}$ is the Riemann sphere.

For, if a Riemann surface X is of type O_G, we can find a non-positive harmonic funcion $u(x)$ on $X - \bar{X}_o$ for an arbitrarily small suitable relatively compact open set X_o in X such that $\lim_{\nu \to \infty} u(x_\nu) = -\infty$ for any sequence $\{x_\nu\}$ in X converging to the ideal boundary (c.f. M. Nakai [5], Theorem, p. 624).

(iii) Let D be a domain in C^n and S be an at most k dimensional analytic set in D. If A is an irreducible analytic set in $D - S$ and x is a point in $\bar{A} \cap S$, we can find polydiscs $U_1 := \{|z_i| < r_i,\ 1 \leqq i \leqq k\}$, $U_2 := \{|z_i| < r_i,\ k+1 \leqq i \leqq n\}$ $(r_i > 0)$ for a suitable system of local coordinates $z_1, \cdots, z_n$ on a neighborhood of x with $x = (0)$ such that the Riemann domain $(X, \pi \cdot \mu, U_1)$ has the boundary of capacity zero, where $\pi : (z_1, \cdots, z_n) \to (z_1, \cdots, z_k)$ is the canoncal projection and X is the normalization of the locally analytic set $A \cap (U_1 \times U_2)$ with projection map μ.

To see this, we take a system of local coordinates $z_1, \cdots, z_n$ in a neighborhood V of x with $x = (0)$ such that the map $\pi : (z_1, \cdots, z_n) \to (z_1, \cdots, z_k)$ is discrete on $V \cap (A \cup S)$. Then we can find easily a sufficiently small polydisc $U_2 := \{|z_i| < r_i,\ k+1 \leqq i \leqq n\}$ $(r_i > 0)$ with the property $(\{z_1 = \cdots = z_k = 0\} \times \partial U_2) \cap (A \cup S) = \phi$ and hence a polydisc $U_1 := \{|z_i| < r_i,\ 1 \leqq i \leqq k\}$ $(r_i > 0)$ such that $(U_1 \times \partial U_2) \cap (A \cup S) = \phi$ because $A \cup S$ is closed, where $U := U_1 \times U_2 \Subset V$. On the other hand, by the assumption we may assume that $U \cap S \subset \{f=0\}$ with a suitable holomorphic function $f(z)$ on U which does not vanish identically on A. Using the plurisubharmonic function $u(x) := \log|(f\cdot\mu)(x)|$ on X, we can easily conclude that these polydiscs U_1 and U_2 satisfy the desired condition.

§ 5. The projection image of a Riemann domain with boundary of capacity zero.

The following theorem is a generalization of a result in [4] (c.f. Tsuji, [10]. p. 437) to the case of Riemann domains of arbitrary dimension.

THEOREM 5.1. *If (X, π, M) is a Riemann domain with boundary of capacity zero, then $M-\pi(X)$ is a closed set of capacity zero in M.*

For the proof, we give the following

LEMMA 5.2. *Let (X, π, M) be a Riemann domain such that π is proper and $u(x)$ be a plurisubharmonic function on X. If we put*

$$w(z) := \max\{u(x);\ \pi(x)=z,\ x\in X\}$$

for each $z\in M$, then the function $w(z)$ is plurisubharmonic on M.

Proof. By the assumption, there is a thin analytic set N in M such that $(X-\pi^{-1}(N),\ \pi|X-\pi^{-1}(N),\ M-N)$ is an unramified proper covering space. If we put $\pi^{-1}(z)=\{x_1(z),\cdots,x_m(z)\}$ $(z\in M-N)$, then $w(z)=\max_{1\leqq i\leqq m} u(x_i(z))$ is plurisubharmonic on $M-N$ by (2.2). To prove Lemma 5.2, it suffices to show $\overline{\lim}_{z\to z_0,\ z\in M-N} w(z)=w(z_0)$ for any $z_0\in N$ because of (2.3).

Take an arbitrary sequence $\{z_\nu\}$ in $M-N$ such that $\lim_{\nu\to\infty} z_\nu = z_0$ $(z_0\in N)$ and $\lim_{\nu\to\infty} w(z_\nu)=\overline{\lim}_{z\to z_0,\ z\in M-N} w(z)$. By the definition of $w(z)$, there is a sequence $\{x_\nu\}$ with the properties $\pi(x_\nu)=z_\nu$ and $w(z_\nu)=u(x_\nu)$, which may be assumed to have a limit point x_0 in X by the properness of π. Then we get

$$\overline{\lim_{z\to z_0,\ z\in M-N}} w(z)=\lim_{\nu\to\infty} u(x_\nu)\leqq \overline{\lim_{x\to x_0}} u(x)\leqq w(z_0).$$

On the other hand, if we take a point $x_0\in X$ such that $w(z_0)=u(x_0)$ and $\pi(x_0)=z_0$, there is a sequence $\{x_\nu\}$ in $X-\pi^{-1}(N)$ satisfying the condition $\lim_{\nu\to\infty} x_\nu = x_0$ and $\lim_{\nu\to\infty} u(x_\nu)=u(x_0)$ because of (2.4). Hence we have

$$w(z_0)=\lim_{\nu\to\infty} u(x_\nu)\leqq \lim_{\nu\to\infty} w(\pi(x_\nu))\leqq \overline{\lim_{z\to z_0,\ z\in M-N}} w(z).$$

This completes the proof.

LEMMA 5.3. *Let $u(x)$ be a plurisubharmonic function on a Riemann domain (X, π, M) such that* (i) $0 \geqq u(x) \not\equiv -\infty$ *and* (ii) $\lim_{\nu\to\infty} u(x_\nu) = -\infty$ *for any sequence $\{x_\nu\}$ in X which converges to the boundary. If we put*

$$w(z) = \begin{cases} \sup\{u(x) \ ; \ \pi(x) = z, \ x \in X\} & \text{for any } z \in \pi(X) \\ -\infty & \text{for any } z \notin \pi(X), \end{cases}$$

it is plurisubharmonic on M.

Proof. Firstly, we shall show that $w(z)$ is upper semicontinuous on M. For any $z_0 \in M$, take a sequence $\{z_\nu\}$ in M such that $\lim_{\nu\to\infty} z_\nu = z_0$ and $K := \lim_{\nu\to\infty} w(z_\nu)$ exists. We want to prove $K \leqq w(z_0)$. Let $K \neq -\infty$ because, if not, the proof is trivial. Choosing a subsequence and changing indices if necessary, we may assume that there is a sequence $\{x_\nu\}$ in X such that $\pi(x_\nu) = z_\nu$, $\lim_{\nu\to\infty} u(x_\nu) = K$ and, moreover, $\{x_\nu\}$ has a limit x_0 in X by the assumption (iii) in Lemma 5.3. Since $\pi(x_0) = z_0$ and $u(x)$ is upper semicontinuous, it holds that

$$K = \lim_{\nu\to\infty} u(x_\nu) \leqq \overline{\lim_{x\to x_0}}\, u(x) \leqq u(x_0) \leqq w(z_0).$$

This shows that $w(z)$ is upper semi-continuous.

To complete the proof, taking an open set W in $\boldsymbol{C}$ and a holomorphic map $\phi : W \to M$, we shall prove that

$$(w \cdot \phi)(t_0) \leqq \frac{1}{2\pi} \int_0^{2\pi} (w \cdot \phi)(t_0 + re^{i\theta}) d\theta$$

for any $t_0 \in W$ and a sufficiently small arbitrary positive real number r. We may assume $(w \cdot \phi)(t_0) \neq -\infty$. Consider the set $E := \{x \in X \ ; \ u(x) = w(z_0), \pi(x) = z_0\}$, where $z_0 = \pi(t_0)$. By the properties of π and u, E is a finite set. We can find easily neighborhoods V of E and U of z_0 such that the map $\pi' := \pi | V : V \to U$ is proper. Then the Riemann domain (V, π', U) satisfies the assumption in Lemma 5.2. So, the function $\tilde{w}(z) := \max\{u(x);\ \pi(x) = z, \ x \in V\}$ is plurisubharmonic and, obviously, satisfies the conditions $\tilde{w}(z_0) = w(z_0)$ and $\tilde{w}(z) \leqq w(z)$ on U. We obtain

$$w(z_0) = (\tilde{w} \cdot \phi)(t_0) \leqq \frac{1}{2\pi} \int_0^{2\pi} (\tilde{w} \cdot \phi)(t_0 + re^{i\theta}) d\theta$$

$$\leqq \frac{1}{2\pi} \int_0^{2\pi} (w \cdot \phi)(t_0 + re^{i\theta}) d\theta$$

for any sufficiently small $r>0$. This asserts that $w(z)$ is plurisubharmonic on M.

Proof of Theorem 5.1. Let z_0 be an arbitrary point in the boundary $\partial\pi(X)$ of $\pi(X)$. By the assumption, for a suitable neighborhood U of z_0, there is a plurisubharmonic function $u(x)$ on $\pi^{-1}(U)$ satisfying the conditions in Definition 4.5, (i)$\sim$(iii). For an arbitrarily fixed connected component V_ι of $\pi^{-1}(U)$, the Riemann domain $(V_\iota, \pi|V_\iota, U)$ satisfies the assumption in Lemma 5.3. The function

$$w(z):=\begin{cases}\sup\{u(x)\,;\,\pi(x)=z,\ x\in V_\iota\} & \text{for any } z\in\pi(V_\iota)\\ -\infty & \text{for any } z\in U-\pi(V_\iota)\end{cases}$$

is plurisubharmonic on U. Obviously, $U\cap\partial\pi(X)\subset\{z\in U\,;\,w(z)=-\infty\}$, which is of capacity zero. Since $\partial\pi(X)$ is covered by countably many U's with the above properties, it is of capacity zero. Then we have $M-\pi(X)=\partial\pi(X)$ and hence Theorem 5.1. Indeed, if not, $M-\pi(X)$ has a boundary point in $M-\partial\pi(X)$ because $M-\partial\pi(X)$ is connected by Corollary 3.2, which is absurd.

§ 6. Direct boundary points of Riemann domains with boundary of capacity zero.

Using Theorem 5.1, we can prove the following theorem on direct boundary points of a Riemann domain, which is a generalization of A. Mori [4], Corollary 2, p. 288.

THEOREM 6.. *If (X, π, M) is a Riemann domain with boundary of capacity zero, the projection image of the set of all direct boundary points of X is a set of capacity zero in M.*

Proof. Let $\mathfrak{A}=\{U_\nu\}$ be a countable base for connected open sets in M. For each U_ν, each connectd component U_ν^ι of $\pi^{-1}(U_\nu)$ defines a Riemann domain $(U_\nu^\iota, \pi|U_\nu^\iota, U_\nu)$ which has the boundary of capacity zero. According to Theorem 5.1, the set $F_\nu^\iota:=U_\nu-\pi(U_\nu^\iota)$ is of capacity zero in U_ν. So, $F:=\cup_{\iota,\nu}F_\nu^\iota$ is also of capacity zero in M. We need only to show that the $\check{\pi}$-image of an arbitrary direct boundary point of (X, π, M) is contained in F. Let $r\in\partial_a X$ be an arbitrary direct boundary point and put $\check{\pi}(r)=z$. By definition, there is a neighborhood U of z such that a connected component U' of $\pi^{-1}(U)$ with $U'\in r$ satisfies the condition $U'\cap\pi^{-1}(\check{\pi}(r))=\phi$, where we may assume $U = U_{\nu_0} \in \mathfrak{A}$ and $U' = U_{\nu_0}^{\iota_0}$ for some ν_0 and ι_0. Obviously, $z \in F_{\nu_0}^{\iota_0} \subset F$. This completes the proof.

As an application of Theorem 6.1, we can generalize the result of W. Rothstein [8], p. 172 to the case of Riemann domains of arbitrary dimension.

COROLLARY 6.2. *Assume that (X, π, M) has the boundary of capacity zero. If P is a subset of the set of all boundary values of (X, π, M) such that $\pi^{-1}(c)$ is finite for any $c \in P$, it is of capacity zero.*

Proof. Assume that cap $P > 0$. Without loss of generality, we may assume that P contains no direct boundary point because of Theorem 6.1. Consider the set

$$P_\nu = \{z \in P\,;\ \pi^{-1}(z) \text{ consists of at most } \nu \text{ points}\}$$

for any $\nu = 0, 1, 2, \cdots$. Since $P = \cup_\nu P_\nu$, there is some ν_0 such that cap $P_{\nu_0} > 0$ by (2.6), (i). Here, it cannot happen to be $\nu_0 = 0$ in virtue of Theorem 5.1. Moreover, according to (2.6) (iii),

$$P' := \{z \in P_{\nu_0}\,;\ \mathrm{cap}\,(P_{\nu_0} \cap U) > 0 \text{ for any neighborhood } U \text{ of } z\}$$

is of positive capacity. Take a point z_o in P' and an arbitrary neighborhood U of z_o. For any connected component U_ι of $\pi^{-1}(U)$, since cap $(U - \pi(U_\iota)) = 0$, we can choose a point $z_1 \in P' \cap U$ such that $z_1 \notin \cup_\iota (U - \pi(U_\iota))$, whence $z_1 \in \cap\, \pi(U_\iota)$. This shows that $\pi^{-1}(U)$ has at most ν_o connected components. Since z_o is the boundary value of (X, π, M), there is at least one connected component of $\pi^{-1}(U)$ which is not relatively compact in X. This concludes that z_o is the $\check{\pi}$-image of an accessible boundary point of (X, π, M), which is absurd because of (4.4). Hence Corollary 6.2 is proved.

The following theorem is essentially another description of the Bishop's result in [1], Theorem 4, p. 301 (c.f. Theorem 7.3 in the following section).

THEOREM 6.3. *Let (X, π, M) be a Riemann domain with boundary of capacity zero. If there is at least one point in M which is not a boundary value of (X, π, M), then $\pi^{-1}(z)$ is finite for any $z \in M$ and, moreover, there exists a closed set N of capacity zero in M such that $\pi \,|\, X - \pi^{-1}(N) : X - \pi^{-1}(N) \to M - N$ is proper.*

Proof. Since the set S of all boundary values is closed, we can take a connected open set U in M such that $U \cap S = \phi$. Obviously, the map $\pi \,|\, \pi^{-1}(U) : \pi^{-1}(U) \to U$ is proper. Consider the set G of all points z in M

with the property that $\pi|\pi^{-1}(V):\pi^{-1}(V)\rightarrow V$ is proper for some neighborhood V of z. Then obviously, $\pi|\pi^{-1}(G):\pi^{-1}(G)\rightarrow G$ itself is also proper and $\partial G\subset S$. Moreover, $\pi^{-1}(z)$ is finite for any $z\in\partial G$. In view of Corollary 6.2, ∂G is of capacity zero and so $N:=M-G=\partial G$ as in the proof of Theorem 5.1. This concludes Theorem 6.3.

COROLLARY 6.4. *For a Riemann domain (X,π,M) with boundary of capacity zero, if there is a set P in M such that* $\operatorname{cap} P>0$ *and $\pi^{-1}(z)$ is a finite set for any $z\in P$, the same conclusion in Theorem* 6.3 *is valid* (c.f. W. Rothstein [8], Satz 1, p. 173).

The proof is evident by Corollary 6.2 and Theorem 6.3.

Now, we give another application of Theorem 5.1. The following is a generalization of the well-known Iversen's theorem.

THEOREM 6.5. *Assume that (X,π,M) has the boundary of capacity zero. Take an arbitrary point z_0 in M and a connected neighborhood U of z_0. Then, for any $x_0\in\pi^{-1}(U)$, there exists a continuous curve $\gamma(t)$ $(0<t\leqq 1)$ in $\pi^{-1}(U)$ such that $\gamma(1)=x_0$ and $\lim_{t\to 0}(\pi\cdot\gamma)(t)=z_0$.*

Proof. Let $\{U_\nu\}$ be a countable base for connected open neighborhoods of z_0. It suffices to show that there is a sequence $\{x_\nu\}$ such that $x_\nu\in\pi^{-1}(U_\nu)$ and a suitable continuous curve $\gamma_\nu(t)$ in $\pi^{-1}(U_\nu)$ joins x_ν with $x_{\nu+1}$ for any ν. We proceed by induction on ν. Assume that there exist points x_μ and curves γ_μ $(1\leqq\mu\leqq\nu)$ with the desired properties. Let U' be a connected component of $\pi^{-1}(U_\nu)$ which contains x_ν. Since $(U',\ \pi|U', U_\nu)$ has the boundary of capacity zero, we see $\operatorname{cap}(U_\nu-\pi(U'))=0$. If we choose an arbitrary $x_{\nu+1}\in U'$ with $\pi(x_{\nu+1})\in U_{\nu+1}$, x_ν is joined with $x_{\nu+1}$ by a continuous curve $\gamma_{\nu+1}$ in U'. These $x_{\nu+1}$ and $\gamma_{\nu+1}$ are the desired ones.

§7. Continuations of Riemann domains and analytic sets.

Applying the results of the previous sections, we can give some sufficient conditions for the continuability of Riemann domains and analytic sets by the similar arguments as in W. Rothstein [8].

THEOREM 7.1. *Let (X,π,M) be a Riemann domain with boundary of capacity zero. Assume that there is a set P in M with the properties that* (i) $\operatorname{cap} P>0$, (ii) *for any $z\in P$ $\pi^{-1}(z)$ is finite and* (iii) *for each $z_0\in P$ a suitable bounded holo-*

morphic function f on X satisfies the condition $f(x_i) \neq f(x_j)$ $(i \neq j)$, where $\pi^{-1}(z_o) = \{x_1, \cdots, x_m\}$. Then the space $\check{X} = X \cup \partial_a X$ defined as in § 4 has a structure of a complex space such that X is an open subspace of $\check{X}$ with $\operatorname{cap}(\check{X} - X) = 0$ *and the projection $\check{\pi}$ is a proper discrete holomorphic map.*

For the proof, we need

LEMMA 7.2. *For a Riemann domain (X, π, M), if it has the boundary of capacity zero and $\pi^{-1}(z)$ contains at most finitely many points for any $z \in M$, it holds that* (i) *X is dense in $\check{X}$,* (ii) *$D \cap X$ is connected for any connected open subset D of $\check{X}$,* (iii) *$\check{X}$ is locally compact and* (iv) *$\check{\pi} : \check{X} \to M$ is proper discrete.*

Proof. The properties (i) and (ii) are evident by the definition of the topology of $\check{X}$ and (iii) follows from (iv). It remains only to prove (iv). As is easily seen, there is an integer ν_o (>0) such that $\pi^{-1}(z)$ consists of at most ν_o points for any $z \in M$. Moreover, as in the proof of Corollary 6.2, $\pi^{-1}(U)$ has at most ν_o connected components for any connected open set U in M. This implies that any sequence converging to the boundary has a subsequence converging to an accessible boundary point and so π is proper. The discreteness of $\check{\pi}$ is obvious because $\check{\pi}^{-1}(z)$ $(z \in M)$ contains at most ν_o points.

Proof of Theorem 7.1. By Corollary 6.4, $\pi | X - \pi^{-1}(N) : X - \pi^{-1}(N) \to M - N$ is proper for a suitable closed set N of capacity zero in M, where $(X - \pi^{-1}(N), \pi | X - \pi^{-1}(N), M - N)$ may be assumed to be an unramified proper covering space. Put $X' := X - \pi^{-1}(N)$ and $\pi' := \pi | X - \pi^{-1}(N)$. Since $\operatorname{cap}(\pi^{-1}(N)) = 0$, we see easily $\check{X}' = \check{X}$ as topological spaces and $\check{\pi}' = \check{\pi}$, where $\check{X}' = X \cup \partial_a X'$ and $\check{\pi}'$ is an extension of π' to $\check{X}'$ as defined in § 4 for a Riemann domain (X', π', M). There is no harm in assuming that $X = X'$ and $\pi = \pi'$.

Put $\pi^{-1}(z) = \{x_1(z), \cdots, x_m(z)\}$ for any $z \in M - N$. Using the holomorphic function f on X given for some $z_o \in P - N$, we define the pseudopolynomial

$$\Pi_{i=1}^{m}(w - f(x_i(z))) = w^m + a_1(z) w^{m-1} + \cdots + a_m(z)$$

whose coefficients $a_i(z)$ are holomorphic on $M - N$ and locally bounded on N. By virtue of Proposition 3.1, each $a_i(z)$ has a holomorphic continuation $\tilde{a}_i(z)$ to the whole M. Let Y be the analytic set in $M \times \boldsymbol{C}$ defined by the equation

$$w^m + \tilde{a}_1(z)w^{m-1} + \cdots + \tilde{a}_m(z) = 0$$

and take the normalization $\mu : \tilde{Y} \to Y$ of Y. For the canonical projection $\pi_1 : (z, w) \to z$, putting $\tilde{\pi} = \pi_1 \cdot \mu$, we have proper finite covering space $(\tilde{Y}, \tilde{\pi}, M)$. By the assumption of f, $\tilde{\pi}^{-1}(z)$ contains exactly m points for any $z \in M$ except a thin analytic set. Then there is a homeomorphism τ of $\tilde{Y} - \tilde{\pi}^{-1}(N)$ onto $X - \pi^{-1}(N)$ such that $\pi\tau = \tilde{\pi}$ on $\tilde{Y} - \tilde{\pi}^{-1}(N)$. In this situation, by virtue of Lemma 7.2, we can easily prove that τ has an extension $\tilde{\tau} : \tilde{Y} \to \check{X}$ with $\check{\pi}\tilde{\tau} = \tilde{\pi}$ which gives a homeomorphism between $\tilde{Y}$ and $\check{X}$ by the analogous argument as in the proof of the uniqueness of the normalization of a not necessarily normal reduced complx space (c.f. [3], Satz 2, p. 250). We can define a structure of a complex space on $\check{X}$ such that $\tilde{\tau}$ is biholomorphic. The Riemann domain $(X, \check{\pi}, M)$ obtained in this manner satisfies obviously the conditions in Theorem 7.1.

Now, we shall prove Theorem III stated in §1 under slightly weaker assumptions.

THEOREM 7.3. *Let M be a complex manifold of dimension n. Assume that an irreducible n-dimensional analytic set A in some open subset of $M \times \boldsymbol{C}^k$ satisfies the following conditions;*

(i) *there is a plurisubharmonic function $u(x)$ on A such that $0 \geqq u(x) \not\equiv -\infty$ and $\lim_{\nu\to\infty} u(x_\nu) = -\infty$ for any sequence $\{x_\nu\}$ in A without accumulation points if $\lim_{\nu\to\infty} \pi_1(x_\nu)$ exists,*

(ii) *$\pi_2(A)$ is a bounded subset of $\boldsymbol{C}^k$,*

(iii) *there is a set P in M such that $\operatorname{cap}(P) > 0$ and $\pi_1^{-1}(z) \cap A$ is finite for any $z \in P$,*

where $\pi_1 : M \times \boldsymbol{C}^k \to M$ and $\pi_2 : M \times \boldsymbol{C}^k \to \boldsymbol{C}^k$ are the canonical projections.

Then, $\bar{A}$ is analytic in $M \times \boldsymbol{C}^k$.

Proof. Let $\mu : X \to A$ be the normalization of A and put $\pi := \pi_1 \cdot \mu$. The Riemann domain (X, π, M) has the boundary of capacity zero. By Corollary 6.4, $\pi | X - \pi^{-1}(N) : X - \pi^{-1}(N) \to M - N$ is proper if we take a suitable closed subset N of capacity zero in M. Then, $\pi_1 | A - \pi_1^{-1}(N) : A - \pi_1^{-1}(N) \to M - N$ is also proper. As usual, for each coordinate $w_i \, (1 \leqq i \leqq k)$ in $\boldsymbol{C}^k$, we take the equation

$$P_i(z\,;\,w_i)=w_i^m+a_1^{(i)}(z)w_i^{m-1}+\cdots+a_m^{(i)}(z)=0$$

on $A-\pi_1^{-1}(N)$, where $a_j^{(i)}$ are bounded holomorphic on $M-N$ and so have holomorphic continuations $\tilde{a}_j^{(i)}$ to M. If we take the analytic set $A':=\{w_i^m+\tilde{a}_1^{(i)}w_i^{m-1}+\cdots+\tilde{a}_m^{(i)}=0,\ 1\leqq i\leqq k\}$, it can be easily seen that the irreducible component of A' including $A\cap((M-N)\times\boldsymbol{C}^k)$ coincides with the set $\bar{A}$. This shows that $\bar{A}$ is analytic in $M\times\boldsymbol{C}^k$.

Lastly we note that the Thullen-Remmert-Stein's theorem on the essential singularities of analytic sets in [9] and [7] is an immediate consequence of Theorem 7.3 and Example 4.6, (iii).

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