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CHARACTERISTIC CLASSES FOR PL MICRO BUNDLES

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§ 0. Introduction.

Let $BSPL$ be the classifying space of the stable oriented PL micro bundles. The purpose of this paper is to determine $H_*(BSPL : Z_p)$ as a Hopf algebra over Z_p, where p is an odd prime number. In this chapter, p is always an odd prime number.

The conclusions are as follows.

THEOREM 2–22. *As a Hopf algebra over* Z_p, $H_*(BSPL : Z_p) = Z_p[\bar{\bar{b}}_1, \bar{\bar{b}}_2, \cdots] \otimes Z_p[\sigma(\bar{\bar{x}}_I)] \otimes \Lambda(\sigma(\bar{\bar{x}}_J))$. $\Delta(\bar{\bar{b}}_j) = \sum_{i=0}^{j} \bar{\bar{b}}_i \otimes \bar{\bar{b}}_j$, $b_0 = 1$, $\sigma(\bar{\bar{x}}_I)$, $\sigma(\bar{\bar{x}}_J)$ *are primitive.*

THEOREM 3–1. *As a Hopf algebra over* $Z[1/2]$,

i) $H^*(BSPL : Z[1/2])/_{Torsion} = Z[1/2][R_1, R_2, \cdots]$

ii) $\Delta R_j = \sum_{i=0}^{j} R_i \otimes R_{j-i}$, $R_0 = 1$. $\deg R_j = 4j$.

iii) *In* $H^*(BSPL : Q) = Q[p_1, p_2, \cdots]$, R_j *are expressed as follows.*

$$R_j = 2^{a_j}(2^{2j-1} - 1)\, Num(B_j/4j) \cdot p_j + dec, \text{ for some } a_j \in Z.$$

Let $MSPL$ denote the spectrum defined by the Thom complex of the universal PL micro bundle over $BSPL(n)$, and $A = A_p$ denote the mod p Steenrod algebra. And $\phi : A \to H^*(MSPL : Z_p)$ is defined by $\phi(a) = a(u)$, where $u \in H^0(MSPL : Z_p)$ is the Thom class.

THEOREM 4–1. *The kernel of* ϕ *is* $A(\underline{\underline{Q}}_0, \underline{\underline{Q}}_1)$, *the left ideal generated by Milnor elements* $\underline{\underline{Q}}_0$, $\underline{\underline{Q}}_1$.

This is the conjecture of Peterson [12].

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The method is to compute the Serre spectral sequence associated to the fibering $F/PL \to BSPL \to BSF$. The structure of $H_*(BSF: Z_p)$ is determined in [9] and [16]. The homotopy type of F/PL is the consequence of the deep results of Sullivan [15]. In §1 we study the H space structure of F/PL and the inclusion map $SF \to F/PL$. The main tool is the result of Sullivan and its extention that tells the existence of the KO_P^* theory Thom classes for oriented PL disk bundle.

PROPOSITION 1–4. *For a oriented PL disk bundle $\pi: E \to X$ over a finite CW complex of fiber dim m. Then there is a Thom class $u(\pi) \in KO^m(E, \partial E)_P$ with the following properties.*

i) *functorial*

ii) $\varphi_H^{-1} ph u(\pi) = L(\pi)^{-1}$.

iii) $u(\pi \oplus 1) = \sigma u(\pi)$.

iv) *Multiplicative mod Torsion i.e* $u(\pi_1 \oplus \pi_2) = u(\pi_1) \cdot u(\pi_2)$. *mod torsions.*

The proof of this is in §6.

§1. *H* space structure on *F/PL*.

1–1. Let $F/PL(N)$ denote the classifying space of PL disk bundle of fiber dim N with homotopy trivialization. And F/PL denote the limit space $\varinjlim F/PL(N)$. Denote by BO, the classifying space of stable real vector bundle. F/PL and BO are homotopy commutative H-spaces defined by Whitney products. BO_P denotes the space obtained by localizing BO at odd primes P i.e. the space which represents the functor $[\ , BO] \otimes_Z Z[1/2]$. Let C_P denote the class of abelian groups consisting of 2-torsion group, i.e abelian group G with $G \otimes_Z Z[1/2] = 0$. Then the following proposition is due to Sullivan [15].

PROPOSITION 1–1. *There exists a continuous map $\sigma: F/PL \to BO_P$, with the following properties.*

i) *σ is C_P homotopy equivalence.*

ii) $\sigma^{**}(ph_1 + ph_2 + \cdots) = \frac{1}{8}(L_1 + L_2 + \cdots) \in H^{**}(F/PL, Q)$, *where $ph = 1 + ph_1 + ph_2 + \cdots \in H^{**}(BO_P, Q)$ is the Pontrjagin character and $L = 1 + L_1 + L_2 + \cdots \in H^{**}(F/PL, Q)$ is L-polynomial of Hirzebruch.*

iii) *The map σ is uniquely determined by the property* ii) *up to homotopy.*

Since the C_P homotopy equivalence σ is not a H space map. We introduce another H space structure $\mu_{\otimes}$ on BO. $\mu_{\otimes} : BO \times BO \to BO$ is defined by the following diagram.

$$\text{(1-1)} \qquad \mu_{\otimes} : BO \times BO \xrightarrow{\Delta \times \Delta} (BO \times BO) \times (BO \times BO) \xrightarrow{id \times T \times id}$$
$$BO \times BO \times BO \times BO \xrightarrow{\mu_{\oplus} \times \mu_{\wedge}} BO \times BO \xrightarrow{\mu_{\oplus}} BO.$$

where $\mu_{\wedge} : BO \times BO \to BO$ denotes the map representing $(\xi_m - m) \cdot (\xi_n - n)$ in $KO^0(BO(m) \times BO(n))$, where $\xi_m \to BO(m)$, and $\xi_n \to BO(n)$ denote the universal bundles. Then the H-space $(BO, \mu_{\otimes})$ is a homotopy commutative H-space. We denote this H space by $BO_{\otimes}$ simply. Denote by $BO_{\otimes P}$, the localizing space of $BO_{\otimes}$ at odd primes P. Then identity map $i : BO \to BO_{\otimes}$ can be uniquely extended to the map $i_P : BO_P \to BO_{\otimes P}$, and i_P is a homotopy equivalence.

Define a continuous map $\bar{\sigma} : F/PL \to BO_{\otimes P}$ by the following diagram.

$$\text{(1-2)} \qquad \bar{\sigma} : F/PL \xrightarrow{\sigma} BO_P \xrightarrow{8} BO_P \xrightarrow{i_P} BO_{\otimes P}.$$

PROPOSITION 1-2. *The C_P homotopy equivalence $\bar{\sigma}$ is a H space map, and* $\bar{\sigma}^{**}(1 + ph_1 + ph_2 + \cdots) = 1 + L_1 + L_2 + \cdots \in H^{**}(F/PL\,;\,Q)$.

Proof. Since $\bar{\sigma}^{**}(1 + ph_1 + ph_2 + \cdots) = 1 + L_1 + L_2 + \cdots$ follows easily from proposition 1-1, ii) and (1-2), it is sufficient to prove that the following diagram is homotopy commutative.

$$\begin{array}{ccc} F/PL \times F/PL & \xrightarrow{\bar{\sigma} \times \bar{\sigma}} & BO_{\otimes P} \times BO_{\otimes P} \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu_{\otimes P}} \\ F/PL & \xrightarrow{\bar{\sigma}} & BO_{\otimes P} \end{array}$$

But by proposition 1-1, any map $f : F/PL \times F/PL \to BO_{\otimes P}$ is uniquely determined by $f^{**}(1 + ph_1 + ph_2 + \cdots) \in H^{**}(F/PL \times F/PL\,;\,Q)$. On the other hand, $\mu^{**} \cdot \sigma^{**}(1 + ph_1 + ph_2 + \cdots) = \mu^{**}(1 + L_1 + L_2 + \cdots) = (1 + L_1 + L_2 + \cdots) \otimes (1 + L_1 + L_2 + \cdots)$. And $(\bar{\sigma} \times \bar{\sigma})^{**}(\mu_{\otimes P})^{**}(1 + ph_1 + ph_2 + \cdots) = (\bar{\sigma} \times \bar{\sigma})^{**} \times (ph \otimes ph) = (1 + L_1 + \cdots) \otimes (1 + L_1 + \cdots)$. This showes the proposition.

1-2. Let $BO\langle 8N \rangle$ denote the space obtained by killing the homotopy group $\pi_i(BO)$, $i < 8N$. Let $f_N : S^{8N} \to BO\langle 8N \rangle$ be the canonical generator of $\pi_{8N}(BO\langle 8N \rangle) \cong Z$. Then by Bott periodicity, the map $S^{8(N-1)} \xrightarrow{i} \Omega^8 S^{8N} \xrightarrow{\Omega^8 f_N}$

$\Omega^8 BO\langle 8N\rangle = BO\langle 8(N-1)\rangle$ coincide with f_{N-1}. So we can take a limit and obtain a map.

$$g = \Omega^\infty f_\infty : \varinjlim \Omega^{8N} S^{8N} = QS^0 \to \varinjlim \Omega^{8N} BO\langle 8N\rangle = Z\times BO. \tag{1-3}$$

The spaces $BO\langle 8N\rangle$ have product $\mu_{M,N}$.

$$\mu_{M,N} : BO\langle 8M\rangle \times BO\langle 8N\rangle \to BO\langle 8(M+N)\rangle. \tag{1-4}$$

These products define product μ on $\Omega^{8N} BO\langle 8N\rangle = Z\times BO$, i.e. $\mu : \Omega^{8M}\times BO\langle 8M\rangle \times \Omega^{8N} BO\langle 8N\rangle \to \Omega^{8(M+N)} BO\langle (M+N)\rangle$. By Bott periodicity, the following diagram is homotopy commutative.

$$\begin{array}{ccc}
\Omega^{8N} BO\langle 8M\rangle \times \Omega^{8N} BO\langle 8N\rangle & \longrightarrow & \Omega^{8(M+N)} BO\langle 8(M+N)\rangle \\
\downarrow & & \downarrow \\
\Omega^{8(M+1)} BO\langle 8(M+1)\rangle \times \Omega^{8(N+1)} BO\langle 8(N+1)\rangle & \longrightarrow & \Omega^{8(M+N+2)} BO\langle 8(M+N+2)\rangle
\end{array}$$

And the reduced join product $\mu_\wedge : \Omega^{8M} S^{8M} \times \Omega^{8N} S^{8N} \to \Omega^{8(M+N)} S^{8(M+N)}$ is compatible with the product $\Omega^{8M} BO\langle 8M\rangle \times \Omega^{8M} BO\langle 8N\rangle \to \Omega^{8(M+N)} BO\langle 8(M+N)\rangle$. Passing to limit we obtain a product $\mu_\wedge$ on $QS^0 = \lim \Omega^{8N} S^{8N}$. And we have the following commutative diagram.

$$\begin{array}{ccc}
QS^0 \times QS^0 & \xrightarrow{g\times g} & (Z\times BO)\times(Z\times BO) \\
\downarrow{\scriptstyle \mu_\wedge} & & \downarrow{\scriptstyle \mu} \\
QS^0 & \xrightarrow[g]{} & Z\times BO
\end{array} \tag{1-5}$$

Consider the 1 component $Q_1 S^0$ of QS^0, then $\mu_\wedge : Q_1 S^0 \times Q_1 S^0 \to Q_1 S^0 \subset QS^0$ is the H space SF, where $SF = \varinjlim SG(n)$, $SG(n) = \{f : S^{n-1} \to S^{n-1}$, degree $1\}$. And it is easy to show that 1 component $1\times BO$ of $Z\times BO$ with product $\mu : (1\times BO)\times(1\times BO) \to 1\times BO$ is the H space $(BO_\otimes, \mu_\otimes)$ defined in (1-1).

So that we have a H map $g_1 : SF = Q_1 S^0 \to 1\times BO = BO_\otimes$.

PROPOSITION 1-3. *The map* $g_1 : SF \to BO_\otimes \to BO_{\otimes P}$, *and* $\bar{\sigma}\cdot k$; $SF \xrightarrow{k} F/PL \xrightarrow{\bar{\sigma}} BO_{\otimes P}$ *coincide.*

Before proving this proposition, we prepare some results.

1-3. Let $KO^*(\ )$ denote 8 graded cohomology theory defined by using Grothendieck group of real vector bundle. Construct a 4 graded cohomology theory $KO^*(\ )_P$ by $KO^q(\ )_P = KO^q(\ ) \otimes_Z Z[1/2]$. Consider the generator $\eta_4 \in$

$KO^{-4}(S^0) \cong Z$, then $\eta_4^2 = 4\eta_8 \in KO^{-8}(S^0)$, $\eta_8 \in KO^{-8}(S^0) \cong Z$, generator. $\bar{\eta}_4$ is by definition $\bar{\eta}_4 = \frac{1}{2}\eta_4 \in KO^{-4}(S^0)_P$. And define Bott map $\beta : KO^q(X, A)_P \xrightarrow{\cong} KO^{q-4}(X, A)_P$ by the following.

$$\beta : KO^q(X, A)_P \xrightarrow{\otimes \bar{\eta}_4} KO^q(X, A)_P \otimes KO^{-4}(S^0)_P \xrightarrow{\wedge} KO^{q-4}(X, A)_P. \tag{1–6}$$

This Bott map makes $KO^*(\ )_P$, 4 graded cohomology theory.

Let $\pi : E \to X$ be a oriented PL disk bundle over finite complex X of fiber dim m. Then we can define a fundamental Thom class $u(\pi) \in KO^m(E, \partial E)_P$ as the following proposition.

PROPOSITION 1–4. *There is a fundamental Thom class* $u(\pi) \in KO^m(E, \partial E)_P$ *with following properties.*

i) *functorial i.e. for* $f : Y \to X$, $u(f!\pi) = f!(u(\pi))$.

ii) $\varphi_H^{-1} phu(\pi) = L(\pi)^{-1} \in H^*(X, Q)$, *where* φ_H *is Thom isomorphism, and* $L(\pi)$ *is the L polynomial of Hirzebruch for* $\pi : E \to X$.

iii) $u(\pi \oplus 1) = \sigma(u(\pi))$, *where* $\sigma : KO^m(E, \partial E)_P \xrightarrow{\sigma} KO^{m+1}((E/\partial E) \wedge S^1)_P = KO^{m+1}(E \oplus 1, \partial(E \oplus 1))_P$ *is suspension isomorphism.*

iv) *Multiplicative mod torsion i.e* $u(\pi_1 \oplus \pi_2) = u(\pi_1) \cdot u(\pi_2)$ *mod torsion elements, where* $\pi_1 : E_1 \to X_1$, *and* $\pi_2 : E_2 \to X_2$.

We shall prove this proposition in the appendix.

1–4. Now we prove proposition 1–3. At first we analyse the map $g_1 : Q_1S^0 \to BO_\otimes$. Consider the following mapping $t : SG(N) \times (D^N, \partial D^N) \to (D^N, \partial D^N)$ defined by $t(f, x) = cf(x)$, where $cf : (D^N, \partial D^N) \to (D^N, \partial D^N)$ be a map defined by cone of $f : \partial D^N = S^{N-1} \to \partial D^N = S^{N-1}$. Consider the case $N = 8M$. And consider the canonical generator $\eta_{8M} \in KO^{8M}(D^{8M}, \partial D^{8M}) \cong Z$, then $t^*(\eta_{8M}) \in KO^{8M}(SG(8M) \times (D^{8M}, \partial D^{8M})) \cong KO^0(SG(8M)) \otimes_Z KO^{8M}(D^{8M}, \partial D^{8M})$. So that there is unique element $l_{8M} \in KO^0(SG(8M))$ such that $l_{8M} \otimes \eta_{8M} = t^*(\eta_{8M})$. It is easy to show that for $i : SG(8M) \to SG(8(M+1))$, $i^*(l_{8(M+1)}) = l_{8M}$. And $\varepsilon(l_{8M}) = 1$, where $\varepsilon : KO^0(SG(8M)) \to KO^0(p.t) \cong Z$ be the augmentation. So passing to the limit, we obtain $l \in KO^0(SG) = KO^0(Q_1S^0)$. And since $\varepsilon(l) = 1$, l is represented by a map $l : SG = Q_1S^0 \to 1 \times BO = BO_\otimes \subsetneq Z \times BO$.

LEMMA 1–5. *The map* l *coincides with* $g_1 : Q_1S^0 \to BO_\otimes$ *defined in* 1–2.

It is easy to prove this lemma so we omit its proof.

Proof of proposition 1–3. Let $\pi : E \to X$ be a PL disk bundle of fiber dimension $8N$ over a finite complex X with homotopy trivialization $t : (E, \partial E) \to (D^{8N}, \partial D^{8N})$. Consider the element $t^*(\eta_{8N}) \in KO^{8N}(E, \partial E)_P$. By proposition 1–4, there is a Thom isomorphism $\varphi_{KO_P} : KO^0(X)_P \to KO^{8N}(E, \partial E)_P$ defined by $\varphi_{KO_P}(x) = i^*(x) \cdot u(\pi)$, $i : X \to E$. Then $\bar{l}(E)$ is by definition $\varphi_{KO_P}^{-1}(t^*(\eta_{8N})) \in KO^0(X)_P$. It is easy to see $\bar{l}(E \oplus 8) = \bar{l}(E)$. Since $KO^0(F/PL(8N))_P = \varprojlim_\alpha KO^0(X_\alpha)_P$, where X_α runs through all finite subcomplexes of $F/PL(8N)$, the universal bundle $\pi_{8N} : E_{8N} \to F/PL(8N)$, with $t_{8N} : (E_{8N}, \partial E_{8N}) \to (D^{8N}, \partial D^{8N})$ defines the element $\bar{l}(E_{8N}) \in KO^0(F/PL(8N))_P$. It is easy to see $i^*(\bar{l}(E_{8(N+1)})) = \bar{l}(E_{8N})$, where $i : F/PL(8N) \to F/PL(8(N+1))$. Passing to limit, we obtain the element $\bar{l} \in KO^0(F/PL)_P$. The natural inclusion $k_{8N} : SG(8N) \to F/PL(8N)$ is defined by the classifying map for the F/PL bundle over $SG(8N)$ defined by $t : SG(8N) \times (D^{8N}, \partial D^{8N}) \to (D^{8N}, \partial D^{8N})$. Since the fundamental Thom class of this bundle is $1 \otimes \eta_{8N} \in KO^{8N}(SG(8N) \times (D^{8N}, \partial D^{8N}))_P = KO^0(SG(8N))_P \underset{Z[1/2]}{\otimes} KO^{8N}(D^{8N}, \partial D^{8N})_P$. So that $k_{8N}^*(\bar{l}(E_{8N})) = \bar{l}_{8M} \in KO^0(SG(8N))_P$. So that to prove the proposition, it is sufficient to prove $\bar{l} = \bar{\sigma}$ as elements $KO^0(F/PL)_P$. By proposition 1–2, it is sufficient to prove $ph(\bar{l}) = ph(\bar{\sigma})$. This follows from proposition 1–4, ii).

§ 2. Determination of $H_*(BSPL : Z_p)$.

2–1. At first we determine the Hopf algebra over Z_p, $H_*(F/PL : Z_p)$. By proposition 1–2, $H_*(F/PL : Z_p) \underset{\to}{\simeq} H_*(BO_{\otimes P} : Z_p) = H_*(BO_\otimes : Z_p)$, it is sufficient to determine $H_*(BO_\otimes : Z_p)$.

PROPOSITION 2–1. *As a Hopf algebra over* Z_p, $H_*(BO_\otimes : Z_p) = Z_p[a_1, a_2, \cdots]$, *for some* $a_j \in H_{4j}(BO_\otimes : Z_p)$. *And* $\Delta a_j = \sum_{i=0}^{j} a_i \otimes a_{j-i}$, $a_0 = 1$.

Proof. It is sufficient to prove that for any non zero element $x \in H_r(BO_\otimes : Z_p)$, $x^p \neq 0$. By the same method as $(BO_\otimes, \mu_\otimes)$, c.f. (1–1), we obtain a H space $(BU_\otimes, \mu_\otimes)$ as the 1 component of $Z \times BU$, where $Z \times BU$ is the representation space of complex K theory. Let $j : BO_\otimes \to BU_\otimes$ denote the natural H map defined by complexifying vector bundle. Since $j_* : H_*(BO_\otimes : Z_p) \to H_*(BU_\otimes : Z_p)$ is monomorphism, it is sufficient to prove $(j_*(x))^p \neq 0$ for $x \in H_r(BO_\otimes : Z_p)$, $x \neq 0$. Let $B = H_*(BU_\otimes : Z_p)$ and B^* denote dual Hopf algebra $\mathrm{Hom}_{Z_p}(B, Z_p)$, So that $B^* = H^{**}(BU_\otimes : Z_p) = Z_p[[c_1, c_2, \cdots]]$, c_i is i-th Chern class. Let $\alpha : B \to B$ denote the Hopf algebra homomorphism defined by $\alpha(x) = x^p$, and $\alpha^* : B^* \to B^*$ denote dual of α. We compute $\alpha^*(1 + c_1 + c_2 + \cdots)$. Let $\xi \in K(BU_\otimes) = K(BU)$ denote the universal element with augmentation. $\varepsilon(\xi) = 0$. Then it is easy to show $[\alpha^*(c)]^p = c((1+\xi)^p)$ $= c(\xi)^p \cdot c(\xi^2)^{\binom{p}{2}} \cdots c(\xi^{p-1})^{\binom{p}{p-1}} c(\xi^p)$ in $H^{**}(BU_\otimes : Z_p)$. So that $\alpha^*(c) = c(\xi) \cdot c(\xi^2)^{\frac{1}{p}\binom{p}{2}}$ $\cdots c(\xi^{p-1})^{\frac{1}{p}\binom{p}{p-1}} \cdot c(\xi^p)^{\frac{1}{p}}$. Using Chern character it is easy to show that $c(\xi^j) = 1 +$ decomposable in c_r in $H^{**}(BU_\otimes : Z)$, $j \geqq 2$. And the same argument show that the coefficient of c_n^p in $c(\xi^p)$ is zero in $H^{**}(BU_\otimes : Z_p)$, when $n \geqq 2$. So that $\alpha^*(c) = 1 + c_2 + c_3 + \cdots$, mod {decomposable $+ c_1$}. This shows that $\bar{\alpha}^* : H^{**}(BU_\otimes : Z_p)/(c_1) \to H^{**}(BU_\otimes : Z_p)/(c_1)$ is onto mapping, where (c_1) denote the ideal generated by c_1, and as $\alpha^*(c_1) = 0$, $\bar{\alpha}^*$ is well defined. Since $j^{**}(c_1) = 0$ where $j^* : H^{**}(BU_\otimes : Z_p) \to H^*(BU_\otimes : Z_p)$, this shows that for any $x \neq 0$, $[j_*(x)]^p \neq 0$.

Remark 2-2. Indeed we can show that $H_*(BU_\otimes : Z_p) \cong \Gamma_p[b_1] \otimes Z_p[b_2', b_3', \cdots]$, where $\deg b_1 = 2$, $\deg b_j' = 2j$.

2-2. Now we study the map $k_* : H_*(SF : Z_p) \to H_*(F/PL : Z_p)$. By proposition 1-3 it is sufficient to study $g_{1*} : H_*(Q_1S^0 : Z_p) \to H_*(BO_\otimes : Z_p)$. Since $g : QS^0 \to Z \times BO$ is a infinite loop map, g is a H_p^∞ map in the sense of Dyer-Lashof [8]. So that the following diagram is commutative, where $W(\pi_p) = W$ is a acyclic free π_p CW complex, and π_p is the cyclic group of order p.

(2-1)
$$\begin{array}{ccc} W \underset{\pi_p}{\times} (QS^0)^p & \xrightarrow{id \times (g)^p} & W \underset{\pi_p}{\times} (Z \times BO)^p \\ \downarrow \theta & & \downarrow \theta \\ QS^0 & \xrightarrow{\quad g \quad} & Z \times BO \end{array}$$

At first we analyes the map $\theta : W \underset{\pi_p}{\times} (Z \times BO)^p \to Z \times BO$ defined by infinite loop structure $Z \times BO = \varinjlim \Omega^{8n} BO\langle 8n \rangle$. Let X be a finite CW complex, for any element $x \in KO(X)$, we define a element $P(x) \in KO(W \underset{\pi_p}{\times} (X)^p)$ as follows. Represent x as $x = \xi - \eta$ where ξ and η are vector bundles over X, and define $P(x) = P(\xi) - P(\eta)$. Where $P(\xi)$ and $P(\eta)$ are defined by $P(\xi) : W \underset{\pi_p}{\times} E_\xi^p \to W \underset{\pi_p}{\times} X^p$, $P(\eta) : W \underset{\pi_p}{\times} E_\eta^p \to W \underset{\pi_p}{\times} X^p$. Then $P(x)$ is independent to the expression $x = \xi - \eta$. And the construction P has the following properties.

(2-2) i) $P : KO(X) \to KO(W \underset{\pi_p}{\times} X^p)$ is abelian group homomorphism.

ii) P is natural, i.e. for a map $f : X \to Y$ the following diagram is commutative.

$$\begin{array}{ccc} KO(Y) & \xrightarrow{P} & KO(W \underset{\pi_p}{\times} Y^p) \\ \downarrow f^! & P & \downarrow (id \underset{\pi_p}{\times} f^p)^! \\ KO(X) & \longrightarrow & KO(W \underset{\pi_p}{\times} X^p) \end{array}$$

iii) Let $L_p = W/\pi_p$ be the mod p lens space. And $\underset{\approx}{N} \in KO(L_p)$ denote the element defined by regular representation $\pi_p \to SO(p)$. Then $\Delta^* P(x) = \underset{\approx}{N} \otimes x$ in $KO(L_p \times X)$ where $\Delta : L_p \times X \to W \underset{\pi_p}{\times} X^p$.

Since $KO(W \underset{\pi_p}{\times} (Z \times BO)^p) = \underset{\underset{\alpha}{\leftarrow}}{\lim} KO(W \underset{\pi_p}{\times} X_\alpha^p)$, where X_α runs all finite complexes of $Z \times BO$, the above construction P define a map $P : W \underset{\pi_p}{\times} (Z \times BO)^p \to Z \times BO$.

CONJECTURE 2-3. *The two maps θ and $P : W \underset{\pi_p}{\times} (Z \times BO)^p \to Z \times BO$ coincide.*

Since we can not prove this conjecture, we can prove more weak form of the conjecture.

PROPOSITION 2-4. *$\theta(1) = P(1)$ as an element of $KO(L_p) = KO(W \underset{\pi_p}{\times} (*)^p)$, where $1 \in KO((*))$.*

Proof. The Dyer-Lashof map $\theta : W^{(n-1)} \underset{\pi_p}{\times} (\Omega^n X)^p \to \Omega^n X$ is reconstructed in [18] as follows. Let S_p^n denote $S_p^n = S^n \vee \cdots \vee S^n$, the one point union of p sheres. Define $\mu : \Omega^n S_p^n \times (\Omega^n X)^p \to \Omega^n X$ by $\mu(\omega, l_1, \cdots, l_p) = (l_p \vee \cdots \vee l_p) \cdot \omega : S^n \xrightarrow{\omega} S^n \vee \cdots \vee S^n \xrightarrow{l_1 \vee \cdots \vee l_p} X$. The cyclic group π_p operates on $\Omega^n S_p^n$, by induced action of π_p on S_p^n, defined by $\sigma((x, i)) = (x, \sigma((i))$, $\sigma \in \pi_p$, $(x, i) \in S_p^n$. And π_p acts on $(\Omega^n X)^p$ by permutation. Then μ is a π_p equivariant map and define $\mu : \Omega^n S_p^n \underset{\pi_p}{\times} (\Omega^n X)^p \to \Omega^n X$. On the other hand, there is a π_p equivariant map $\theta_n : W^{[(n-1)(p-1)]} \to \Omega^n S_p^n$, such that the image is in the connected component represented by $1 + \cdots + 1 \in \pi_0(\Omega^n S_p^n) \cong Z + \cdots Z$, $n \geq 2$. The Dyer-Lashof map $\theta : W^{[(n-1)(p-1)]} \underset{\pi_p}{\times} (\Omega^n X)^p \to \Omega^n X$ is defined by $\mu \cdot (\theta_n \times id)$: $W^{[(n-1)(p-1)]} \underset{\pi_p}{\times} (\Omega^n X)^p \to \Omega^n S_p^n \underset{\pi_p}{\times} (\Omega^n X)^p \to \Omega^n X$.

Now consider the element $\theta(1) \in KO(L_p)$. Let $\eta_{8N} \in K\tilde{O}^{8N}(S^{8N})$, and $\bar{\eta}_{8N} \in K\tilde{O}^0(S^{8N})$ be the canonical generators. Then $\theta(1) \otimes \eta_{8N} \in K\tilde{O}^{8N}(L_p \ltimes S^{8N})$ is, by Bott periodicity, defined by the adjoint map of $\theta(1) : L_p \to Z \times BO = \Omega^{8N} BO\langle 8N \rangle$, where $X \ltimes Y = X \times Y / X \times (*)$. By the definition of $\theta(1)$, on $(8N-1)(p-1)$ skelton of L_p, $\theta(1) \otimes \eta_{8N}$ is defined by the following π_p equivariant map.

$$W^{[(8N-1)(p-1)]}\ltimes S^{8N} \xrightarrow{\theta_{8N}} S^{8N}\vee\cdots\vee S^{8N} \xrightarrow{\eta_{8N}\vee\cdots\vee\eta_{8A}} BO\langle 8N\rangle.$$

On the other hand the mapping $P: W\ltimes(0\times BO)^p \to (0\times BO)$ can be liftable on $P: W\underset{\pi_p}{\ltimes}(BO\langle 8N\rangle)^p \to BO\langle 8N\rangle$. And denfie a π_p equivariant map $P: W\underset{\pi_p}{\ltimes}(BO\langle 8N\rangle)^p \to BO\langle 8N\rangle$. Then the following diagram is π_p equivariantly homotopy commutative.

$$\begin{tikzcd}
S^{8N}\vee\cdots\vee S^{8N} \arrow[rr, "\eta_{8N}\vee\cdots\vee\eta_{8N}"] \arrow[d, "\bar{i}"] & & BO\langle 8N\rangle \arrow[d] \\
W\ltimes(S^{8N})^p \arrow[r, "id\ltimes(\eta_{8N})^p"] \arrow[dr, "id\times(\bar{\eta}_{8N})^p"] & W\ltimes(BO\langle 8N\rangle^p \arrow[r, "P"] \arrow[d, "id\times(\pi)^p"] & BO\langle 8N\rangle \arrow[d, "\pi"] \\
& W\ltimes(0\times BO)^p \arrow[r, "P"] & 0\times BO
\end{tikzcd}$$

where $\bar{i}: S^{8N}\vee\cdots\vee S^{8N} \to W\ltimes(S^{8N})^p$ is defined by $\bar{i}((x,j))=(\sigma^j(\omega_0);\overbrace{*\times\cdots\times*\times x}^{j}\times*\cdots\times*)$, where $\sigma\in\pi_p$: generator s,t $\sigma(i)=\sigma(i+1)$ mod p, and $\omega_0\in W$: fixed element.

On the other hand, by equivariant cohomology theory due to Bredon [4], the following diagram is π_p equivariantly homotopy commutative, c.f. the argument in [18].

$$\begin{tikzcd}
W^{[8N]}\ltimes S^{8N} \arrow[r, "\theta_N"] \arrow[dr, "id\ltimes(\Delta_p)"'] & S^{8N}\vee\cdots\vee S^{8N} \arrow[d, "\bar{i}"] \\
& W\underset{\pi_p}{\ltimes}(S^{8N})^p
\end{tikzcd}$$

So that $\pi\cdot(\theta(1)\otimes\eta_{8N}): L_p^{[8N]}\ltimes S^{8N} \to BO\langle 8N\rangle \to 0\times BO$ is by Bott periodicity $\theta(1)\otimes\bar{\eta}_{8N}$ in $K\tilde{O}^0(L_p^{[8N]}\ltimes S^{8N})$ on the other hand the above two commutative diagrams show that $\pi\cdot(\theta(1)\otimes\eta_{8N})$ is represented by $\Delta^*(P(\bar{\eta}_{8N}))$ in $K\tilde{O}^0(L_p^{[8N]}\ltimes S^{8N})$. On the other hand by (2–2) iii) shows that $\Delta^*(P(\bar{\eta}_{8N}))=N\otimes\bar{\eta}_{8N}$. This shows $\theta(1)=\underset{\approx}{N}$ in $KO^0(L_p^{[8N]})$, so limiting to $N\to\infty$ we obtain $\theta(1)=\underset{\approx}{N}$ in $KO^0(L_p)$. On the other hand $P(1)=\underset{\approx}{N}$ in $KO^0(L_p)$. This shows the proposition.

PROPOSITION 2–5. *The Dyer Lashoj operations on $H_*(Z\times BO:Z_p)$ defined by θ and P coincide.*

Proof. Let $\mu:(Z\times BO)\times(Z\times BO)\to Z\times BO$ denote the product defined by tensor product. Then the two diagrams are homotopy commutative.

$$\begin{array}{ccc} W \underset{\pi_p}{\times} (Z\times BO)^p & \xrightarrow{P} & Z\times BO \\ \Big\downarrow id\times\Delta_p & & \Big\uparrow \mu \\ W/\pi_p \times (Z\times BO) & \xrightarrow[=]{N\times id} & (p\times BO)\times(Z\times BO) \end{array}$$

$$\begin{array}{ccc} W \underset{\pi_p}{\times} (Z\times BO)^p & \xrightarrow{\theta} & Z\times BO \\ \Big\downarrow id\times\Delta_p & & \Big\uparrow \mu \\ W/\pi_p \times (Z\times BO) & \xrightarrow[=]{N\times id} & (p\underset{\pi_p}{\times} BO)\times(Z\times BO) \end{array}$$

On the other hand any element of $H_*(W\underset{\pi_p}{\times}(Z\times BO)^p : Z_p)$ of the form $e_i\underset{\pi}{\otimes}(x)^p$ is in the image of $(id\times\Delta_p)_* : H_*(W/\pi_p\times(Z\times BO) : Z_p)\rightarrow H_*(W\underset{\pi_p}{\times}(Z\times BO)^p : Z_p)$, c.f. Lemma 2-1 of [17]. This proves the proposition.

2.3. Now we determine the map $g_{1*} : H_*(Q_1S^0 : Z_p)\rightarrow H_*(BO_{\otimes} : Z_p)$. We remember the result of [17] about the Pontrjagin ring $H_*(Q_1S^0 : Z_p) = H_*(SF : Z_p)$. Let $H = \{J = (\varepsilon_1, j_1, \varepsilon_2, j_2, \cdots, \varepsilon_r, j_r)\}$ be the set of sequences J satisfying,

(2-3) i) $r\geq 1$

ii) $j_i\equiv 0 \mod (p-1)$, $i = 1, \cdots, r$.

iii) $j_r\equiv 0 \mod 2(p-1)$.

iv) $(p-1)\leq j_1\leq\cdots\leq j_r$.

v) $\varepsilon_i = 0$ or 1.

vi) if $\varepsilon_{i+1} = 0$, then $j_i/(p-1)$ and $j_{i+1}/(p-1)$ are even parity.

if $\varepsilon_{i+1} = 1$, then $j_i/(p-1)$ and $j_{i+1}/(p-1)$ are odd parity.

And $h : L_p\rightarrow Q_pS^0$ is defined by $h : W/\pi_p\rightarrow W\underset{\pi_p}{\times}(id)^p\rightarrow W\underset{\pi_p}{\times}(Q_1S^0)^p \xrightarrow{\theta} Q_pS^0$ And $h_0 : L_p\rightarrow Q_0S^0$ is by definition $h_0 = h\vee(-pid)$. Then $x_j = h_{0*}(e_{2j(p-1)})$ $\in H_{2j(p-1)}(Q_0S^0 : Z_p)$. And for $J = (\varepsilon_1, j_1, \cdots, \varepsilon_r, j_r)\in H$, x_J is by definition $x_J = \beta_p^{\varepsilon_1}Q_{j_1}\cdots\beta_p^{\varepsilon_{r-1}}Q_{j_{r-1}}\beta_p^{\varepsilon_r}x_{j_r/2(p-1)}\in H_*(Q_0S^0 : Z_p)$. And $\tilde{x}_J = i_*(x_J)\in H_*(SF : Z_p)$, $i : Q_0S^0\rightarrow SF$. Then Theorem I of [17] is as follows,

(2-4) *$H_*(SF : Z_p)$ is free commutative algebra generated by $\tilde{x}_J$, $J\in H$.*

LEMMA 2-6. *For $J=(\varepsilon_1, j_1, \cdots, \varepsilon_r, j_r)\in H$ with $\varepsilon_i=1$ for some i, $g_{1*}(\tilde{x}_J)=0$.*

Proof. Since the following diagram is commutative.

$$\begin{array}{ccc} Q_0S^0 & \xrightarrow{g_0} & O\times BO \\ \downarrow i & & \downarrow i \\ Q_1S^0 & \xrightarrow{g_1} & 1\times BO \end{array}$$

$g_{1*}(\tilde{x}_J) = g_{1*}i_*(\beta_p^{\varepsilon_1}Q_{j_1}\cdots\beta_p^{\varepsilon_r}x_{j_r/2(p-1)}) = i_*(\beta_p^{\varepsilon_1}Q_{j_1}\cdots\beta_p^{\varepsilon_r}g_{0*}(x_{j_r/2(p-1)}))$. On the other hand in $H_*(BO:Z_p)$, the Bockstein map β_p is zero map, so the lemma follows.

PROPOSITION 2-7. *The elements $g_{1*}(\tilde{x}_j)$ are indecomposable in $H_*(BO_\otimes:Z_p)$. And the image of $H_*(SF:Z_p)$ by g_{1*} coincides with the subalgebra generated by $g_{1*}(\tilde{x}_j)$.*

Proof. Since $j_*:H_*(BO_\otimes:Z_p)\to H_*(BU_\otimes:Z_p)$ is monomorphism of Hopf algebra, it is sufficient to prove analog proposition for $\bar{g}_{1*}=(j\cdot g_1)_*:H_*(Q_1S^0:Z_p)\to H_*(BU_\otimes:Z)$. By lemma 2-6, the kernel of $\bar{g}_1^*$ contains ideal generated by c_j, $j\not\equiv 0\ (p-1)$. Let $A=Z_p[\tilde{x}_1,\tilde{x}_2,\cdots]\subseteq H_*(Q_1S^0:Z_p)$ denote the subalgebra generated by $\tilde{x}_j$, then this is a subHopf algebra. A^* denotes the dual Hopf algebra of A, and $\bar{i}:H^*(Q_1S:Z_p)\to A^*$ denotes the dual of inclusion. Then to prove the proposition, it is sufficient to prove $\bar{i}\circ\bar{g}_1^*:H^*(BU_\otimes:Z_p)\to A^*$ is onto. We construct A^* and $i\circ\bar{g}_1^*$ concretely as follows. Let $h_1=h_0\vee id:L_p\to Q_1S^0$, and consider $\bar{h}_1:L_p\to Q_1S^0\to BU_\otimes\to BU_\otimes$. Then, by Proposition 2-4, $\bar{h}_1$ determines the element $1+\underline{\underline{\tilde{N}}}\in K(L_p)$, where $\underline{\underline{N}}$ is the element determined by regular representation, and $\underline{\underline{\tilde{N}}}=\underline{\underline{N}}-p$. For large l consider $H_l:L_p^l=L_p\times\cdots\times L_p\xrightarrow{\bar{h}\times\cdots\times\bar{h}_1}BU_\otimes\times\cdots\times BU_\otimes\xrightarrow{\mu_\otimes}BU_\otimes$. And consider $H_l^*:H^*(BU_\otimes:Z_p)\to H^*(L_p^l:Z_p)=Z_p[\beta_1,\cdots,\beta_l]\otimes\Lambda(\alpha_1,\cdots,\alpha_l)$. Then the image of H_l^* is contained in $SZ_p[\beta_1^{p-1},\cdots,\beta_l^{p-1}]$, where $SZ_p[\beta_1^{p-1},\cdots,\beta_l^{p-1}]$ means invariant subHopf algebra of $Z_p[\beta_1^{p-1},\cdots,\beta_l^{p-1}]$ by the action of permutation group $\sum_l$. $SZ_p[\beta_1^{p-1},\cdots,\beta_l^{p-1}]=Z_p[\sigma_1,\cdots,\sigma_l]$, where σ_i is the i-th elementary symmetric function of $\beta_1^{p-1},\cdots,\beta_l^{p-1}$. And up to dim $2l(p-1)$, A^* and $\bar{i}\cdot\bar{g}_1^*$ is represented by $SZ_p[\beta_1^{p-1},\cdots,\beta_l^{p-1}]=Z_p[\sigma_1,\cdots,\sigma_l]$ and H_l^*. Consider the element $H_l^*(1+c_1+\cdots)$, and we shall show, for $1\leq s\leq l$, the coefficient of σ_s in $H_l^*(1+c_1+\cdots)$ is $(-1)^s$. Then this shows the proposition, since H_l^* is algebra homomorphism, and $\{c_i\}$ and $\{\sigma_i\}$ are algebra generator of $H^*(BU_\otimes:Z_p)$ and $SZ_p[\beta_1^{p-1},\cdots\beta_l^{p-1}]$. By definition $H_l^*(1+c_1+\cdots)=c((1+\underline{\underline{\tilde{N}}}_1)\cdots(1+\underline{\underline{\tilde{N}}}_l))$, where $\underline{\underline{\tilde{N}}}_i\in K(L_p^l)$ is the element defined by $1\otimes\cdots\otimes 1\otimes\underline{\underline{\tilde{N}}}\otimes 1\otimes\cdots\otimes 1\in K(L_p^l)=K(L_p)\otimes\cdots\otimes K(L_p)$, where $\underline{\underline{\tilde{N}}}$ is in the i-th factor.

$$c((1+\underline{\underline{\tilde{N}_1}})\cdots(1+\underline{\underline{\tilde{N}_l}}))$$
$$=\prod_i c(\underline{\underline{\tilde{N}_i}})\cdot\prod_{i<j}c(\underline{\underline{\tilde{N}_i}}\underline{\underline{\tilde{N}_j}})\cdots\prod c(\underline{\underline{\tilde{N}_1}}\cdots\underline{\underline{\tilde{N}_l}}).$$

And

$$\prod_i c(\underline{\underline{\tilde{N}_i}})=\prod_i(1-\beta_i^{p-1})$$
$$=1-\sigma_1+\cdots+(-1)^l\sigma_l.$$

Then the following lemma show the proposition.

LEMMA 2-8. *In the above situation, for* $2\leq t\leq l$, *the coefficient of* σ_s, $1\leq s\leq l$, *in* $\prod_{1\leq i_1<\cdots<i_t\leq l} c(\underline{\underline{\tilde{N}_{i_1}}}\cdots\underline{\underline{\tilde{N}_{i_t}}})$ *is zero.*

Proof. We prove in the case $t=2$, since proof is analog for the case $t>2$, since it is tediously long.

$$\prod_{1\leq i<j\leq l}c(\underline{\underline{\tilde{N}_i}}\underline{\underline{\tilde{N}_j}})=\prod_{1\leq i<j\leq l}c(\underline{\underline{(N_i-p)(N_j-p)}})$$
$$=[\prod_{1\leq i<j\leq l}c(\underline{\underline{\tilde{N}_i}}\underline{\underline{\tilde{N}_j}})]\cdot[\prod_{1\leq i<j\leq l}(c(\underline{\underline{\tilde{N}_i}})c(\underline{\underline{\tilde{N}_j}}))]^{-p}$$
$$\equiv\prod_{1\leq i<j\leq l}c(\underline{\underline{\tilde{N}_i}}\underline{\underline{\tilde{N}_j}})\ \text{mod decomposable}$$
$$=[\prod_{\substack{i=1\cdots l\\ j=1\cdots l}}c(\underline{\underline{\tilde{N}_i}}\underline{\underline{\tilde{N}_j}})]^{1/2}\cdot[\prod_{i=1\cdots l}c(\underline{\underline{\tilde{N}_i}}\underline{\underline{\tilde{N}_i}})]^{-1}$$
$$=[\prod_{\substack{i=1\cdots l\\ j=1\cdots l}}\prod_{\substack{a_i=0\cdots p-1\\ a_j=0\cdots p-1}}(1+a_i\beta_i+a_j\beta_j)]^{1/2}\cdot[\prod_{i=1\cdots l}\prod_{\substack{a=0\cdots p-1\\ b=0\cdots p-1}}(1+(a+b)\beta_i)]^{-1}$$
$$=[\prod_{\substack{i=1\cdots l\\ a_i=0\cdots p-1}}\prod_{i=1\cdots l}((1+a_i\beta_i)^p-\beta_j^{p-1}(1+a_i\beta_i))]^{1/2}[\prod_{i=1\cdots l}\prod_{a=0\cdots p-l}(1+a\beta_i)]^{-p}$$
$$\equiv[\prod_{\substack{i=1\cdots l\\ a_i=0\cdots p-1}}((1+a_i\beta_i)^{pl}-\sigma_1(1+a_i\beta_i)^{p(l-1)+1}+\cdots+(-1)^l\sigma_l\cdot(1+a_i\beta_i)^l]^{1/2}$$
$$\text{mod dec.}$$
$$\equiv[\prod_{\substack{i=1\cdots l\\ a_i=0\cdots p-1}}((1+a_i\beta_i)^{pl}-\sigma_i+\cdots+(-1)^l\sigma_l)]^{1/2}\ \text{mod dec.}$$
$$\equiv[(\prod_{\substack{i=1\cdots l\\ a_i=0\cdots p-1}}(1+a_i\beta_i)^{pl})+pl(-\sigma_1+\cdots+(-1)^l\sigma_l)]^{1/2},\ \text{mod dec.}$$
$$\equiv 1\ \text{mod dec.}$$

where mod decomposable means in $SZ_p[\beta_1^{p-1},\cdots,\beta_l^{p-1}]=Z_p[\sigma_1,\cdots,\sigma_l]$. This proves the lemma.

2.4. Let $y_j \in H_{2j(p-1)-1}(SO : Z_p)$ denote the unique element defined by the following conditions, $j=1,2,\cdots$, i) $\langle\sigma(q_j),\ y_j\rangle=1$, ii) y_j is a primitive element. Denote $i_*(y_j)$ by $\tilde{y}_j$ for $i_*: H_*(SO : Z_p)\to H_*(SF : Z_p)$.

CONJECTURE 2-9. *$\tilde{y}_j$ is contained in the subalgebra of $H_*(SF : Z_p)$ generated by $\tilde{x}_k$, $\beta_p\tilde{x}_k$, $k=1,2,\cdots$.*

Since we can not prove this conjecture, we prepare the following two lemmas, which are proved in §5.

LEMMA 2-10. *There are continuous maps, $f: L_p\to SF$ and $g: CP^\infty\to F/O$ with the following properties.*

i) *The following diagram is commutative.*

$$\begin{array}{ccc} L_p & \xrightarrow{\quad f \quad} & SF \\ \downarrow & & \downarrow \\ CP^\infty & \xrightarrow{\quad g \quad} & F/O \end{array}$$

ii) *The map $L_p\to SF\to F/PL\xrightarrow{\bar{\sigma}} BO_{\otimes(p)}$ represents in $KO(L_p)_{(p)}$ the element $1+\dfrac{2}{p+1}\tilde{N}$, where $BO_{\otimes(p)}$ denote the localized space of $BO_\otimes$ at prime p and* $KO(L_p)_{(p)}=KO(L_p)\otimes Z[1/2,1/3,\cdots,\widehat{1/p},\cdots]$.

LEMMA 2-11. *The following formula are valid, for some $c\neq 0$.*

(2-5)
$$f_*(e_{2j(p-1)})=cx_j+a_j,\quad a_j\in G_2,\quad j=1,2,\cdots.$$
$$f_*(e_{2j(p-1)-1})=c\beta_p x_j+b_j,\quad b_j\in G_2,\quad j=1,2,\cdots.$$

Now we define the subsets of H as follows.

(2-6)

i) $H_2^+=\{J=(0,\ p-1,\ 1,\ 2j(p-1))\in H,\ j=1,2,\cdots\}$

ii) $H_2^-=\{J=(1,\ p-1,\ 1,2j(p-1))\in H,\ j=1,2,\cdots\}$

iii) $H_{1,1}^+=\{J=(0,\ j_1,\ 0,\ j_2,\cdots,0,j_r)\in H,\ r\geq 2\}$

iv) $H_{1,1}^-=\{J=(1,\ j_1,\ 0,\ j_2,\cdots 0,\ j_r)\in H,\ r\geq 2\}$

v) $H_{1,2}^+=\{J=(\varepsilon_1,\ j_1,\ \varepsilon_2,\ j_2,\ \cdots,\ \varepsilon_r,\ j_r)\in H,\ r\geq 2,$
$j_1\neq p-1,\ \deg x_J=\text{even},\ J\notin H_{1,1}^+\}$

vi) $H_{1,2}^-=\{J=(\varepsilon_1,\ j_1,\cdots,\varepsilon_r,\ j_r)\in H,\ r\geq 2,$
$j_1\neq p-1,\ \deg x_J=\text{odd},\ J\notin H_{1,1}^-\}$.

Now we define the element $x'_j \in H_{2j(p-1)-1}(Q_0S^0 : Z_p)$, $j=1,2,\cdots$, by $x'_j = f_{0*}(e_{2j(p-1)})$ for $f_0 : L_p \to Q_0S^0$, where $L_0 : L_p \to Q_0S^0$ is defined by $f_0 = f \vee (-id)$ for $f : L_p \to SF$ defined in lemma 2-10.

For $J=(\varepsilon_1, j_1, \cdots, \varepsilon_r, j_r) \in H$, we define $\bar{x}_J \in H_*(SF : Z_p)$ by $i_*(\beta_p^{\varepsilon_1} Q_{j_1} \cdots \beta_p^{\varepsilon_r} x'_{j_r/2(p-1)})$, where $i_\infty : H_\infty(Q_0S^0 : Z_p) \to H_\infty(SF : Z_p)$.

LEMMA 2-12. *As the algebraic generators for $H_*(SF : Z_p)$, we can choose the following elements.*

i) $\bar{x}_j, \beta_p \bar{x}_j$, $j=1,2,\cdots$.

ii) $\bar{x}_I$, $I \in H^+_{1,1} \cup H^+_{1,2} \cup H^+_2$.

iii) $\bar{Q}_{p-1} \cdots \bar{Q}_{p-1}(\bar{x}_I)$, $I \in H^-_{1,1} \cup H^-_{1,2} \cup H^-_2$.

iv) $\bar{Q}_{p-2}\bar{Q}_{p-1} \cdots \bar{Q}_{p-1}(\bar{x}_I)$, $I \in H^-_{1,1} \cup H^-_{1,2} \cup H^-_2$.

Where $\bar{Q}_{p-2}$, and $\bar{Q}_{p-1}$ are the Dyer-Lashof operations on $H_*(SF : Z_p)$ defined in [17].

Proof of this lemma is analog of that of proposition 6-8 of [17], so we omit the proof.

PROPOSITION 2-13. *The elements $\tilde{y}_j$ are in the subalgebra of $H_*(SF : Z_p)$ generated by $\bar{x}_k$, $\beta_p \bar{x}_k$, $k=1,2,\cdots$. And $\tilde{y}_j \equiv c_j \beta_p x_j$ mod dec, $c_j \neq 0$.*

Proof. Since $\tilde{y}_j$ is non decomposable element, $\tilde{y}_j \equiv c_j \beta_p \bar{x}_j + \sum c_{k,I} \bar{Q}^k_{p-1}(\bar{x}_I)$, in $QH_*(SF : Z_p)$[1)] the vector space of indecomposable elements. Now consider $\tilde{y}_j$ in $QH_*(F/O : Z_p)$. By lemma 2-10, $\beta_p \bar{x}_j$ is zero in $H_*(F/O : Z_p)$. Since kernel of $QH_{2j(p-1)-1}(SF : Z_p) \to QH_{2j(p-1)-1}(F/O : Z_p)$ is 1 dimensional, other elements $\bar{Q}^k_{p-1}(\bar{x}_I)$ are linear independent. On the other hand, $\tilde{y}_j = 0$ in $H_*(F/O : Z_p)$, this shows that $\tilde{y}_j = c_j \beta_p \bar{x}_j$, $c_j \neq 0$, in $QH_{2j(p-1)-1}(SF : Z_p)$. On the other hand since $\tilde{y}_j$ is a primitive element, and $0 \to PH_{2j(p-1)-1}(SF : Z_p) \to QH_{2j(p-1)-1}(SF : Z_p) \to 0$, and the subalgebra of $H_*(SF : Z_p)$ generated by $\bar{x}_k$, $\beta_p \bar{x}_k$, $k=1,2,\cdots$, is subHopf algebra, so that $\tilde{y}_j$ belongs to the subalgebra generated by $\bar{x}_k$, $\beta_p \bar{x}_k$.

Remark 2-14. By lemma 2-10, $g_{1*}(\bar{x}_j) = c g_{1*}(x_j)$, $j=1,2,\cdots$, for $g_{1*} : H_*(SF : Z_p) \to H_*(BO_\otimes : Z_p)$, for $c \neq 0$.

For $J \in H^\circ_{1,1}$, consider $g_{1*}(x_J)$, by proposition 2-7 and remark 2-14, there is a unique element $\bar{u}_J \in Z_p[\bar{x}_1, \bar{x}_2, \cdots]$ $H_*(SF : Z_p)$ such that $g_{1*}(\bar{x}_J) = g_{1*}(\bar{u}_J)$.

1) $Q(\ \ )$ denotes the space of indecomposable elements.

Define $\bar{x}'_J \equiv \bar{x}_J - \bar{u}_J$. And for $J=(1, j_1, 0, j_2, \cdots, 0, j_r) \in H^-_{1,1}$, define $\bar{x}'_J = \beta_p \bar{x}'_{J'}$, where $J' = (0, j_1, 0, j_2, \cdots, 0, j_r) \in H^+_{1,1}$.

PROPOSITION 2-15. *As algebraic generators for $H_*(SF : Z_p)$, we can choose following elements.*

i) $\bar{x}_j$, $\bar{y}_j$, $j = 1, 2, \cdots$.

ii) $\bar{x}_I$, $I \in H^+_{1,2} \cup H^+_2$ *and* $\bar{x}'_I$, $I \in H^+_{1,1}$.

iii) $\bar{Q}_{p-1} \cdots \bar{Q}_{p-1}(\bar{x}_I)$, $I \in H^-_{1,2} \cup H^-_2$ *and* $\bar{Q}_{p-1} \cdots \bar{Q}_{p-1}(\bar{x}'_I)$, $I \in H^-_{1,1}$.

iv) $\bar{Q}_{p-2}\bar{Q}_{p-1} \cdots \bar{Q}_{p-1}(\bar{x}_I)$, $I \in H^-_{1,2} \cup H^-_2$ *and* $\bar{Q}_{p-2}\bar{Q}_{p-1} \cdots \bar{Q}_{p-1}(\bar{x}'_I)$, $I \in H^-_{1,1}$.

Proof. For a basis of $QH_*(SF : Z_p)$, we can choose elements in lemma 2-12. By proposition 2-13, $\bar{y}_j = c_j \beta_p \bar{x}_j$, $c_j \neq 0$, in $QH_*(SF : Z_p)$. For $I \in H^-_{1,1}$, $\bar{x}'_I = \bar{x}_I + c_I y_{|I|}$, in $QH_*(SF : Z_p)$, where $|I| = (\deg \bar{x}_I) + 1/2(p-1)$, by definition of $\bar{x}'_I$ and by proposition 2-13. Since the construction of §4 of [17], defining the H^∞_p structure on SF can be extended on SO, and define the H^∞_p structure on SO with the following commutative diagram.

$$\begin{array}{ccc} W \underset{\pi_p}{\times} (SO)^p & \longrightarrow & W \underset{\pi_p}{\times} (SF)^p \\ \downarrow \theta & & \downarrow \theta \\ SO & \longrightarrow & SF \end{array}$$

So that we can define the operations $\bar{Q}_j$ on $H_*(SO : Z_p)$ compatible with the operations $\bar{Q}_j$ on $H_*(SF : Z_p)$. So by proposition 2-13 and by the fact that the image of $H_*(SO : Z_p) \to H_*(SF : Z_p)$ is the subalgebra generated by $\bar{y}_j$, $j = 1, 2, \cdots$, we can easily show that $\bar{Q}^k_{p-1}(\bar{y}_j)$ are in $Z_p[\bar{x}_1, \bar{x}_2, \cdots] \otimes \Lambda(\beta_p \bar{x}_1, \beta_p \bar{x}_2, \cdots)$ and $\bar{Q}_{p-2}\bar{Q}^k_{p-1}(y_j) = 0$. So that for $I \in H^-_{1,1}$, $\bar{Q}^k_{p-1}(\bar{x}'_I) \equiv \bar{Q}^k_{p-1}(\bar{x}_I) + c_{(p,I)} y_{(p,I)}$ in $QH_*(SF : Z_p)$, where $y_{(p,I)} = y_{j'}$ for $2j'(p-1)-1 = \deg(\bar{Q}^k_{p-1}(\bar{x}_I))$, and $\bar{Q}_{p-2}\bar{Q}^k_{p-1}(\bar{x}'_I) \equiv \bar{Q}_{p-2}\bar{Q}^k_{p-1}(\bar{x}_I)$ in $QH_*(SF : Z_p)$. This shows the proposition.

2-5. At first we consider the homology spectral sequence associated to $SPL \to SF \to F/PL$, and determine the Pontrjagin ring $H_*(SPL : Z_p)$.

PROPOSITION 2-16. *As a Hopf algebra over Z_p, $H_*(\Omega(F/PL) : Z_p) \cong \Lambda(d_1 d_2, \cdots)$, $\deg d_j = 4j-1$, $j = 1, 2, \cdots$. d_j are primitive elements.*

PROPOSITION 2-17. *There are elements* $\bar{\bar{x}}_J \in H_*(SPL : Z_p)$ *for* $J \in H^{\pm}_{1,1} \cup H^{\pm}_{1,2} \cup H^{\pm}_2$, *such that* $j_*(\bar{\bar{x}}_J) = \bar{x}_J + dec$, *for* $J \in H^{\pm}_{1,2} \cup H^{\pm}_2$, *and* $j_*(\bar{\bar{x}}_J) = \bar{x}'_J + dec$, *for* $J \in H^{\pm}_{1,1}$. *Where* $j_* : H_*(SPL : Z_p) \to H_*(SF : Z_p)$.

Proof. Since $i_*(\bar{x}_J) = 0$, for $J \in H^{\pm}_{1,2} \cup H^{\pm}_2$, and $i_*(\bar{x}'_J) = 0$ for $J \in H^{\pm}_{1,1}$, where $i_* : H_*(SF : Z_p) \to H_*(F/PL : Z_p)$. Proposition follows from the homology spectral sequences associated to the following two fibering.

$$\begin{array}{ccccc} \Omega(F/PL) \longrightarrow & SPL & \longrightarrow & * & \longrightarrow \Omega(F/PL) \\ & \downarrow & & \downarrow & \\ & SF & \longrightarrow & F/PL & \end{array}$$

Remark 2-18. For $\bar{\bar{x}}_I$, $I \in H^{\pm}_{1,1} \cup H^{\pm}_{1,2} \cup H^{\pm}_2$, we can choose the pair $\bar{\bar{x}}_J$ and $\beta_p \bar{\bar{x}}_J$.

As in the proof of proposition 2-15, the H^{∞}_p structure on SO and SF can be extended on SPL with the following commutative diagram

(2-7)
$$\begin{array}{ccccc} W \underset{\pi_p}{\times} (SO)^p & \longrightarrow & W \underset{\pi_p}{\times} (SPL)^p & \longrightarrow & W \underset{\pi_p}{\times} (SF)^p \\ \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta} \\ SO & \longrightarrow & SPL & \longrightarrow & SF \end{array}$$

Next define elements $\bar{\bar{d}}_j \in H_{4j-1}(SPL : Z_p)$ by $j_*(d_j)$ for $j_* : H_*(\Omega(F/PL) : Z_p) \to H_*(SPL : Z_p)$, for $j \not\equiv 0 \ (p-1)/2$. And define $\bar{\bar{y}}_j \in H_{2j(p-1)-1}(SPL : Z_p)$ by $j_*(y_j)$, $j_* : H_*(SO : Z_p) \to H_*(SPL : Z_p)$.

PROPOSITION 2-19. $H_*(SPL : Z_p)$ *is a free commutative algebra generated by the following elements.*

i) $\bar{\bar{y}}_j$, $j = 1, 2, \cdots$. $\bar{\bar{d}}$, $j \not\equiv 0 \ (p-1)/2$.

ii) $\bar{\bar{x}}_I$, $I \in H^{+}_{1,1} \cup H^{+}_{1,2} \cup H^{+}_2$.

iii) $\bar{Q}^k_{p-1}(\bar{\bar{x}}_I)$. $I \in H^{-}_{1,1} \cup H^{-}_{1,2} \cup H^{-}_2$.

iv) $\bar{Q}_{p-2}\bar{Q}^k_{p-1}(\bar{\bar{x}}_I)$, $I \in H^{-}_{1,1} \cup H^{-}_{1,2} \cup H^{-}_2$.

Proof of this proposition is by using homology spectral sequence associated to $SPL \to SF \to F/PL$.

2-6. Next we define the elements of $H_*(BSPL : Z_p)$.

Let $\underline{\underline{N}} : L_p \to BSO$ denote the map defined by the regular representation of π_p. Define $z_j = \underline{\underline{N}}_*(e_{2j(p-1)}) \in H_{2j(p-1)}(BSO : Z_p)$. Then z_j are non decomposable elements, $j=1,2,\cdots$. Define the element $\bar{\bar{z}}_j \in H_{2j(p-1)}(BSPL:Z_p)$ by $\bar{\bar{z}}_j = j_*(z_j)$, $j_*: H_*(BSO:Z_p) \to H_*(BSPL:Z_p)$.

And define $\bar{a}_j \in H_{4j}(BSPL:Z_p)$, $j \not\equiv 0\ (p-1)/2$, by $\bar{a}_j = i_*(a_j)$, $i_*: H_*(F/PL:Z_p) \to H_*(BSPL:Z_p)$.

Our main proposition is as follows.

Proposition 2–20. *$H_*(BSPL:Z_p)$ is a free commutative algebra generated by the following elements.*

i) $\bar{\bar{z}}_j$, $j=1,2,\cdots$

ii) $\bar{a}_j$, $j \not\equiv 0\ (p-1)/2$

iii) $\sigma(\bar{x}_J)$, $J \in H^{+}_{1,1} \cup H^{+}_{1,2} \cup H^{+}_{2}$.

Proof. In the spectral sequence $E^2_{**} \cong H_*(F/PL:Z_p) \otimes H_*(\Omega F/PL:Z_p)$, $E^\infty_{**} \cong Z_p$, the following relations hold.

$$d_{4jp^k}(a_j^{p^k}) = c_j d_{p^k j},\quad c_j \neq 0,\quad (j,p)=1,\quad r \geq 0.$$

$$d_{4jp^{k-1}(p-1)}(a_{jp^k}) = c_{jp^k}(a_j)^{p^{k-1}} \otimes d_{jp^{k-1}},\quad (j,p)=1,\quad k \geq 1,\quad c_{jp^k} \neq 0.$$

And in the spectral sequence $E^2_{**} \cong H_*(BSO:Z_p) \otimes H_*(SO:Z_p)$, $E^\infty_{**} \cong Z_p$, the following relations hold.

$$d_{2j(p-1)p^k}(z_i^{p^k}) = c_j y_{p^k j},\quad c_j \neq 0,\quad (j\ p)=1,\quad k \geq 0.$$

$$d_{2j(p-1)p^{k-1}(p-1)}(z_{jp^k}) = c_{jp^k}(z_j)^{p^{k-1}(p-1)} \otimes y_{jp^{k-1}},\quad (j\ p)=1,\quad k \geq 1,\quad c_{jp^k} \neq 0.$$

And since H_p^∞ structure on SPL can be extended on the fibering $SPL \to ESPL \to BSPL$ as that of $SF \to ESF \to BSF$, c.f. (4–15) of [17]. So that Kudo's transgresion theorem holds on the spectral sequence $E^2_{**} = H_*(BSPL:Z_p) \otimes H_*(SPL:Z_p)$, c.f. proposition 6–1 of [17]. These date determine the differential of the spectral sequence for $E^2_{**} \cong H_*(BSPL:Z_p) \otimes H_*(SPL:Z_p)$. And we obtain the proposition by the same method of the proof of Theorem 2 in [17].

Corollary 2–21. *Kernel of the $i_*: H_*(F/PL:Z_p) \to H_*(BSPL:Z_p)$ is ideal generated by $j_*(\bar{x}_j)$, $j=1,2,\cdots$, for $j_*: H_*(SF:Z_p) \to H_*(F/PL:Z_p)$.*

By corollary 2–21, the subalgebra $Z_p[\bar{a}_j]$, $j \not\equiv 0\ (p-1)/2$ of $H_*(BSPL:Z_p)$ is the image of $i_*: H_*(F/PL:Z_p) \to \dot{H}_*(BSPL:Z_p)$, so that this subalgebra is subHopf algebra. And dual algebra of this subHopf algebra is a polynomial algebra, since this subalgebra is realized as a subalgebra of $H^*(F/PL:Z_p)$.

By definition of $\bar{\bar{z}}_j$, $\Delta(\bar{\bar{z}}_j) = \sum_{i=0}^{j} \bar{\bar{z}}_i \otimes \bar{\bar{z}}_{j-i}$, $\bar{\bar{z}}_0 = 1$. These two remarks show that subalgebra generated by $\bar{\bar{z}}_j$, and $\bar{\bar{a}}_j$ of $H_*(BSPL : Z_p)$ is a subHopf algebra and there are elements $\bar{\bar{b}}_k \in Z_p[\bar{\bar{z}}_1, \bar{\bar{z}}_2, \cdots] \otimes Z_p[\bar{\bar{a}}_j]$, $j \not\equiv 0 \ (p-1)/2$, deg $b_k = 4k$, such that

$$Z_p[\bar{\bar{z}}_1, \bar{\bar{z}}_2, \cdots] \otimes Z_p[\bar{\bar{a}}_j] = Z_p[\bar{\bar{b}}_1, \bar{\bar{b}}_2, \cdots]$$

and

$$\Delta(\bar{\bar{b}}_j) = \sum_{i=0}^{j} \bar{\bar{b}}_i \otimes \bar{\bar{b}}_{j-i}, \ \bar{\bar{b}}_0 = 1.$$

THEOREM 2-22. *As a Hopf algebra*

i) $H_*(BSPL : Z_p) \cong Z_p[\bar{\bar{b}}_j] \otimes Z_p[\sigma(\bar{\bar{x}}_I)] \otimes \Lambda(\sigma(\bar{\bar{x}}_J))$, *where*

$$I \in H^-_{1,1} \cup H^-_{1,2} \cup H^-_2, \quad J \in H^+_{1,1} \cup H^+_{1,2} \cup H^+_2.$$

ii) $\Delta(\bar{\bar{b}}_j) = \sum_{i=0}^{j} \bar{\bar{b}}_i \otimes \bar{\bar{b}}_{j-i}$, $\sigma(\bar{\bar{x}}_I)$, $\sigma(\bar{\bar{x}}_J)$ *are primitive elements.*

§ 3. $H^*(BSPL : Z[1/2])$/Torsion.

3-1. The purpose of this section is to prove the following theorem.

THEOREM 3-1. *As a Hopf algebra over* $Z[1/2]$,

i) $H^*(BSPL : Z[1/2])/Torsion = Z[1/2][R_1, R_2, \cdots]$

ii) $\Delta R_j = \sum_{i=0}^{j} R_i \otimes R_{j-i}$, $R_0 = 1$, $deg\, R_j = 4j$.

iii) *In* $H^*(BSPL, Q) = Q[p_1, p_2, \cdots]$, R_j *are expressed as follows.*

$$R_j = 2^{a_j}(2^{2j-1} - 1)\, Num\,(B_j/4j) \cdot p_j + decomposable \ for \ some \ a_j \in Z.$$

At first we study the Bockstein spectral sequence.

PROPOSITION 3-2. *In the Bockstein homology spectral sequence,* $E^1 = H_*(BSPL : Z_p)$, $E^\infty = (H_*(BSPL : Z)/Torsion) \otimes Z_p$, *the following formula holds.*

If $x \in E^r_{2n}$, $y \in E^r_{2n-1}$ *are such that* $d^r(x) = y$, *then* $d^{r+1}(x^p) = x^{p-1}y$.

Proof. For $r > 1$, this is theorem 5-3 of [5], and using H^p_∞ structure $\theta : W \underset{\pi_p}{\times} (BSPL)^p \to BSPL$, it is easy to show that this holds for $r = 1$.

Remark 3-3. The above spectral sequence is a Hopf algebra spectral sequence over Z_p.

PROPOSITION 3-4. *As a Hopf algebra over* Z_p, $E^\infty = (H_*(BSPL : Z)/Torsion) = Z_p[(\bar{\bar{b}}_1), (\bar{\bar{b}}_2), \cdots]$, $\Delta((\bar{\bar{b}}_1)) = \sum (\bar{\bar{b}}_i) \otimes \bar{\bar{b}}_{j-i})$, *where* $(\bar{\bar{b}}_i)$ *is the class which is represented by* $\bar{\bar{b}}_i$ *in Theorem 2-22.*

Proof. By Theorem 2–22, as a Hopf algebra over Z_p, $H_*(BSPL : Z_p) = Z_p[\bar{\bar{b}}_j]\otimes Z_p(\sigma(\bar{\bar{x}}_I))\otimes\Lambda(\sigma(\bar{\bar{x}}_J))$. By remark 2–18, in $\sigma(\bar{\bar{x}}_I)$ and $\sigma(\bar{\bar{x}}_J)$, if $\sigma(\bar{\bar{x}}_J)$ appears then $\alpha(\beta_p\bar{\bar{x}}_J) = \beta_p\sigma(\bar{\bar{x}}_J)$ also appears. So that $Z_p[\sigma(\bar{\bar{x}}_I)]\otimes\Lambda[\sigma(\bar{\bar{x}}_J)]$ is decomposed following two types of Hopf algebras. $Z_p[\sigma(\bar{\bar{x}}_I)]\otimes\Lambda(\beta_p\sigma(\bar{\bar{x}}_I))$ and $Z_p[\beta_p\sigma(\bar{\bar{x}}_J)]\otimes\Lambda(\sigma(\bar{\bar{x}}_J))$. So that the proposition follows from proposition 3–2, remark 3–3, and the fact that $d^1 = \beta_p$.

Proof of Theorem 3–1. Since p is any odd prime, proposition 3–4 shows that $H^*(BSPL : Z[1/2])/\text{Torsion} = Z[1/2][R_1, R_2, \cdots]$, $\Delta(R_j) = \sum_{i=0}^{j} R_i\otimes R_{j-i}$, for some R_j. Since $P(H_{4j}(BSPL : Z)/\text{Torsion}\otimes Z_p)$[1] is 1-dimensional, over Z_p, and spanned by the image of $PH_{4j}(BSO : Z_p)$ and $PH_{4j}(F/PL : Z_p)$, so that $P(H_{4j}(BSPL : Z[1/2]/\text{Torsion})\cong Z[1/2]$ and spanned over $Z[1/2]$ by the image of $PH_{4j}(BSO : Z)\cong Z$, and $PH_{4j}(F/PL : Z[1/2])\cong Z[1/2]$. On the other hand there is a generator $m_j\in PH_{4j}(BSO : Z)\cong Z$, such that $\langle p_j, m_j\rangle = 1$, and $\tilde{m}_j\in PH_{4j}(F/PL, Z[1/2])\cong Z[1/2]$ such that $\langle L_j, \tilde{m}_j\rangle = \frac{1}{(2j-1)!}$. But since $L_j = \frac{2^{2j+1}(2^{2j-1}-1)\,\text{Num}\,(B_j/4j)}{(2j-1)!\,\text{denom}\,(B_j/4j)}\,p_j + \text{dec}$, so that $\langle p_j, \tilde{m}_j\rangle = \frac{\text{denom}\,(B_j/4j)}{2^{2j+1}(2^{2j-1}-1)\text{Num}\,(B_j/4j)}$. So that in $PH_{4j}(BSPL : Q)$, $P(H_{4j}(BSPL, Z[1/2]/\text{Torsion}))\cong Z[1/2]$ is generated over $Z[1/2]$ by m_j and $\frac{\text{denom}\,(B_j/4j)}{2^{2j+1}(2^{2j-1}-1)\text{Num}\,(B_j/4j)}\,m_j$. But odd prime factor of denom$\,(B_j/4j)$ and $(2^{2j-1}-1)\,\text{Num}\,(B_j/4j)$ are relatively prime, so that $P(H_{4j}(BSPL : Z[1/2])/\text{Torsion})$ is spanned over $Z[1/2]$ by $\frac{m_j}{(2^{2j-1}-1)\,\text{Num}\,(B_j/4j)}$. So that we can take R_j by $R_j = 2^{a_j}(2^{2j-1}-1)\,\text{Num}\,(B_j/4j)p_j + \text{dec}$ in $H^*(BSPL : Q)$, for some $a_j\in Z$.

§ 4. Determination of $\phi : A\to H^*(MSPL : Z_p)$.

4–1. Let $A = A_p$ denote the mod p Steenrod algebra over Z_p, and $\phi : A\to H^*(MSPL : Z_p)$ is defined by the following, where $u\in H^0(MSPL : Z_p)$ is the Thom class.

$$\phi(a) = a(u). \tag{4–1}$$

The object of this section is to prove the following theorem.

THEOREM 4–1. *The kernel of* ϕ *is the left ideal generated by* $\underline{\underline{Q}}_0$, $\underline{\underline{Q}}_1$. *Where* $\underline{\underline{Q}}_j$ *is the element defined by Milnor.*

The following lemma is proved in 4–2.

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1) P ( ) denote the space of primitive elements.

LEMMA 4-2. $\phi(\underline{\underline{Q}}_j)\neq 0$ *for* $j\geqq 2$.

Proof of the Theorem. Since $\phi(\underline{\underline{Q}}_0)=\phi(\underline{\underline{Q}}_1)=0$, $\ker\phi\supseteq A(\underline{\underline{Q}}_0, \underline{\underline{Q}}_1)$, where $A(\underline{\underline{Q}}_0, \underline{\underline{Q}}_1)=$ the left ideal generated by $\underline{\underline{Q}}_0$, and $\underline{\underline{Q}}_1$. $MSPL$ has the product $\mu: MSPL\wedge MSPL\to MSPL$, defined by Whitney sum. So that $H^*(MSPL: Z_p)$ has the coalgebra structure over Z_p. And it is well known that ϕ is a coalgebra homomorphism. Let $\chi: A\to A$ denote the canonical anti-automorphism of A. And define $\bar{\phi}: A\to H^*(MSPL: Z_p)$ by $\bar{\phi}(a)=\chi(a)\cdot u$. To prove the theorem, it is sufficient to prove that, kernel of $\bar{\phi}$ is the right ideal generated by $\chi(\underline{\underline{Q}}_0)=-\underline{\underline{Q}}_0$, $\chi(\underline{\underline{Q}}_1)=-\underline{\underline{Q}}_1$. Let A_* denote the dual algebra of A, then by Milnor $A_*=Z_p[\xi_1,\xi_2,\cdots]\otimes\Lambda(\tau_0,\tau_1,\cdots)$. It is easy to show the following.

$$(\chi(A/A(Q_0, Q_1))^*=Z_p[\xi_1,\xi_2,\cdots]\otimes\Lambda(\tau_2,\tau_3,\cdots)\subset A_*.$$

Consider the algebra homomorphism, $\bar{\phi}_*: H_*(MSPL: Z_p)\to A_*$. Since dual basis of $\xi_1^{r_1}\xi_2^{r_2}\cdots\tau_0^{\varepsilon_0}\tau_1^{\varepsilon_1}$ is $\underline{\underline{Q}}_0^{\varepsilon_1}\underline{\underline{Q}}_1^{\varepsilon_1}\cdots P^R$, where $R=(r_1, r_2,\cdots)$. So it is sufficient to prove $\bar{\phi}(P^R)\neq 0$, and $\bar{\phi}(\underline{\underline{Q}}_j)\neq 0$ for $j\geqq 2$. But since in $H^*(MSO: Z_p)$, $\bar{\phi}(P^R)=\phi(\chi(P^R))=\chi(P^R)(u)\neq 0$. And by lemma 4-2, $\bar{\phi}(\underline{\underline{Q}}_j)=\phi(\chi(\underline{\underline{Q}}_j))=-\phi(\underline{\underline{Q}}_j)=-\underline{\underline{Q}}_j(u)\neq 0$ for $j\geqq 2$. This proves the theorem.

4.2. *Proof of lemma* 4-2. Let K is a CW complex of the form.

$$K=S^{pr-1}\underset{p}{\cup}e^{pr}\underset{\alpha_1}{\cup}e^{(p+1)r}\underset{p}{\cup}e^{(p+1)r+1},\quad r=2(p-1).$$

And let $f: K\to BSPL$ be the map which represents β_1 in $j\circ f\circ i: S^{pr-1}\to K\to BSPL\to BSF$. Then f is represented by a PL disk bundle E_f over K of fiber dim N, $N\gg 0$. And $X=X_N$ denotes the Thom complex of E_f. Then X_N is the following form,

$$X_N=S^N\underset{\beta_1}{\cup}e^{N+pr-1}\underset{p}{\cup}e^{N+pr}\underset{\alpha_1}{\cup}e^{N+(p+1)r}\underset{p}{\cup}e^{N+(p+1)r+1}.$$

Then the action of A on $H^*(X_N: Z_p)$ is the following, for $s\in H^N(X_N)$, $e_{pr-1}\in H^{N+pr-1}(X_N)$, $e_{pr}\in H^{N+pr}(X_N)$, $e_{(p+1)r}\in H^{N+(p+1)r}(X_N)$ and $e_{(p+1)r+1}\in H^{N+(p+1)r+1}(X_N)$.

i) $P^p(s)=e_{pr}$

ii) $P^1P^p(s)=P^{p+1}(s)=e_{(p+1)r}$, $P^pP^1(s)=0$

iii) $\delta P^{p+1}(s)=\delta P^1P^p(s)=e_{(p+1)r+1}$.

$P^{p+1}\delta(s)=P^pP^1\delta(s)=\delta P^pP^1(s)=P^p\delta P^1(s)=0$.

$P^1\delta P^p(s)=0$.

iv) $\delta(e_{pr-1})=e_{pr}$,

v) $P^1(e_{pr})=e_{(p+1)r}$, $\delta P^1(e_{pr})=e_{(p+1)r+1}$

vi) $\delta(e_{(p+1)r})=e_{(p+1)r+1}$.

So that the Milnor homomorphism $\lambda: H^*(X_N: Z_p)\to H^*(X_N: Z_p)\otimes A_*$ is given by the following.

i) $\lambda(s)=e\otimes 1+e_{pr}\otimes\xi_1^p+e_{(p+1)r}\otimes(\xi^{p+1}-\xi_2)$
$+e_{(p+1)r+1}\otimes(\xi_1^{p+1}\tau_0-\xi_2\tau_0-\xi_1^p\tau_1+\tau_2)$.

ii) $\lambda(e_{pr-1})=e_{pr-1}\otimes 1+e_{pr}\otimes\tau_0+e_{(p+1)r}\otimes\tau_1+e_{(p+1)r+1}\otimes\tau_1\tau_0$

iii) $\lambda(e_{pr})=e_{pr}\otimes 1+e_{(p+1)r}\otimes\xi_1+e_{(p+1)r+1}\otimes\xi_1\tau_0$

iv) $\lambda(e_{(p+1)r})=e_{(p+1)r}\otimes 1+e_{(p+1)r+1}\otimes\tau_0$

v) $\lambda(e_{(p+1)r+1})=e_{(p+1)r+1}\otimes 1$.

Now consider the following construction. Let $\pi: W\to B$ be a oriented PL disk bundle over B of fiber dim N. Then $W\underset{\pi_p}{\times}(E)^p\to W\underset{\pi_p}{\times}B^p$ is a PL disk bundle of fiber dim pN. Then the Thom complex of this bundle is of the form,

$$W\underset{\pi_p}{\ltimes}(ME\wedge\overset{p}{\cdots}\wedge ME)=W\underset{\pi_p}{\times}(ME\wedge\cdots\wedge ME)/W\underset{\pi_p}{\times}*,$$

where ME is the Thom complex of $\pi: E\to X$. If $u\in H^N(ME: Z_p)$ is the Thom class of $\pi: E\to X$, then $P(u)\in H^{pN}(W\underset{\pi_p}{\ltimes}(ME)^{(p)}: Z_p)$ is the Thom class of $W\underset{\pi_p}{\times}(E)^p\overset{p}{\to}W\underset{\pi_p}{\times}X^p$, where $P(u)$ is the Steenrod construction of u, c.f. Steenrod cohomology operations, *ch* VII.

Now consider the case $\pi_f: E=E_f\to K$. And consider the twisted diagonal map,

$$\Delta_1=\Delta\underset{\pi_p}{\ltimes}\Delta_p: W/\pi_p\ltimes X_N\longrightarrow W\underset{\pi_p}{\ltimes}(X_N)^{(p)}.$$

Then by the definition of the Steenrod reduced powers,

$$\Delta_1^*(P(s))=\sum_{j=0}(-1)^{N+j+mN(N+1)/2}(m!)^N\beta^{\frac{(N-2j)(p-1)}{2}}\otimes P^i(s),$$

$$+\sum_{j}(-1)^{N+j+mN(N+1)/2}(m!)^N\alpha\cdot\beta^{\frac{(N-2j)(p-1)}{2}-1}\otimes\delta P^j(s).$$

where $m=\frac{p-1}{2}$, $\alpha\in H^1(W/\pi_p:Z_p)$, $\beta\in H^2(W/\pi_p:Z_p)$.

By Milnor $\lambda(\alpha)=\alpha\otimes 1+\beta\otimes\tau_0+\cdots+\beta^{p^r}\otimes r_r+\cdots$. $\lambda(\beta)=\beta\otimes 1+\beta^p\otimes\xi_1+\cdots$. And $\Delta_1^*(P(s))=((-1)^{N+mN(N+1)/2}(m!)^N)[\beta^{\frac{1}{2}N(p-1)}\otimes s+\beta^{\frac{1}{2}N(p-1)-p(p-1)}\otimes e_{pr}+\beta^{\frac{1}{2}N(p-1)-(p+1)(p-1)}\otimes e_{(p+1)r}+\alpha\beta^{\frac{1}{2}N(p-1)-(p+1)(p-1)-1}\otimes(e_{(p+1)r+1})]$. Applying λ and using the fact that λ is a ring homomorphism we obtain,

$$\lambda(\Delta_1^*(P(s))=(-1)^{N+mN(N+1)/2}(m!)^N)[2\beta^{\frac{1}{2}N(p-1)}\otimes e_{(p+1)r+1}\otimes\tau_2$$
$$+\sum_{j\geq 3}\beta^{p^j}\cdot\beta^{\frac{1}{2}N(p-1)-p^2}\otimes e_{(p+1)r+1}\otimes\tau_j]$$
$$+\text{other term with respect to the last term}\cdots\otimes\xi_1^{r_1}\cdots\xi_s^{r_s}\tau_0^{\varepsilon_1}\tau_1^{\varepsilon_1}\cdots$$

So that $\underline{\underline{Q}}_j(\Delta_1^*(P(s))\neq 0$, so that $\underline{\underline{Q}}_jP(s)\neq 0$, for $j\geq 2$. Using naturality of Thom class, $\underline{\underline{Q}}_j(u)\neq 0$ for $u\in H^0(MSPL:Z_p)$. This proves the lemma.

§5. Proof of Lemma 2-10 and 2-11.

5-1. The main idea of this section is come from the work of Adames [1], and we use his results freely in this section.

Let $\pi: E\to X$ be a spin $(8n)$ bundle over a finite complex, then it is well known the existence of the fundamental Thom class in KO theory in the following form, [3].

(5-1) *There exists a Thom class $a(\pi)\in KO^{8n}(E, E-X)$ with the following property.*

i) *functorial*

ii) *multiplicative.*

iii) $\varphi_H^{-1}pha(\pi)=A(\pi)^{-1}$, *where $A(\pi)$ is the A polynomial of π.*

Now consider $\pi: E\to X$, a oriented real vector bundle with homotopy trivialization, $t:(E,E-X)\to X\times(R^{8n},R^{8n}-O)$. Consider the following element $\bar{\tau}(\pi)\in KO^0(X)$, defined by $\bar{\tau}(\pi)\otimes\eta_{8n}=(t^{-1})^*(a(\pi))\in KO^{8n}(X\times(R^{8n},R^{8n}-O))=KO^0(X)\otimes KO^{8n}(R^{8n},R^{8n}-O)$. Then it is easy to show that i) $\varepsilon(\bar{\tau}(\pi))=1\in K^0(p,t)$ ii) $\bar{\tau}(\pi\oplus 8)=\bar{\tau}(\pi)$ iii) $\bar{\tau}$ is functorial iv) $Ph(\bar{\tau}(\pi))=A(\bar{u})$. And passing to the limit we obtain a universal element $\bar{\tau}\in KO^0(F/O)$, $\varepsilon(\bar{\tau})=1$.

Now for any integer k, we define the H-map $\delta^k : BO_{\otimes} \to BO_{\otimes}$ by the formula, $\delta^k(1+\xi) = \Psi^k(1+\xi)/1+\xi$, where $1+\xi \in 1+\widetilde{KO}(BO_{\otimes})$ denotes the universal element.

Next for any integer k with $(k,p)=1$, we define a H-map $\varphi^k : BSO_{\oplus} \to BO_{\otimes(p)}$ by the following way. The isomorphism,

$$P^* : KO^{8n}(ESO(8n),\ ESO(8n)-BSO(8n))_P \to KO^{8n}(ESpin(8n),\ ESpin(8n)-BSpin(8n))_P.$$

define the Thom class $(p^{-1})^*(a(ESO(8n)) \in KO^{8n}(ESO(8n),\ ESO(8n)-BSO(8n))_P$, and we also write this Thom class by $a(ESO(8n))$. Then this element defines the Thom isomorphism $\varphi_{KO} : KO^0(BSO(8n))_P \to KO^{8n}(ESO(8n),\ ESO(8n)-BSO(8n))_P$ defined by $\varphi_{KO}(x) = \pi^*(x)\cdot a(ESO(8n))$. Then define $\varphi^k_{8n} : BSO(8n) \to BO_{\otimes(p)}$ by $\varphi^k_{8n} = \frac{1}{4n}\varphi^{-1}_{KO}\Psi^k(a(ESO(8n))$, then it is easy to show that $i^*\varphi^k_{8(n+1)} = \varphi^k_{8n}$ for $i : BSO(8n) \to BSO(8(n+1))$. So passing to the limit we obtain $\varphi^k : BSO \to BO_{\otimes(p)}$. Then it is easy to show the following, cf Adames [1].

PROPOSITION 5–2. *The following two diagrams are homotopy commutative.*

i)
$$\begin{tikzcd} F/O \arrow[r, "\bar{\tau}"] \arrow[d] & BO_{\otimes} \arrow[d, "\delta^k"] \\ BSO \arrow[r, "\varphi^k"] & BO_{\otimes(p)} \end{tikzcd}$$

ii)
$$\begin{tikzcd} BSO \arrow[r, "\varphi^k"] \arrow[d, "\Psi^k-1"] & BO_{\otimes(p)} \arrow[d, "\delta^k"] \\ BSO \arrow[r, "\varphi^k"] & BO_{\otimes(p)} \end{tikzcd}$$

Let $\gamma \to L_p$ and $\gamma \to CP^\infty$ denote the canonical complex line bundle and $\gamma_R \to L_p$, $\gamma_R \to CP^\infty$ denote the corresponding real vector bundle of dim 2, and $\xi_R \in KO(L_p)$ or $KO(CP^\infty)$ is the element $\xi_R = \gamma_R - 2$.

PROPOSITION 5–2. *In* $KO(L_p)_{(p)}$, $\varphi^{p+1}(\xi_R)$ *represent the element* $1+\frac{2}{p+1}\underline{\underline{\tilde{N}}}$, *where* $\underline{\underline{\tilde{N}}} \in \widetilde{KO}(L_p)_{(p)}$ *is the class corresponding the regular representation.*

Proof of this is due to the Theorem 5–9 of [1].

5–2. *Proof of lemma* 2–10. For $\xi_R \in KO(CP^\infty)$, consider the element $\varphi^{p+1}(\xi_R) \in 1+\widetilde{KO}(CP^\infty)_{(p)}$. And consider $(\Psi^{p+1}-1)(\xi_R)$, then by Adames conjecture, there is a map $g : CP^\infty \to F/O$ with the following commutative diagram.

$$\begin{tikzcd}
CP^{\infty} \arrow[r, "g"] \arrow[d, "\xi_R"] & F/O \arrow[r] \arrow[d, dash] & BSO_{\otimes} \arrow[d, "\delta^{p+1}"] \\
BSO \arrow[r, "\psi^{p+1}-1"] & BSO \arrow[r, "\varphi^{p+1}"] & BSO_{\otimes(p)}
\end{tikzcd}$$

Since $[CP^{\infty}, BO_{\otimes(p)}] \xrightarrow{\delta^{p+1}} [CP^{\infty}, BO_{\otimes(p)}]$ is monomorphism, the above commutative diagram and the following commutative diagram

$$\begin{tikzcd}
CP^{\infty} \arrow[r, "\xi_R"] & BSO \arrow[r, "\varphi^{p+1}"] \arrow[d, "\psi^{p+1}-1"] & BSO_{\otimes(p)} \arrow[d, "\delta^{p+1}"] \\
 & BSO \arrow[r, "\varphi^{p+1}"] & BSO_{\otimes(p)}
\end{tikzcd}$$

show that the two maps φ^{p+1}, ξ_R and $\bar{\tau}\circ g : CP^{\infty} \to BO_{\otimes(p)}$ is homotopic. So that $\bar{\tau}\circ g\circ \pi\ L_p \to CP^{\infty} \to F/O \to BO_{\otimes(p)}$ represents $1+\frac{2}{p+1}\underline{\underline{\tilde{N}}}$ by proposition 5–2. And since $L_p \xrightarrow{\pi} CP^{\infty} \xrightarrow{g} F/O \to BSO$ is homotopic to $L_p \xrightarrow{\pi} CP^{\infty} \xrightarrow{\xi_R} BSO \xrightarrow{\psi^{p+1}-1} BSO$, so that this map is trivial. So that $g\circ\pi : L_p \to F/O$ factors $L_p \xrightarrow{f} SF \to F/O$. And it is easy to show the following commutative diagram.

$$\begin{tikzcd}
SF \arrow[r, "i"] \arrow[d, "j"] & F/O \arrow[d, "\bar{\tau}"] \\
F/PL \arrow[r, "\bar{\sigma}"] & BO_{\otimes p}
\end{tikzcd}$$

So that $\bar{\sigma}\circ j\circ f : L_p \to BO_{\otimes(p)}$ is equal to $\bar{\tau}\circ i\circ f$, and $\bar{\tau}\circ i\circ f$ is equal to $\bar{\tau}\circ g\circ\pi$: $L_p \to CP^{\infty} \to F/O \to BO_{\otimes(p)}$ and this element represent $1+\frac{2}{p+1}\underline{\underline{\tilde{N}}}$. This shows the lemma.

5–3. *Proof of lemma* 2–11. We prove this lemma by induction on j. For $j=1$. Since $\bar{\sigma}\circ j\circ f : L_p \to SF \to F/PL \to BO_{\otimes(p)}$ represents $1+\underline{\underline{\tilde{N}}}$, so that $(\bar{\sigma}\circ j\circ f)^{*}(P_{\frac{p-1}{2}}) \neq 0$. So that $f_{*}(e_{2(p-1)}) = cx_1$ for some non zero $c \in Z_p$. So that $f_{*}(e_{2(p-1)-1}) = f_{*}(\beta_p e_{2(p-1)}) = c\beta_p x_1$. Suppose we can prove the lemma for $j < j_0$, $j_0 \geq 2$, we prove the case of j_0. Put $f_{*}(e_{2j_0(p-1)}) = c_{j_0}x_{j_0} + a_{j_0}$ and $f_{*}(e_{2j_0(p-1)-1}) = c_{j_0}\beta_p x_{j_0} + b_{j_0}$ for some $c_{j_0} \in Z_p$ and a_{j_0}, $b_{j_0} \in G_2$. We prove $c_{j_0} = c = c_1 = \cdots = c_{j_0-1}$. But the following lemma 5–4 shows that for some $1 \leq 1 < j_0$, $P_{*}^{k} e_{2j_0(p-1)} = de_{2(j_0-k)(p-1)}$, or $P_{*}^{k} e_{2j_0(p-1)-1} = de_{2(j_0-k)(p-1)-1}$ for some $0 \neq d \in Z_p$. Then for example $P_{*}^{k} f(e_{2j_0(p-1)}) = c_{j_0}P_{*}^{k}x_{j_0} + P_{*}^{k}(a_{j_0}) = c_{j_0}dx_{j_0-k} + P_{*}^{k}(a_{j_0})$ $P_{*}^{k} f(e_{2j_0(p-1)}) = f(P_{*}^{k}(e_{2j_0(p-1)})) = f(de_{2(j_0-k)(p-1)}) = d\,cx_{(j_0-k)} + da_{j_0-k}$.

But $P_*^k(a_{j_0}) \in G_2$ by definition of G_2 in [17] and by Nishida [11], so that $c_{j_0}d = dc$ and $c_{j_0} = c$. This prove the lemma.

LEMMA 5-3. *In $H_*(L_p, Z_p)$ and for any $j_0 > 1$, there is a integer $1 \leq k < j_0$ such that $P_*^k(e_{2j_0(p-1)}) \neq 0$ or $P_*^k(e_{2j_0(p-1)-1}) \neq 0$.*

Proof is easy.

§ 6. Appendix.

6-1. The object of this section is to prove propostion 1-4, the existence theorem for KO theory fundamental Thom class for oriented PL disk bundles. The essential idea of this section depends on the work of Sullivan [15].

At first we remember the result of Sullivan [15]. Let $\pi : E \to X$ be a oriented real vector bundle over a finite complex of fiber dim m. Then there is a fundamental Thom class $u(\pi) \in KO^m(X^E, *)_P$ with the following properties, where X^E is Thom complex of $\pi : E \to X$.

(6-1) i) *functorial.*

ii) *multiplicative.*

iii) $\varphi_H^{-1} phu(\pi) = L(\pi)^{-1} \in H^*(X, O)$.

Let $KO_*(\ )_P$ denote the homology KO theory localized at odd primes P, and make 4-graded by the same method (1-6). And $\Omega^*(\ )$, and $\Omega_*(\ )$ denote the oriented real cobordism and bordism theory. Then above Thom class induces following multiplicative cohomology and homology operations.

(6-2)
$$u : \Omega^*(\ ) \to KO^*(\ )_P$$
$$u : \Omega_*(\ ) \to KO_*(\ )_P.$$

By (6-1) iii) and Index theorem of Hirzebruch. The map $u : \Omega_*(p, t) = \Omega^*(p, t) \to KO_*(p, t)_P = KO^*(p, t) = Z[1/2]$ is the map defined by associating to each represented manifold its index. And we consider $Z[1/2]$ as a $\Omega_* = \Omega^*$ module by this map. Then the natural transformations in (6-2) define the following natural transformations.

(6-3)
$$u : \Omega^*(\ ) \underset{\Omega}{\otimes}{}_* Z[1/2] \to KO^*(\ )_P.$$
$$u : \Omega_*(\ ) \underset{\Omega_*}{\otimes} Z[1/2] \to KO_*(\ )_P.$$

Then the following proposition is due to Sullivan [15].

PROPOSITION 6-1. *The natural transformations in* (6-3) *give equivalence of functors.*

Now let $\pi : E \to X$ be a oriented real vector bundle of fiber dim m. Then we define the following map $\bar{u}$ by taking Kronecher index $\langle\ , u(\pi)\rangle$.

$$\text{(6-4)} \qquad \bar{u} : \Omega_p(E, \partial E) \xrightarrow{u} KO_p(E, \partial E)_P \xrightarrow{\langle\ , u(\pi)\rangle} KO_{p-m}(S^0)_P$$

$$\text{where } KO_{p-m}(S^0)_P = \begin{cases} Z[1/2] & \text{if} \quad p-m \equiv 0(4) \\ 0 & \text{if} \quad p-m \not\equiv 0(4). \end{cases}$$

Another map $\bar{\bar{u}}$ is defined by the following

$$\text{(6-5)} \qquad \bar{\bar{u}} : \Omega_p(E, \partial E) \to \begin{cases} Z[1/2] & p-m \equiv 0(4) \\ 0 & p-m \not\equiv 0(4). \end{cases}$$

If $x = (M^p, \partial M^p : f) \in \Omega_p(E, \partial E)$, we can take f satisfying the condition that f is t-regular to the zero section X of E. Then $\bar{\bar{u}}(x)$ is by definition index of $(f^{-1}(X))$. Then $\bar{\bar{u}}$ is well defined. And it is easy to prove the following proposition.

PROPOSITION 6-2. *The above two homomorphism $\bar{u}$ and $\bar{\bar{u}}$ coincide*

6-2. For any odd integer $q > 0$ introduce the mod q homology theories $\Omega_*(\ : Z_q)$ and $KO_*(\ : Z_q)$ as follows. Let $M_q = S^1 \cup_q e^2$ be the mod q Moore space, for a finite CW-pair (X, A), we define,

$$\text{(6-6)} \qquad \Omega_m(X, A : Z_q) = \varinjlim_N [M_q \wedge S^{N+m-2}, (X/A) \wedge MSO(N)]_0.$$

$$KO_m(X, A : Z_q) = \varinjlim_N [M_q \wedge S^{8N+m-2}, \ (X/A) \wedge (Z \times BO)]_0.$$

As in the case of $KO_*(\ )_P$, the homology theory $KO_*(\ : Z_q)$ is considered 4-graded by $\bar{\eta}_4 \in KO_4(S^0)_P$.

Since q is odd integer, by Araki-Toda [2], these modules $\Omega_*(X, A : Z_q)$ and $KO_*(X, A : Z_q)$ are Z_q modules.

And by the method of [2], the Bochstein homomorphism β_q, the reduction mod q homomorphism φ_q, and for $\alpha : Z_q \to Z_r$, an abelian group homomorphism, the reduction homomorphism φ_α can be defined.

$$\text{(6-7)} \qquad \beta_q : \Omega_m(X, A : Z_p) \to \Omega_{m-1}(X, A), \ KO_m(X, A : Z_q) \to KO_{m-1}(X, A).$$

$$\varphi_q : \Omega_m(X, A) \to \Omega_m(X, A : Z_q), \ KO_m(X, A) \to KO_m(X, A : Z_q)$$

$$\varphi_\alpha : \Omega_m(X, A : Z_p) \to \Omega_m(X, A : Z_r),\ KO_m(X, A : Z_p) \to KO_m(X, A : Z_r).$$

The homology operation u defined in 6-2 can be naturaly extendable to the following homology operation u_q.

(6-8) $$u_q : \Omega_*(\ \ : Z_q) \to KO_*(\ \ : Z_q).$$

And this homology operation u_q induces the following natural transformation.

(6-9) $$u_q : \Omega_*(\ \ : Z_q) \underset{\Omega_*}{\otimes} Z[1/2] \to KO_*(\ \ : Z_q).$$

Then proposition 6-1 induces,

PROPOSITION 6-3. *The natural transformation u_q in* (6-9) *is an equivalence of functors.*

6-3. Now we show the geometric interpretation of the homotopically defined homology theory $\Omega_*(\ \ : Z_q)$.

For finite CW-pair (X, A), a singular Z_q manifold of dimension m for (X, A) means the following system $(Q, f) = (Q, f, \varphi, \bar{M}_1)$ satisfying the following condition.

(6-10) i) $(Q, \partial Q)$ is a compact oriented differentiable manifold of dim m.

ii) $\partial Q = Q_0 \cup Q_1$, where M_0 and M_1 are regular $(m-1)$ submanifolds, and $Q_0 \cap Q_1 = \partial Q_0 = \partial Q_1$.

iii) $(\bar{M}_1, \partial \bar{M})$, compact oriented $(m-1)$ differentiable manifold, $\varphi : (\underset{q}{\cup} \bar{M}_1, \underset{q}{\cup} \partial \bar{M}_1) \to (Q_1, \partial Q_1)$ is an orientation preserving diffeomorphism. Where $\underset{q}{\cup}$ means disjoint union of q elements.

iv) $f : (Q, Q_0) \to (X, A)$, continuous map

v) For any inclusion $i : \bar{M}_1 \to \underset{q}{\cup} \bar{M}_1$, the composite map $f \circ \varphi \circ i$ is independent of this inclusion.

Then as in the usual case, the equivalence relation "bordant" can be defined. And we denote the set of equivalence classes of singular Z_q manifolds of dim m for (X, A) by $\Omega'_m(X, A : Z_q)$. Then this becomes an abelian group, and $\Omega'_*(X, A : Z_q)$ becomes a right $\Omega_*(p, t)$ module by defining the product of manifold.

PROPOSITION 6-4. *The functor $\Omega'_*(\ \ : Z_q)$ constitutes a generalized homology theory, and $\Omega'_*(p, t : Z_p) \cong \Omega_*(p, t) \underset{Z}{\otimes} Z_q$.*

Then by the same method in the case of $\Omega_*(\ \ )$, constructed in Conner-Floyd [7], we have the following.

PROPOSITION 6–5. *There is a natural equivalence,* $\tau : \Omega_*'(\ \ : Z_q) \to \Omega_*(\ \ : Z_q)$.

The reduction mod q homomorphism, $\varphi_q' : \Omega_m'(X, A) \to \Omega_m'(X, A : Z_q)$ can be defined by considering the ordinary singular manifolds as Z_q singular manifolds. And for the homomorphism $\alpha : Z_q \to Z_{qs}$ defined by $\alpha(1) = (s)$, the reduction homomorphism $\varphi_\alpha' : \Omega_m'(X, A : Z_q) \to \Omega_m'(X, A : Z_{qs})$ is defined by $\varphi_\alpha'((Q, f)) = ((\bigcup_s Q, \bigcup_s f))$. And the Bockstein homomorphism $\beta_q' : \Omega_m'(X, A : Z_q) \to \Omega_{m-1}(X, A)$ is defined by $\beta_q((Q, f, \varphi, \bar{M}_1)) = (\bar{M}_1, f\circ\varphi\circ i)$. Then φ_q' and φ_α' is compatible with φ_q and φ_α in (6–7), and β_q' and β_q are compatible up to sign.

6–4. Now we define the mod q index homomorphism $I_q : \Omega_*(p, t : Z_q) \to Z_q$ by the following way. Let $(M^m, \partial M)$ is a Z_q manifold, then we define $I_q(M^m)$ by

$$I_q(M^m) = \begin{cases} 0 & \text{if } m \not\equiv 0(4) \\ p_+ - p_-, \text{ mod } q & \text{if } m \equiv 0(4). \end{cases} \tag{6–11}$$

Where p_+ and p_- are the following numbers. Consider the following symmetric pairing,

$$H^{2n}(M, \partial M : R) \otimes H^{2n}(M, \partial M : R) \xrightarrow{u} H^{4n}(M, \partial M : R) \xrightarrow{\langle\ \ , u_M\rangle} R.$$

where $4n = \dim M$. Then $p_+ =$ the number of the positive eigen values of the above pairing, and p_- is the number of the negative eigen values.

PROPOSITION 6–6. *I_q is not depend on the representative, and define a map $I_q : \Omega_*(p, t : Z_q) \to Z_q$ and has the following property.*

i) $I_q(x + y) = I_q(x) + I_q(y)$

ii) $I_q(x, y) = I_q(x) \cdot I(y)$ *for* $x \in \Omega_*(p, t : Z_q)$, $y \in \Omega_*(p, t)$.

iii) $I_{qs}(\varphi_\alpha(x)) = \alpha I_q(x)$, *for* $x \in \Omega_*(p, t : Z_q)$ *and* $\alpha : Z_q \to Z_{qs}$ *defined by* $\alpha(1) = (s)$.

Let $\pi : E \to X$ be an oriented *PL* disk bundle over a finite complex of fiber dim m. We define the following homomorphism $\bar{u}_q$, $\bar{u}$, for odd integer $q > 1$.

$$\bar{u} : \Omega_n(E, \partial E) \to \begin{cases} Z & n - m \equiv 0(4) \\ 0 & n - m \not\equiv 0(4) \end{cases} \tag{6–12}$$

$$\bar{u}_q : \Omega_n(E, \partial E : Z_q) \to \begin{cases} Z_q & n-m \equiv 0(4) \\ 0 & n-m \not\equiv 0(4). \end{cases}$$

Let $(Q, f) \in \Omega_n(E, \partial E : Z_q)$, we can suppose f is t-regular to the zero-section X of E. Then $f^{-1}(X)$ define a element of $\Omega_{n-m}(p, t : Z_q)$. Define $\bar{u}_q((Q, f)) = I_q(f^{-1}(X))$. The same for $\bar{u}$. Then it is easy to show that $\bar{u}(x, y) = \bar{u}(x) \cdot I(y)$ for $x \in \Omega_*(E, \partial E)$, $y \in \Omega_*(p, t)$, and $\bar{u}_q(x, y) = \bar{u}_q(x) \cdot I(y)$, $x \in \Omega_*(E, \partial E : Z_q)$, $y \in \Omega_*(p, t)$. So that $\bar{u}_0$ and $\bar{u}_q$ define the following homomorphism.

(6–13)
$$\bar{u} : \Omega_*(E, \partial E) \underset{\Omega_*}{\otimes} Z[1/2] = KO_*(E, \partial E)_P \to \begin{cases} Z[1/2] & * - m \equiv 0(4) \\ 0 & * - m \not\equiv 0(4) \end{cases}$$

$$\bar{u}_q : \Omega_*(E, \partial E : Z_q) \underset{\Omega_*}{\otimes} Z[1/2] = KO_*(E, \partial E : Z_q) \to \begin{cases} Z_q & * - m \equiv 0(4) \\ 0 & * - m \not\equiv 0(4). \end{cases}$$

Then these $\bar{u}$ and $\bar{u}_q$ satisfy the following relations.

(6–14)
$$\bar{u}_q \circ \varphi_q = \alpha_q \cdot \bar{u} \qquad \alpha_q : Z \to Z_q = Z/qZ$$

$$\bar{u}_{qs} \circ \varphi_\alpha = \alpha \cdot \bar{u}_q \qquad \alpha : Z_q \to Z_{qs}, \ \alpha(1) = (s).$$

6–5. Now remember the following duality law for $KO^*(\ )_P$ and $KO_*(\ )_P$.

PROPOSITION 6–7. *For any finite CW-pair, There is a correspondence between the following set* i) *and* ii)

i) $u \in KO^m(X, A)_P$

ii) $\bar{u} \in Hom_{Z[1/2]}(KO_m(X, A)_P, \ Z[1/2])$,

$\bar{u}_q \in Hom_{Z_q}(KO_m(X, A : Z_q), Z_q)$, q : *odd integers satisfying the following relations.*

$$\bar{u}_q \circ \varphi_q = \alpha_q \circ \bar{u}_q \qquad \alpha_q : Z \to Z_q = Z/qZ$$

$$\bar{u}_{qs} \circ \varphi_\alpha = \alpha \cdot \bar{u}_q \qquad \alpha : Z_q \to Z_{qs}, \ \alpha(1) = (s),$$

And the correspondence is given by

$$u \to \begin{cases} \langle\ , u\rangle : KO_m(X, A)_P \to KO_0(S^0)_P = Z[1/2] \\ \langle\ , u\rangle : KO_m(X, A : Z_q) \to KO_0(S^0 : Z_q) = Z_q. \end{cases}$$

And these correspondence is functorial.

Proof of proposition 1–4. For PL disk bundle $\pi : E \to X$ of fiber dim m, consider $\bar{u}$, and $\bar{u}_q$ defined in (6–13). Then by (6–14) and proposition 6–7, there is an unique element $u(\pi) \in KO^m(E, \partial E)_P$. This element is what we want.

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