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ON ELLIPTIC CURVES WITH COMPLEX MULTIPLICATION AS FACTORS OF THE JACOBIANS OF MODULAR FUNCTION FIELDS

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1. As Hecke showed, every L-function of an imaginary quadratic field K with a Grössen-character λ is the Mellin transform of a cusp form $f(z)$ belonging to a certain congruence subgroup Γ of $SL_2(\boldsymbol{Z})$. We can normalize λ so that

$$\lambda((\alpha)) = \alpha^{\nu} \quad \text{for} \quad \alpha \in K, \ \alpha \equiv 1 \ \mathrm{mod}^{\times} \mathfrak{c}$$

with a positive integer ν, where $\mathfrak{c}$ is the conductor of λ, and $\mathrm{mod}^{\times} \mathfrak{c}$ means the multiplicative congruence modulo $\mathfrak{c}$. Then $f(z)$ is of weight $\nu+1$, i.e.,

$$f((az+b)/(cz+d)) = f(z)(cz+d)^{\nu+1} \quad \text{for} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma,$$

and Γ is given by

$$\Gamma = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\boldsymbol{Z}) \,\middle|\, a \equiv d \equiv 1, \ c \equiv 0 \ \mathrm{mod}\ (D \cdot N(\mathfrak{c})) \right\},$$

where $-D$ is the discriminant of K. If $\nu = 1$, $f(z)dz$ is a differential form of the first kind on the compactification $(H/\Gamma)^*$ of the quotient H/Γ, where H denotes the upper half complex plane. Denote by $\mathrm{Jac}(H/\Gamma)$ the jacobian variety of $(H/\Gamma)^*$, and identify the tangent space of $\mathrm{Jac}(H/\Gamma)$ at the origin with the space of all differential forms of the first kind on $(H/\Gamma)^*$. Let A be the smallest abelian subvariety of $\mathrm{Jac}(H/\Gamma)$ that has $f(z)dz$ as a tangent at the origin. Then the first main result of this paper can be stated as follows:

The abelian variety A is a product of copies of an elliptic curve whose endomorphism algebra is isomorphic to K.

Hecke [3] proved this fact in the case where $K = \boldsymbol{Q}(\sqrt{-q})$ with a prime $q > 3$, $\equiv 3 \ \mathrm{mod}\ (4)$ and $\mathfrak{c} = (\sqrt{-q})$. In the general case, he showed only that

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the periods of $f(z)dz$ belong to a certain class field over K. His proof requires rather deep arithmetic results of complex multiplication. Ours is simpler, and based on the following

LEMMA 1. *Let X be an abelian variety of dimension n defined over $\boldsymbol{C}$, and h an injective homomorphism of K into $End_{\boldsymbol{Q}}(X)$. Suppose that the representation of K, through h, on the tangent space of X at the origin is equivalent to n copies of the identity injection of K into $\boldsymbol{C}$. Then X is isogenous to a product of n copies of an elliptic curve E such that $End_{\boldsymbol{Q}}(E)$ is isomorphic to K.*

Here and henceforth we denote by $\mathrm{End}(X)$ the ring of all endomorphisms of X over $\boldsymbol{C}$, and put $\mathrm{End}_{\boldsymbol{Q}}(X) = \mathrm{End}(X) \otimes \boldsymbol{Q}$.

Our next purpose is to show that every elliptic curve E defined over $\boldsymbol{Q}$ with complex multiplication is isogenous over $\boldsymbol{Q}$ to a factor of $\mathrm{Jac}(H/\Gamma')$ for some Γ' in the following way. By virtue of Deuring's result [1], if K is isomorphic to $\mathrm{End}_{\boldsymbol{Q}}(E)$, the zeta-function of E over $\boldsymbol{Q}$ is exactly the L-function of a certain Grössen-character λ of K. Then we obtain an abelian variety A by the procedure described above, i.e.,

elliptic curve $E \to$ zeta-function with a Grössen-character λ
$\to$ cusp form $f(z) \to$ abelian subvariety A of $\mathrm{Jac}(H/\Gamma')$.

In this situation, we shall prove:

A is an elliptic curve isogenous to E over $\boldsymbol{Q}$.

This is an easy consequence of the results in the previous articles [7], [8]. If $-D$ is the discriminant of K, and $\mathfrak{c}$ is the conductor of λ, the group Γ' is of the form

$$\Gamma' = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\boldsymbol{Z}) \,\middle|\, c \equiv 0 \bmod (D \cdot N(\mathfrak{c})) \right\}.$$

2. Let us first prove the above lemma. Although it is a special case of [6, Prop. 14], we give here a direct proof for the reader's convenience.

Identify X with a complex torus $\boldsymbol{C}^n/L$ with a lattice L. Let $\boldsymbol{Q} \cdot L$ denote the $\boldsymbol{Q}$-linear span of L. Then K acts, through h, on $\boldsymbol{Q} \cdot L$, so that there exists a K-linear isomorphism p of K^n onto $\boldsymbol{Q} \cdot L$, where K^n is the submodule of $\boldsymbol{C}^n$ consisting of the vectors whose components belong to K. Since $\boldsymbol{C}^n = K^n \otimes_{\boldsymbol{Q}} \boldsymbol{R} = (\boldsymbol{Q} \cdot L) \otimes_{\boldsymbol{Q}} \boldsymbol{R}$, we can extend p to an $\boldsymbol{R}$-linear automorphism of $\boldsymbol{C}^n$, which we denote again by p. By our assumption, we may assume that the action of an element α of K on X is represented by the complex linear transformation $u \longrightarrow \alpha u$ $(u \in \boldsymbol{C}^n)$ of $\boldsymbol{C}^n$. We can find a real number r and an element α of K so that $r \cdot \alpha = \sqrt{-1}$. Now p is K-linear and $\boldsymbol{R}$-linear, hence p commutes with the map $u \to \sqrt{-1} \cdot u$, i.e., p is $\boldsymbol{C}$-linear. Take any free $\boldsymbol{Z}$-submodule $\mathfrak{a}$ of rank 2 in K. Then p gives an isogeny of $\boldsymbol{C}^n/\mathfrak{a}^n = (\boldsymbol{C}/\mathfrak{a})^n$ onto $\boldsymbol{C}^n/L$. This proves the lemma, since $\boldsymbol{C}/\mathfrak{a}$ is an elliptic curve with K as its endomorphism algebra.

3. For a function $f(z)$ on H and $\xi = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\boldsymbol{R})$ with $\det(\xi) > 0$, we define a function $f|[\xi]_k$ on H by

$$(f|[\xi]_k)(z) = \det(\xi)^{k/2} \cdot (cz+d)^{-k} \cdot f((az+b)/(cz+d)).$$

For an arbitrary positive integer N, put

$$\Gamma_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\boldsymbol{Z}) \,\middle|\, c \equiv 0 \mod (N) \right\},$$

$$\Gamma_1(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N) \,\middle|\, a \equiv 1 \mod (N) \right\}.$$

Further, for a complex-valued character ε of $(\boldsymbol{Z}/N\boldsymbol{Z})^\times$,[1] we denote by $S_k(N, \varepsilon)$ the vector space of all the cusp forms $f(z)$ satisfying

$$f|[\gamma]_k = \varepsilon(d) \cdot f$$

for every $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$.

LEMMA 2. *Let* $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$ *be an element of* $S_k(N, \varepsilon)$, r *a positive integer,* M *a common multiple of* Nr *and* r^2, *and let*

$$g(z) = \sum_{(n,r)=1} a_n e^{2\pi i n z}.$$

Then $g \in S_k(M, \varepsilon')$, *where* ε' *is the restriction of* ε *to* $(\boldsymbol{Z}/M\boldsymbol{Z})^\times$.

Proof. Put $\zeta = e^{2\pi i/r}$, $\eta_u = \begin{bmatrix} r & u \\ 0 & r \end{bmatrix}$ for $u \in \boldsymbol{Z}$, and $\Gamma = \Gamma_1(N)$. We see easily that $\Gamma\eta_u = \Gamma\eta_v$ if and only if $u \equiv v \mod (r)$. We can find numbers x_u of $\boldsymbol{Q}(\zeta)$ for $u \in \boldsymbol{Z}$ such that

$$x_u = x_v \quad \text{if} \quad u \equiv v \mod (r),$$

$$\sum_{u=0}^{r-1} x_u \zeta^{un} = \begin{cases} 1 & \text{if} \quad (n, r) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

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[1] If S is an associative ring with the identity element, $S^\times$ denotes the group of all invertible elements in S.

We see easily that $g(z) = \sum_{u=0}^{r-1} x_u \cdot f|[\eta_u]_k$. Further, it can be seen that

(1) $$x_u = x_{au} \quad \text{if} \quad (a, r) = 1,$$

and x_u is invariant under $\mathrm{Gal}(\boldsymbol{Q}(\zeta)/\boldsymbol{Q})$, hence $x_u \in \boldsymbol{Q}$. Now $g(z)$ is a cusp form of level Nr^2 (see for example [7, Prop. 2.4, Lemma 3.9]). Therefore, to prove our assertion, it is sufficient to check the behavior of g under an element $\gamma = \begin{bmatrix} a & b \\ Mc & d \end{bmatrix}$ of $\Gamma_0(M)$. We have

$$\begin{bmatrix} r & u \\ 0 & r \end{bmatrix} \begin{bmatrix} a & b \\ Mc & d \end{bmatrix} = \begin{bmatrix} a' & b' \\ Mc & d' \end{bmatrix} \begin{bmatrix} r & d^2u \\ 0 & r \end{bmatrix}$$

with $a' = a + cuM/r$, $b' = b + du(1 - a'd)/r$, $d' = d - cd^2uM/r$. Note that $a' \equiv a$, $d' \equiv d \mod (N) \cap (r)$, and $a'd \equiv ad \equiv 1 \mod (r)$. Therefore, putting $v = d^2u$, we have $f|[\eta_u \gamma]_k = \varepsilon(d) \cdot f|[\eta_v]_k$. In view of (1), we obtain $g|[\gamma]_k = \varepsilon(d) \cdot g$, q.e.d.

4. For our purpose, it is necessary to consider Grössen-characters which are not necessarily "primitive". To define them, let $\mathfrak{m}$ be an integral ideal in K, and $I_{\mathfrak{m}}$ the group of all fractional ideals in K prime to $\mathfrak{m}$. Let $W_{\mathfrak{m}}$ denote the group of all elements α of $K^\times$ such that $\alpha \equiv 1 \mod^\times \mathfrak{m}$, i.e., $\alpha - 1$ is $\mathfrak{p}$-integral and divisible by $\mathfrak{m}_{\mathfrak{p}}$ for all prime factors $\mathfrak{p}$ of $\mathfrak{m}$, where $\mathfrak{m}_{\mathfrak{p}}$ is the $\mathfrak{p}$-closure of $\mathfrak{m}$. Further let $P_{\mathfrak{m}}$ denote the subgroup of $I_{\mathfrak{m}}$ consisting of all principal ideals (α) with $\alpha \in W_{\mathfrak{m}}$. For a positive integer ν, let $\Lambda_{\mathfrak{m}}^{\nu}$ denote the set of all homomorphisms λ of $I_{\mathfrak{m}}$ into $\boldsymbol{C}^\times$ such that $\lambda((\alpha)) = \alpha^\nu$ for every $\alpha \in W_{\mathfrak{m}}$. Such a λ is called a Grössen-character of K defined modulo $\mathfrak{m}$. Obviously, $\Lambda_{\mathfrak{m}}^{\nu}$ is not empty if and only if the following condition is satisfied:

(2) *If ζ is a root of unity in K and $\zeta \equiv 1$ mod $\mathfrak{m}$, then $\zeta^\nu = 1$.*

For each $\lambda \in \Lambda_{\mathfrak{m}}^{\nu}$, there is a unique divisor $\mathfrak{c}$ of $\mathfrak{m}$ such that: (i) λ is the restriction of an element of $\Lambda_{\mathfrak{c}}^{\nu}$; (ii) no proper divisor of $\mathfrak{c}$ has the property (i). Then $\mathfrak{c}$ is called the *conductor* of λ. We call λ *primitive* if $\mathfrak{m}$ is the conductor of λ.

We can associate with every $\lambda \in \Lambda_{\mathfrak{m}}^{\nu}$ an L-function $L(s, \lambda)$ and a function $f_\lambda(z)$ on H by

$$L(s, \lambda) = \sum_{\mathfrak{x}} \lambda(\mathfrak{x}) N(\mathfrak{x})^{-s} \qquad (s \in \boldsymbol{C}),$$

$$f_\lambda(z) = \sum_{\mathfrak{x}} \lambda(\mathfrak{x}) e^{2\pi i N(\mathfrak{x}) z} \qquad (z \in H),$$

where each sum is taken over all integral ideals $\mathfrak{x}$ in $I_{\mathfrak{m}}$. Under the assumption (2), let $V_{\mathfrak{m}}^{\nu}$ be the vector space spanned by the f_{λ} over $\boldsymbol{C}$ for all $\lambda \in \Lambda_{\mathfrak{m}}^{\nu}$. For λ, $\mu \in \Lambda_{\mathfrak{m}}^{\nu}$, we see easily that $f_{\lambda} = f_{\mu}$ if and only if $\lambda = \mu$. Moreover, we shall see later that the f_{λ} for $\lambda \in \Lambda_{\mathfrak{m}}^{\nu}$ are linearly independent over $\boldsymbol{C}$. Therefore $V_{\mathfrak{m}}^{\nu}$ is of dimension $[I_{\mathfrak{m}} : P_{\mathfrak{m}}]$.

Fix any set S of representatives for $I_{\mathfrak{m}}$ modulo $P_{\mathfrak{m}}$, whose members are prime to $\mathfrak{m}$, and put, for each $\mathfrak{a} \in S$,

$$g_{\mathfrak{a}}(z) = \textstyle\sum_{(\alpha)} \alpha^{\nu} \cdot e^{2\pi i N(\alpha) z / N(\mathfrak{a})}, \tag{3}$$

where the sum is taken over all ideals (α) such that $\alpha \in W_{\mathfrak{m}} \cap \mathfrak{a}$. We have then

$$f_{\lambda} = \textstyle\sum_{\mathfrak{a} \in S} \lambda(\mathfrak{a})^{-1} \cdot g_{\mathfrak{a}},$$

so that the functions $g_{\mathfrak{a}}$, for $\mathfrak{a} \in S$, form a basis of $V_{\mathfrak{m}}^{\nu}$ over $\boldsymbol{C}$. Hecke [2] proved that $g_{\mathfrak{a}}$ is a cusp form belonging to a certain congruence subgroup. We can state this fact in the following form.

LEMMA 3. *Let $-D$ be the discriminant of K, and let $\lambda \in \Lambda_{\mathfrak{m}}^{\nu}$, $M = D \cdot N(\mathfrak{m})$. Then f_{λ} is an element of $S_{\nu+1}(M, \varepsilon)$, where ε is the character of $(\boldsymbol{Z}/M\boldsymbol{Z})^{\times}$ defined by*

$$\varepsilon(a) = \left(\frac{-D}{a}\right) \cdot \frac{\lambda((a))}{a^{\nu}} \qquad (a \in \boldsymbol{Z},\ (a, M) = 1).$$

Proof. If λ is primitive, our assertion can be proved by examining the functional equations of $L(s, \lambda)$ and

$$L(s, \lambda, \chi) = \textstyle\sum_{\mathfrak{x}} \lambda(\mathfrak{x}) \chi(N(\mathfrak{x})) N(\mathfrak{x})^{-s}$$

with primitive characters χ of $(\boldsymbol{Z}/p\boldsymbol{Z})^{\times}$ for all rational primes p not dividing M, and applying the principle of Weil [9]. Although [9, Satz 2] is concerned with $S_k(M, \varepsilon)$ for real characters ε, the result can easily be extended to the case of an arbitrary character ε. Let us now prove the general case by induction on $N(\mathfrak{c}^{-1}\mathfrak{m})$, where $\mathfrak{c}$ is the conductor of λ. Suppose that $\mathfrak{c}^{-1}\mathfrak{m}$ has a prime factor $\mathfrak{p}$, and put $\mathfrak{n} = \mathfrak{p}^{-1}\mathfrak{m}$. Let μ be the element of $\Lambda_{\mathfrak{n}}^{\nu}$ whose restriction to $\Lambda_{\mathfrak{m}}^{\nu}$ is λ. By the induction assumption, f_{μ} belongs to $S_{\nu+1}(D \cdot N(\mathfrak{n}), \varepsilon)$. Put $q = N(\mathfrak{p})$. Then

$$f_{\mu}(qz) = \textstyle\sum_{(\mathfrak{x}, \mathfrak{n})=1} \mu(\mathfrak{x}) e^{2\pi i N(\mathfrak{p}\mathfrak{x}) z},$$

hence

$$f_\mu(z) - \mu(\mathfrak{p}) f_\mu(qz) = \textstyle\sum_{(\mathfrak{x},\mathfrak{m})=1} \mu(\mathfrak{x}) e^{2\pi i N(\mathfrak{x})z} = f_\lambda(z), \tag{4}$$

where we understand that $\mu(\mathfrak{p}) = 0$ if $\mathfrak{p}$ divides $\mathfrak{n}$. Since we have

$$\begin{bmatrix} q & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ qc & d \end{bmatrix} = \begin{bmatrix} a & qb \\ c & d \end{bmatrix} \begin{bmatrix} q & 0 \\ 0 & 1 \end{bmatrix},$$

it can easily be verified that $f_\mu(qz) \in S_{\nu+1}(q \cdot D \cdot N(\mathfrak{n}), \varepsilon)$. Therefore the equality (4) implies that $f_\lambda \in S_{\nu+1}(q \cdot D \cdot N(\mathfrak{n}), \varepsilon)$, q.e.d.

The symbols λ, M, and ε being as above, put $f_\lambda(z) = \sum_n a_n e^{2\pi i n z}$. Then the L-function $L(s, \lambda)$ has an Euler product:

$$L(s, \lambda) = \Pi_p (1 - a_p p^{-s} + \varepsilon(p) p^{\nu - 2s})^{-1},$$

where the product is taken over all rational primes p ; $\varepsilon(p) = 0$ for every prime factor p of M. Therefore, by Hecke [4, II, Satz 42] (see also [7, Th. 3.43]), f_λ must be a common eigen-function of all Hecke operators. Thus the functions f_λ, for $\lambda \in \Lambda^\nu_\mathfrak{m}$, are distinct eigen-functions whose first Fourier coefficients are 1. Therefore they are linearly independent over $\boldsymbol{C}$.

5. Let us now consider a projective non-singular curve C_M biregularly isomorphic to the compactification of the quotient $H/\Gamma_1(M)$ for a positive integer M. There is a "standard" way to define C_M rational over $\boldsymbol{Q}$, up to biregular isomorphisms over $\boldsymbol{Q}$. (One can define, for instance, the function field of C_M to be the field of all $\Gamma_1(M)$-invariant modular functions whose Fourier expansions with respect to $e^{2\pi i z}$ have rational coefficients. See also [5], [7, §6.7, §6.3].) Then the jacobian variety $\mathrm{Jac}(C_M)$ of C_M can naturally be defined over $\boldsymbol{Q}$. We denote by τ_n the endomorphism of $\mathrm{Jac}(C_M)$ corresponding to the Hecke operator of degree n.

Let $\lambda \in \Lambda^1_\mathfrak{m}$, $M = D \cdot N(\mathfrak{m})$, and $f_\lambda(z) = \sum_n a_n e^{2\pi i n z}$. Further let k_λ denote the field generated over $\boldsymbol{Q}$ by the numbers a_n for all n. Since f_λ is a common eigen-function of all Hecke operators, we obtain, by virtue of [7, Th. 7.14], a couple $(A_\lambda, \theta_\lambda)$ satisfying the following three conditions:

(i) *A_λ is an abelian subvariety of $\mathrm{Jac}(C_M)$ of dimension $[k_\lambda : \boldsymbol{Q}]$.*

(ii) *θ_λ is an isomorphism of k_λ into $\mathrm{End}_{\boldsymbol{Q}}(A_\lambda)$ such that $\theta_\lambda(a_n)$ is the restriction of τ_n to A_λ for all n.*

(iii) *A_λ is rational over $\boldsymbol{Q}$.*

Moreover, $(A_\lambda, \theta_\lambda)$ is unique for f_λ under the conditions (i) and (ii).

For an automorphism σ of the algebraic closure of $\boldsymbol{Q}$, we define an element λ_σ of $\Lambda^1_{\mathfrak{m}^\sigma}$ by $\lambda_\sigma(\mathfrak{x}) = \lambda(\mathfrak{x}^\sigma)^\sigma$. If $f_\lambda(z) = \sum_n a_n e^{2\pi i n z}$, we see that $f_{\lambda_\sigma}(z) = \sum_n a_n^\sigma e^{2\pi i n z}$. Now identify the tangent space of Jac(C_M) at the origin with the space of all cusp forms of weight 2 with respect to $\Gamma_1(M)$. Then the proof of [7, Th. 7.14] shows that the tangent space of A_λ at the origin can be identified with the vector space spanned by all distinct f_{λ_σ}. Therefore our result mentioned at the beginning of this paper follows from the following

THEOREM 1. *The abelian variety A_λ is isogenous to a product of copies of an elliptic curve whose endomorphism algebra is isomorphic to K.*

Proof. (I) First let us assume that $\mathfrak{m}$ is divisible by $\sqrt{-D}$, and $\mathfrak{m}=\mathfrak{m}^\rho$, where ρ denotes the complex conjugation. Put

$$\Gamma = \Gamma_1(M), \quad \delta = \begin{bmatrix} 1 & 1/d \\ 0 & 1 \end{bmatrix}.$$

We can let $\Gamma\delta\Gamma$ act on the vector space of cusp forms with respect to Γ (see [7, §3.4]). Denote the action by $[\Gamma\delta\Gamma]_2$. Take a disjoint coset decomposition $\Gamma\delta\Gamma = \cup_{i=1}^{\kappa}\Gamma\delta\gamma_i$ with $\gamma_i \in \Gamma$. Let $g_\mathfrak{a}$ be as in (3). Then, by definition,

$$g_\mathfrak{a}|[\Gamma\delta\Gamma]_2 = \cup_{i=1}^{\kappa} g_\mathfrak{a}|[\delta\gamma_i]_2.$$

If $\alpha, \beta \in W_\mathfrak{m} \cap \mathfrak{a}$, we have

$$N(\alpha)/N(\mathfrak{a}) \equiv N(\beta)/N(\mathfrak{a}) \bmod (D),$$

so that, if $\zeta_D = e^{2\pi i/D}$,

$$g_\mathfrak{a}|[\delta]_2 = \zeta_D^{N(\alpha)/N(\mathfrak{a})} \cdot g_\mathfrak{a}$$

with any fixed α contained in $W_\mathfrak{m} \cap \mathfrak{a}$. Therefore

$$g_\mathfrak{a}|[\Gamma\delta\Gamma]_2 = \kappa \cdot \zeta_D^{N(\alpha)/N(\mathfrak{a})} \cdot g_\mathfrak{a}. \tag{5}$$

Thus $[\Gamma\delta\Gamma]_2$ maps $V^1_\mathfrak{m}$ onto itself. Let A' be the abelian subvariety of Jac(C_M) generated by the A_λ for all $\lambda \in \Lambda^1_\mathfrak{m}$. Since $\mathfrak{m} = \mathfrak{m}^\rho$, $V^1_\mathfrak{m}$ can be identified with the tangent space of A' at the origin. Let ω denote the endomorphism of A' obtained from $[\Gamma\delta\Gamma]_2$. The relation (5) shows that the representation of ω on the tangent space has characteristic roots $\kappa \cdot \zeta_D^{N(\alpha)/N(\mathfrak{a})}$, where α must be fixed for each $\mathfrak{a} \in S$. Put $\chi(r) = \left(\frac{-D}{r}\right)$. Then we see that

$N(\alpha)/N(\mathfrak{a})$ is prime to D, and $\chi(N(\alpha)/N(\mathfrak{a}))=1$. We can define an embedding h of $\boldsymbol{Q}(\zeta_D)$ into $\mathrm{End}_{\boldsymbol{Q}}(A')$ by $h(\zeta_D)=\kappa^{-1}\omega$. If σ is an automorphism of $\boldsymbol{Q}(\zeta_D)$ such that $\zeta_D^\sigma=\zeta_D^r$ with $\chi(r)=1$, then the restriction of σ to K is the identity map. Therefore applying Lemma 1 to A', we see that A' is isogenous to a product of copies of an elliptic curve with K as its endomorphism algebra.

(II) Next assume that λ is primitive, and put $\mathfrak{m}'=\mathfrak{m}\mathfrak{m}^\rho\cdot(\sqrt{-D})$, $M'=N(\mathfrak{m}')\cdot D$, $\eta_u=\begin{bmatrix} M & u \\ 0 & M\end{bmatrix}$ for $u\in\boldsymbol{Z}$. Then $M'=M^2$ and $\mathfrak{m}'=\mathfrak{m}'^\rho$. Define, as in the proof of Lemma 2, rational numbers x_u so that

$$\sum_{u=0}^{M-1} x_u\zeta_M^{un}=\begin{cases}1 & \text{if } (n,M)=1,\\ 0 & \text{otherwise,}\end{cases}$$

where $\zeta_M=e^{2\pi i/M}$. Take a positive integer t so that tx_u is an integer for every u. Put $\xi=\sum_{u=0}^{M-1} tx_u\cdot[\eta_u]_2$. For every

$$f(z)=\sum_n a_n e^{2\pi i n z}\in S_2(M,\varepsilon),$$

we have, by Lemma 2 and its proof,

$$f|\xi=t\cdot\sum_{(n,M)=1}a_n e^{2\pi i n z}\in S_2(M',\varepsilon).$$

Especially $f_\lambda|\xi=t\cdot f_\mu$ if μ is the restriction of λ to $I_{\mathfrak{m}'}$. Let V_λ be the subspace of $V_{\mathfrak{m}}^1+V_{\mathfrak{m}^\rho}^1$ spanned by all distinct f_{λ_σ} with automorphisms σ of the algebraic closure of $\boldsymbol{Q}$. Since λ is primitive, we see that ξ maps V_λ *injectively* into $V_{\mathfrak{m}'}^1$. (This is not necessarily true if λ is not primitive.) Since $\eta_u\cdot\Gamma_1(M')\eta_u^{-1}\subset\Gamma_1(M)$, the action $[\eta_u]_2$ defines a homomorphism of $\mathrm{Jac}(C_M)$ into $\mathrm{Jac}(C_{M'})$, hence ξ defines a homomorphism ξ^* of $\mathrm{Jac}(C_M)$ into $\mathrm{Jac}(C_{M'})$. Then the restriction of ξ^* to A_λ is an isogeny onto an abelian subvariety of A'', where A'' is the sum of A_μ for all $\mu\in\Lambda_{\mathfrak{m}'}^1$. By the result in the case (I), A'' is isogenous to a product of copies of an elliptic curve with K as its endomorphism algebra. Therefore A_λ has the same property.

(III) Finally let us consider the general case with no assumption on $\mathfrak{m}$. Let $\mathfrak{c}$ be the conductor of λ. To prove our assertion by induction on $N(\mathfrak{c}^{-1}\mathfrak{m})$, suppose that $\mathfrak{c}^{-1}\mathfrak{m}$ has a prime factor $\mathfrak{p}$, and put $\mathfrak{n}=\mathfrak{p}^{-1}\mathfrak{m}$, $q=N(\mathfrak{p})$, $L=q^{-1}M$, $\beta=\begin{bmatrix} q & 0 \\ 0 & 1\end{bmatrix}$. Since $\beta\Gamma_1(M)\beta^{-1}\subset\Gamma_1(L)$, $[\beta]_2$ defines an endomorphism ψ of $\mathrm{Jac}(C_L)$ into $\mathrm{Jac}(C_M)$. Let φ be the natural map of $\mathrm{Jac}(C_L)$ into $\mathrm{Jac}(C_M)$ corresponding to $[1]_2$. If μ is the element of $\Lambda_{\mathfrak{n}}^1$ whose restriction to $I_{\mathfrak{m}}$ is λ, we have $f_{\lambda_\sigma}=f_{\mu_\sigma}-s\cdot f_{\mu_\sigma}|[\beta]_2$ with a constant s, by virtue of (4), for every automorphism σ of the algebraic closure of $\boldsymbol{Q}$. This shows that $A_\lambda \subset \varphi(A_\mu) + \psi(A_\mu)$. Therefore our assertion about A_λ follows from that about A_μ, which is ensured by induction.

Remark. We have thus shown that the center $\mathfrak{Z}$ of $\mathrm{End}_{\boldsymbol{Q}}(A_\lambda)$ is isomorphic to K. It should be noted here that $\mathfrak{Z}$ *is not contained in* $\theta_\lambda(k_\lambda)$. This follows from either of the following two facts:

(i) The elements of $\theta_\lambda(k_\lambda) \cap \mathrm{End}(A_\lambda)$ are rational over $\boldsymbol{Q}$ (see [7, pp. 182–183]), while K is the smallest field of definition for any generator of $\mathfrak{Z}$ contained in $\mathrm{End}(A_\lambda)$.

(ii) The representation of k_λ, through θ_λ, on the tangent space of A_λ at the origin is equivalent to a regular representation over $\boldsymbol{Q}$.

6. Let E be an elliptic curve defined over $\boldsymbol{Q}$ such that $\mathrm{End}_{\boldsymbol{Q}}(E)$ is isomorphic to K. (This can happen if and only if the class number of K is one.) By the result of Deuring [1], the zeta-function of E over $\boldsymbol{Q}$ coincides exactly with $L(s, \lambda)$ with some primitive Grössen-character λ of K. Let $\mathfrak{c}$ be the conductor of λ, and $M = D \cdot N(\mathfrak{c})$. Then we obtain an element f_λ of $S_2(M, \varepsilon)$ as before. If $f_\lambda(z) = \sum_n a_n e^{2\pi i n z}$, we have

$$L(s, \lambda) = \Pi_p (1 - a_p p^{-s} + \varepsilon(p) p^{1-2s})^{-1}. \tag{6}$$

Since E is defined over $\boldsymbol{Q}$, we see that $a_n \in \boldsymbol{Q}$, and ε is the trivial character, so that f_λ is a cusp form invariant under $\Gamma_0(M)$. Therefore we can take $\mathrm{Jac}(H/\Gamma_0(M))$ (of course defined over $\boldsymbol{Q}$) instead of $\mathrm{Jac}(H/\Gamma_1(M))$ in the above discussion, and define A_λ as an abelian subvariety of $\mathrm{Jac}(H/\Gamma_0(M))$. Since $k_\lambda = \boldsymbol{Q}$, A_λ is an elliptic curve defined over $\boldsymbol{Q}$.

THEOREM 2. *The elliptic curve A_λ is isogenous to E over $\boldsymbol{Q}$.*

Proof. By [7, Th. 7.15], the zeta-function of A_λ over $\boldsymbol{Q}$ coincides, up to finitely many Euler factors, with (6). On the other hand, by Theorem 1, $\mathrm{End}_{\boldsymbol{Q}}(A_\lambda)$ is isomorphic to K, so that the zeta-function of A_λ over $\boldsymbol{Q}$ is $L(s, \mu)$ with a primitive Grössen-character μ of K. Thus $L(s, \lambda)$ coincides with $L(s, \mu)$ up to finitely many Euler factors. It follows that $\lambda(\mathfrak{p}) = \mu(\mathfrak{p})$ or $\lambda(\mathfrak{p}) = \mu(\mathfrak{p}^\rho)$ for almost all prime ideals $\mathfrak{p}$ in K. If $\mathfrak{m}$ is a common multiple of the conductors of λ and μ, we have $\lambda((\alpha)) = \alpha = \mu((\alpha))$ for $\alpha \in K$, $\alpha \equiv 1 \mod^\times \mathfrak{m}$. Therefore we must have $\lambda(\mathfrak{p}) = \mu(\mathfrak{p})$, so that $\lambda = \mu$. Thus E and

A_λ determine the same Grössen-character of K. By [8, Th. 8], they must be isogenous over $\boldsymbol{Q}$.

It should be noted that E has good reduction modulo a rational prime p if and only if p does not divide $D \cdot N(\mathfrak{c})$. This is due to Deuring [1, IV] (see also [8] for a simpler proof).

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