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CHARACTERISTIC CLASSES FOR SPHERICAL FIBER SPACES

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§ 0. Introduction and statement of results.

Let $SF = SG$ denote the space $\varinjlim SG(n)$, $SG(n) = \{f : S^{n-1} \to S^{n-1}; \text{degree } 1\}$, and BSF be the classifying space of SF. Our purpose is to determine $H_*(BSF : Z_p)$ as a Hopf algebra over Z_p, where p is an odd prime number. We have announced the main result in [14].

Let $Q_0S^0 = \varinjlim \Omega_0^n S^n$, where $\Omega_0^n S$ is the zero component of the n-th loop space of S^n. Then Q_0S^0 has the same homotopy type of SF. Dyer-Lashof [4] determined $H_*(Q_0S^0 : Z_p)$ as an algebra over Z_p, where p is an odd prime. $H_*(Q_0S^0 : Z_p)$ is a free commutative algebra generated by x_J, $J \in H$, where $H = \{J = (\varepsilon_1, j_1, \varepsilon_2, j_2, \cdots, \varepsilon_r, j_r)\}$, J satisfies the following properties.

(0-1) i) $r \geq 1$.

ii) $j_i \equiv 0 \bmod (p-1)$, $i = 1, 2, \cdots, r$.

iii) $j_r \equiv 0 \bmod 2(p-1)$.

iv) $(p-1) \leq j_1 \leq j_2 \leq \cdots \leq j_r$.

v) $\varepsilon_i = 0$ or 1.

vi) if $\varepsilon_{i+1} = 0$ then $j_i/(p-1)$ and $j_{i+1}/(p-1)$ are even parity.
if $\varepsilon_{i+1} = 1$ then $j_i/(p-1)$ and $j_{i+1}/(p-1)$ are odd parity.

The elements x_J are determined as follows. There is a continuous map $h_0 : L_p \to Q_0S^0$, where L_p is the mod p lens space of infinite dimension. Then x_j is by definition $h_{0*}(e_{2j(p-1)})$. And x_J is by definition $\beta_p^{\varepsilon_1} Q_{j_1} \beta_p^{\varepsilon_2} Q_{j_2} \cdots \beta_p^{\varepsilon_{r-1}} Q_{j_{r-1}} \beta_p^{\varepsilon_r} x_{j_r/2(p-1)}$, where $J = (\varepsilon_1, j_1, \cdots, \varepsilon_r, j_r) \in H$, and Q_j are the extended power operations defined by Dyer-Lashof, and β_p is Bockstein operation.

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We identify $H_*(Q_0S^0: Z_p)$ with $H_*(SF: Z_p)$ as Z_p module by i_*, where $i: Q_0S^0 \to SF$ is the homotopy equivalence, and we denote $\tilde{x} = i_*(x)$ for $x \in H_*(Q_0S^0 : Z_p)$.

The space SF becomes an H-space by composition of maps. The homotopy equivalence $i: Q_0S^0 \to SF$ is not an H-space map, so i_* is not an algebra homomorphism.

Our first object is to determine the algebra structure of $H_*(SF: Z_p)$. The result is the following theorem.

THEOREM 1. *$H_*(SF: Z_p)$ is a free commutative algebra generated by $\tilde{x}_J$, $J \in H$, even though i_* is not a ring homomorphism.*

To show this theorem, we proceed as follows. In §1, we study the relationship between the H-structures on Q_0S^0 and SF. And in §2, introducing a filtration on $H_*(Q_0S^0 : Z_p)$, mod this filtration we compute the multiplications on $H_*(Q_0S^0 : Z_p)$ and $H_*(SF: Z_p)$. We obtain the first theorem in §3.

The next object is to determine the Hopf algebra structure of $H_*(BSF: Z_p)$. Let H_1 be the subset of H consisting of $J = (\varepsilon_1, j_1, \cdots, \varepsilon_r, j_r)$ such that $j_1 \neq p-1$, and $r \geq 2$. Let $H_2 = \{(\varepsilon, p-1, 1, j) \in H\}$. And let $H_i^+ = \{J \in H_i, \deg x_J = \text{even}\}$, $H_i^- = \{J \in H_i, \deg x_J = \text{odd}\}$, $i = 1, 2$. Let $j: BSO \to BSF$ be the natural inclusion, then By Peterson-Toda [12], $\operatorname{Im} j_* = Z_p[\tilde{z}_1, \tilde{z}_2, \cdots]$, $\deg \tilde{z}_j = 2j(p-1)$, $\Delta \tilde{z}_j = \sum_{i=0}^{j} \tilde{z}_i \otimes \tilde{z}_{j-i}$.

THEOREM 2. i) $H_*(BSF: Z_p) = Z_p[\tilde{z}_1, \tilde{z}_2, \cdots] \otimes \Lambda(\sigma\tilde{x}_1, \sigma\tilde{x}_2, \cdots) \otimes C_*$. *$C_*$ is a free commutative algebra generated by $\sigma\tilde{x}_J$, $J \in H_1 \cup H_2$. $\sigma\tilde{x}_j$, $\sigma\tilde{x}_J$ are primitive elements, and $\Delta(\tilde{z}_j) = \sum_{i=0}^{j} \tilde{z}_i \otimes \tilde{z}_{j-i}$.*

ii) $H^*(BSF: Z_p) = Z_p[q_1, q_2, \cdots] \otimes (\Delta q_1, \Delta q_2, \cdots) \otimes C$

$C = \bigotimes_{I \in H_1^+ \cup H_2^+} \Lambda((\sigma(\tilde{x}_I))^*) \bigotimes_{J \in H_1^- \cup H_2^-} \Gamma_p[(\sigma(\tilde{x}_J))^*]$. *where $(\ )^*$ denotes the dual element, and q_j is the j-th Wu class.*

This theorem is proved using the Serre spectral sequence associated to the principal fibering, $SF \to ESF \to BSF$. In §4, we introduce the H_p^∞ structures $\bar{\theta}: W \times \pi_p(SF)^p \to SF$, and $W \times \pi_p(BSF)^p \to BSF$. Using this $\bar{\theta}$, we introduce, in §6, the extended p-th power $\bar{Q}_j$ on $H_*(SF: Z_p)$ and $H_*(BSF: Z_p)$. Related with this $\bar{Q}_j$, we formulate the Kudo's transgression theorem in proposition 6-1.

To compute the operations $\bar{Q}_j$ on $H_*(SF:Z_p)$, we study the map $\bar{\theta}: W\times\pi_p(SF)^p \to SF$, in §5, and using this we compute $\bar{Q}_{p-1}(x)$, $\bar{Q}_{p-2}(x)$ for $x\in H_*(SF:Z_p)$. Using these we obtain Theorem 2.

Peter May [7] independently succeeded to determine $H_*(BSF:Z_p)$.

In a forthcoming paper [15], we shall use the results of this paper to determine the characteristic classes for *PL* micro-bundles.

§1. *H*-space structures on $\Omega_0^n S^n$.

1-1. Let $SF(n)$ be the space of base point preserving continuous maps from S^n to S^n with degree 1, and $SG(n)$ be the space of continuous maps from S^{n-1} to S^{n-1} with degree 1. These spaces are given the compact open topology. Then $SF(n)$ and $SG(n)$ become topological monids by composition of maps. We shall define the suspension homomorphism, $SF(n)\to SF(n+1)$, and $SG(n)\to SG(n+1)$, as follows.

$$\begin{aligned} &f\in SF(n)\to f\wedge id_1\in SF(n+1). \\ (1\text{-}1)\qquad & \\ &g\in SG(n)\to g*id_0\in SG(n+1). \end{aligned}$$

where $\wedge$ and $*$ denote reduced join and join respectively and $id_1\in SF(1)$, $id_0\in SG(1)$ denote identity elements.

We shall introduce another *H*-space structures on $SG(n)$ and $SF(n)$ by join and reduced join respectively.

$$\begin{aligned} &SF(n)\times SF(n)\xrightarrow{\wedge} SF(2n) \\ (1\text{-}2)\qquad & \\ &SG(n)\times SG(n)\xrightarrow{*} SG(2n). \end{aligned}$$

We shall discuss various relations between these maps.

LEMMA 1-1. *The following diagrams are homotopy commutative.*

i)
$$\begin{array}{ccc} SF(n)\times SF(n) & \longrightarrow & SF(n+1)\times SF(n+1) \\ \downarrow\wedge & \wedge(id_2) & \downarrow\wedge \\ SF(2n) & \longrightarrow & SF(2n+2) \end{array}$$

ii)
$$\begin{array}{ccc} SG(n)\times SG(n) & \longrightarrow & SG(n+1)\times SG(n+1) \\ \downarrow * & *(id_1) & \downarrow * \\ SG(2n) & \longrightarrow & SG(2n+2) \end{array}$$

LEMMA 1–2. *The following diagrams are homotopy commutative.*

i)
$$\begin{tikzcd} SF(n)\times SF(n) \arrow[r, "\circ"] \arrow[dr, "\wedge"'] & SF(n) \arrow[d, "\wedge(id_n)"] \\ & SF(2n) \end{tikzcd} \qquad (f, g) \to g \circ f.$$

ii)
$$\begin{tikzcd} SG(n)\times SG(n) \arrow[r, "\circ"] \arrow[dr, "*"'] & SG(n) \arrow[d, "*(id_{n-1})"] \\ & SG(2n) \end{tikzcd} \qquad (f, g) \to g \circ f.$$

Let $i : SF(n) \to SG(n+1)$ be the natural inclusion, and $i : SG(n) \to SF(n)$ be the inclusion defined by $i(f) = f * id_0$ with base point $(0x \oplus 1z_1) \in S^{n-1} * S^0 = S^n$, $S^0 = \{z_1, z_2\}$.

LEMMA 1–3. *The following diagrams are homotopy commutative.*

i)
$$\begin{tikzcd} SF(n)\times SF(n) \arrow[r] \arrow[d, "\wedge"] & & SG(n+1)\times SG(n+1) \arrow[d, "*"] \\ SF(2n) \arrow[r] & SG(2n+1) \arrow[r] & SG(2n+2) \end{tikzcd}$$

ii)
$$\begin{tikzcd} SG(n)\times SG(n) \arrow[r] \arrow[d, "*"] & SF(n)\times SF(n) \arrow[d, "\wedge"] \\ SG(2n) \arrow[r] & SF(2n) \end{tikzcd}$$

LEMMA 1–4. *The following diagrams are homotopy commutative, that is the reduced join and join products on $SF(n)$ and $SG(n)$ are homotopy commutative.*

i)
$$\begin{tikzcd} SF(n)\times SF(n) \arrow[r, "\wedge"] \arrow[d, "T"] & SF(2n) \\ SF(n)\times SF(n) \arrow[ur, "\wedge"'] & \end{tikzcd}$$

ii)
$$\begin{tikzcd} SG(n)\times SG(n) \arrow[r, "*"] \arrow[d, "T"] & SG(2n) \\ SG(n)\times SG(n) \arrow[ur, "*"'] & \end{tikzcd}$$

It is well known that $SG(n)$ and $SF(n)$ have the same homotopy $(n-1)$ type. Therefore $SF = \varinjlim SF(n)$ and $SG = \varinjlim SG(n)$ have the same homotopy type, and $SF = SG$ has three H-space structures defined by composition of maps, reduced join and join, and these three H-structures are homotopic each other.

1–2. Next we shall consider iterated loop spaces. We denote the n-th loop space over X by $\Omega^n X$, where $\Omega^n X=\{l:(I^n,\partial I^n)\to(X,*)$: continuous maps$\}$. And we identify $\Omega^{n+1}X$ and $\Omega(\Omega^n X)$ by the following rule.

$$\text{(1–3)}\qquad \begin{aligned}&\Omega^{n+1}X\ni l,\quad \bar{l}\in\Omega(\Omega^n X)\\ &\bar{l}(t)(t_1,\cdots,t_n)=l(t,t_1,\cdots,t_n),\ (t,t_1,\cdots,t_n)\in I^{n+1}.\end{aligned}$$

We shall define loop product $\vee_j$ on $\Omega^n X$, $1\leq j\leq n$ by the following rule.

$$\text{(1–4)}\qquad \vee_j(l_1,l_2)(t_1,\cdots,t_n)=\begin{cases}l_1(t_1,\cdots,t_{j-1},\ 2t_j,\ t_{j+1},\cdots,t_n), & 0\leq t_j\leq 1/2.\\ l_2(t_1,\cdots,t_{j-1},\ 2t_j-1,\ t_{j+1},\cdots,t_n), & 1/2\leq t_j\leq 1.\end{cases}$$

We write $\vee$ for $\vee_1$. Denote $SX=X\wedge S^1$, and we define the natural inclusion $\Omega^n X\to\Omega^{n+1}SX$ by $l\to l\wedge id_1$

Let $\Omega^n_q S^n$ be the subspace of $\Omega^n S^n$ consisting of elements of degree q, for q any integer. And we shall identify $\Omega^n_1 S^n$ and $SF(n)$ canonically. We shall define the map $i_n:\Omega^n_0 S^n\to SF(n)$ by $l\to l\vee id_n$. It is well known that i_n is a homotopy equivalence, and it is easy to show that the following diagram is commutative.

$$\text{(1–5)}\qquad \begin{array}{ccc}\Omega^n_0 S^n & \xrightarrow{\ i_n\ } & SF(n)\\ \downarrow & & \downarrow\\ \Omega^{n+1}_0 S^{n+1} & \xrightarrow{\ i_{n+1}\ } & SF(n+1).\end{array}$$

Hence, we have a homotopy equivalence

$$\text{(1–6)}\qquad i:Q_0S^0\to SF.$$

We shall define the map $\overline{\wedge}_n:\Omega^n_0 S^n\times\Omega^n_0 S^n\to\Omega^{2n}_0 S^{2n}$ by the following diagram.

$$\text{(1–7)}\qquad \begin{array}{ccc}\Omega^n_0 S^n\times\Omega^n_0 S^n & \xrightarrow{\ i_n\times i_n\ } & SF(n)\times SF(n)\\ \downarrow{\scriptstyle\overline{\wedge}_n} & & \downarrow{\scriptstyle\wedge}\\ \Omega^{2n}_0 S^{2n} & \xleftarrow{\ (\vee(-id_{2n}))\ } & SF(2n),\end{array}$$

where $(-id_n)\in\Omega^n_{-1}S^n$ is the map defined by $(-id_n):(I^n,\partial I^n)\xrightarrow{\ \sigma\ }(I^n,\partial I^n)\xrightarrow{\ \phi_n\ }(S^n,*)$, where $\sigma(t_1,\cdots,t_n)=(1-t_1,t_2,\cdots t_n)$, and ϕ_n is the natural identification map. Then the following diagram is homotopy commutative.

(1-8)
$$\begin{tikzcd}
\Omega_0^n S^n \times \Omega_0^n S^n \arrow[r] \arrow[d, "\overline{\wedge}_n"] & \Omega_0^{n+1} S^{n+1} \times \Omega_0^{n+1} S^{n+1} \arrow[d, "\overline{\wedge}_{n+1}"] \\
\Omega_0^{2n} S^{2n} \arrow[r] & \Omega_0^{2n+2} S^{2n+2}
\end{tikzcd}$$

So that passing to the limit we obtain the map.

(1-9)
$$\overline{\wedge} : Q_0S^0 \times Q_0S^0 \longrightarrow Q_0S^0.$$

Our first proposition is the following structure theorem of $\overline{\wedge}_n$.

PROPOSITION 1.5. *The following diagram is homotopy commutative.*

(1-10)
$$\begin{tikzcd}
\Omega_0^n S^n \times \Omega_0^n S^n \arrow[r, "\overline{\wedge}_n"] \arrow[d, "\triangle \times \triangle"] & \Omega_0^{2n} S^{2n} \\
\Omega_0^n S^n \times \Omega_0^n S^n \times \Omega_0^n S^n \times \Omega_0^n S^n \arrow[d, "id \times T \times id"] & \Omega_0^{2n} S^{2n} \times \Omega_0^{2n} S^{2n} \arrow[u, "\vee"] \\
\Omega_0^n S^n \times \Omega_0^n S^n \times \Omega_0^n S^n \times \Omega_0^n S^n \arrow[r, "\wedge \times \vee"] & \Omega_0^{2n} S^{2n} \times \Omega_0^n S^n \arrow[u, "id \times (\wedge id_n)"]
\end{tikzcd}$$

Passing to the limit we obtain the following corollary.

COROLLARY 1-6. *The following diagram is homotopy commutative.*

(1-11)
$$\begin{tikzcd}
Q_0S^0 \times Q_0S^0 \arrow[r] \arrow[d, "\triangle \times \triangle"] & Q_0S^0 \\
Q_0S^0 \times Q_0S^0 \times Q_0S^0 \times Q_0S^0 \arrow[d, "id \times T \times id"] & \\
Q_0S^0 \times Q_0S^0 \times Q_0S^0 \times Q_0S^0 \arrow[r, "\wedge \times \vee"] & Q_0S^0 \times Q_0S^0 \arrow[uu, "\vee"]
\end{tikzcd}$$

We shall consider the relation between the loop product and the reduced join product. Roughly speaking, it is distributive law.

PROPOSITION 1-7. *The following diagrams are homotopy commutative.*

i)

(1-12)
$$\begin{tikzcd}
\Omega^n K \times (\Omega^m L \times \Omega^m L) \arrow[r, "id \times (\vee)"] \arrow[d, "\triangle \times id"] & \Omega^n K \times \Omega^m L \arrow[d, "\wedge"] \\
(\Omega^n K \times \Omega^n K) \times (\Omega^m L \times \Omega^m L) \arrow[d, "id \times T \times id"] & \Omega^{n+m}(K \wedge L) \\
\Omega^n K \times \Omega^m L \times \Omega^n K \times \Omega^m L \arrow[r, "\wedge \times \wedge"'] & \Omega^{n+m}(K \wedge L) \times \Omega^{n+m}(K \wedge L). \arrow[u, "\vee"]
\end{tikzcd}$$

ii)
$$\begin{array}{ccc}
(\Omega^n K\times\Omega^n K)\times\Omega^m L & \xrightarrow{(\vee)\times id} & \Omega^n K\times\Omega^m L \\
\Big\downarrow{\scriptstyle id\times\triangle} & & \Big\downarrow{\scriptstyle\wedge} \\
\Omega^n K\times\Omega^n K\times\Omega^m L\times\Omega^m L & & \Omega^{n+m}(K\wedge L) \\
\Big\downarrow{\scriptstyle id\times T\times id} & & \Big\uparrow{\scriptstyle\vee} \\
\Omega^n K\times\Omega^m L\times\Omega^n K\times\Omega^m L & \xrightarrow{\wedge\times\wedge} & \Omega^{n+m}(K\wedge L)\times\Omega^{n+m}(K\wedge L)
\end{array}$$

2–3. Let $\bar{\Omega}^n X$ denote the iterated n-th Moore loop space. We can interprete an element $l\in\bar{\Omega}^n X$ as follows. $l:(U_l,\partial U_l)\to(X,*)$, where U_l is a certain closed subset of R^n depending on l. It is well known that the natural inclusion $\Omega^n X\to\bar{\Omega}^n X$ is a homotopy equivalence, and up to homotopy this map preserves the H-space structure defined by the loop product.

We shall define the reduce join product $\wedge:\bar{\Omega}^m X\times\bar{\Omega}^n Y\to\bar{\Omega}^{m+n}(X\wedge Y)$ by the following rule, for $l_1\in\bar{\Omega}^m X$, $l_2\in\bar{\Omega}^n Y$.

(1–13) $$(l_1\wedge l_2):(U_{l_1}\times U_{l_2},\ \partial(U_{l_1}\times U_{l_2}))\to(X\wedge Y,*).$$

Then the natural inclusion $\Omega^n X\to\bar{\Omega}^n X$ is compatible with the reduced join product. We shall define the suspension map $\bar{\Omega}^n X\to\bar{\Omega}^{n+1}(SX)$ as follows, $l\to l\wedge id_1$. Then this is compatible with the natural inclusion $\Omega^n X\to\bar{\Omega}^n X$.

We consider the result of Dyer-Lashof [4] about the iterated loop spaces. Let $\sum_q$ denote the permutation group of q-elements, and $J^n\sum_q$ denote the n-th join of $\sum_q$ with itself. We consider $J^n\sum_q$ as a subset of $J^{n+1}\sum_q$ by the following rule, $J^n\sum_q\ni(t_1\sigma_1\oplus\cdots\oplus t_n\sigma_n)=(0\oplus t_1\sigma_1\oplus\cdots\oplus t_n\sigma_n)\in J^{n+1}\sum_q$. Dyer-Lashof proved that $\bar{\Omega}^n X$ is an H^{n-1}-space in their sense, so that there exists a continuous map.

(1–14) $$\theta_q^{n-1}:J^n\textstyle\sum_q\times(\bar{\Omega}^n X)^q\to\bar{\Omega}^n X$$

with the following properties.

i) $\sum_q$ equivariant i.e. for each $\sigma\in\sum_q$,

(1–15) $$\begin{aligned}&\theta_q^{n-1}(t_1\sigma_1\oplus\cdots\oplus t_n\sigma_n;\ l_1,\cdots,l_q)\\ &=\theta_q^{n-1}(t_1\sigma_1\sigma^{-1}\oplus\cdots\oplus t_n\sigma_n\sigma^{-1},\ l_{\sigma(1)},\cdots,l_{\sigma(q)})\end{aligned}$$

ii) normalized i.e. for each $\sigma\in\sum_q$

$$\theta_q^{n-1}(0\oplus\cdots\oplus 0\oplus 1\cdot\sigma;\ l_1,\cdots,l_q)=l_{\sigma(1)}\vee\cdots\vee l_{\sigma(q)}.$$

We shall consider the relation between θ_q^{n-1} and reduced join, we obtain the following proposition.

PROPOSITION 1–8. *The following diagram is homotopy commutative.*

$$\begin{tikzcd}
(J^n\Sigma_q \times_{\Sigma_q} (\bar{\Omega}^n K)^q)\times \bar{\Omega}^m L \arrow[r, "\theta\times id"] \arrow[d] & \bar{\Omega}^n K\times \bar{\Omega}^m L \arrow[r, "\wedge"] & \bar{\Omega}^{n+m}(K\wedge L) \\
(J^n\Sigma_q \times_{\Sigma_q} ((\bar{\Omega}^n K)^q\times \bar{\Omega}^m L) \arrow[d, "id\times id\times \triangle_q"] & & J^{n+m}\Sigma_q \times_{\Sigma_q} (\bar{\Omega}^{n+m}(K\wedge L))^q \arrow[u, "\theta"'] \\
J^n\Sigma_q \times_{\Sigma_q} ((\bar{\Omega}^n K)^q\times (\bar{\Omega}^m L)^q) \arrow[rr] & & J^n\Sigma_q \times_{\Sigma_q} (\bar{\Omega}^n K\times \bar{\Omega}^m L)^q \arrow[u, "i\times(\wedge)^q"']
\end{tikzcd} \tag{1–16}$$

Proof. At first we shall remark that the following diagram is commutative by the definition of inclusion $J^n\Sigma_q\to J^{n+m}\Sigma_q$ and naturality of θ_q^n with respect to the iterated loop map.

$$\begin{tikzcd}
J^n\Sigma_q \times_{\Sigma_q} (\bar{\Omega}^n(\bar{\Omega}^m(K\wedge L))^q \arrow[r, "i\times id"] \arrow[d, "\theta"] & J^{n+m}\Sigma_q \times_{\Sigma_q} (\bar{\Omega}^{n+m}(K\wedge L))^q \arrow[d, "\theta"] \\
\bar{\Omega}^n(\bar{\Omega}^m(K\wedge L)) \arrow[r] & \bar{\Omega}^{n+m}(K\wedge L))
\end{tikzcd}$$

Fix an element $l\in\bar{\Omega}^m L$, and define the map $l_\sharp : K\to \bar{\Omega}^m(K\wedge L)$ by the following way,

$$l_\sharp(x) : (U_l, \partial U_l)\to K\wedge L, \qquad x\in K.$$
$$l_\sharp(x)(t_1,\cdots,t_m) = (x\wedge l(t_1,\cdots,t_m)).$$

Consider $\bar{\Omega}^n(l_\sharp) : \bar{\Omega}^n K\to\bar{\Omega}^n(\bar{\Omega}^m(K\wedge L))$, Then it is easy to see that $\bar{\Omega}^n(l_\sharp)(l_1)= l_1\wedge l$, $l\in\bar{\Omega}^n K$. Naturality of θ_q^{n-1} under n-th iterated loop map shows that the following diagram is commutative.

$$\begin{tikzcd}
J^n\Sigma_q \times_{\Sigma_q} (\bar{\Omega}^n K)^q \arrow[r, "id\times(\bar{\Omega}^n)(l_\sharp)^q"] \arrow[d, "\theta"] & J^n\Sigma_q \times_{\Sigma_q} (\bar{\Omega}^n(\bar{\Omega}^m(K\wedge L))^q \arrow[d, "\theta"] \\
\bar{\Omega}^n K \arrow[r, "\bar{\Omega}^n(l_\sharp)"] & \bar{\Omega}^n(\bar{\Omega}^m(K\wedge L))
\end{tikzcd}$$

The commutative diagram and the above remarks show the following

$$\begin{aligned}
&\theta_q^{n-1}(\omega;\, l_1,\cdots, l_q)\wedge l \\
&= \bar{\Omega}^n(l_\sharp)(\theta_q^{n-1}(w, l_1;\,\cdots, l_q)) \\
&= \theta_q^{n-1}(\omega, \bar{\Omega}^n(l_\sharp)(l_1);\,\cdots, \bar{\Omega}^n(l_\sharp)(l_q)) \\
&= \theta_q^{n-1}(\omega, l_1\wedge l;\,\cdots, l_q\wedge l) \\
&= \theta_q^{n+m-1}(\omega, l_1\wedge l;\,\cdots, l_q\wedge l).
\end{aligned}$$

This shows the proposition.

Let π_q denote the cyclic group of order q. $Q(X) = \varinjlim \Omega^n(S^n X)$, $\bar{Q}(X) = \varinjlim \bar{\Omega}^n(S^n X)$. $Q_j S^0 = \varinjlim \Omega_j^n S^n$, $\bar{\Omega}_j S^0 = \varinjlim \bar{\Omega}_j^n S^n$. We shall define $h : J^n\pi_q/\pi_q \to \bar{\Omega}_q^n S^n$ by the following rule.

$$h : J^n\pi_q/\pi_q \to J^n\pi_q \times_{\pi_q} (id_n)^q \to J^n\pi_q \times_{\pi_q} (\bar{\Omega}_1^n S^n)^q \xrightarrow{\theta} \bar{\Omega}_q S^n.$$

And passing limit, we obtain $h : J^\infty\pi_q/\pi_q \to \bar{Q}_q S^0$, and define $h_0 : J^n\pi_q/\pi_q \to \bar{\Omega}_0^n S^n$ by the following, $h_0 : J^n\pi_q/\pi_q \xrightarrow{h} \bar{\Omega}_q S^n \xrightarrow{\vee(-q\,id_n)} \bar{\Omega}_0 S^n$, and as a limit, we obtain $h_0 : J^\infty\pi_q/\pi_q \to \bar{\Omega}_0 S^0$.

PROPOSITION 1–9. *The following diagram is commutative.*

$$(1\text{–}17)\qquad \begin{array}{ccccc} (J^n\pi_q/\pi_q)\times\bar{\Omega}^m K & \xrightarrow{h\times id} & \bar{\Omega}_q^n S^n \times \bar{\Omega}^m K & \xrightarrow{\wedge} & \bar{\Omega}^{n+m}(S^n \wedge K). \\ \big\downarrow{\scriptstyle id\times\triangle_q} & & & & \big\uparrow{\scriptstyle\theta} \\ J^n\pi_q \times_{\pi_q} (\bar{\Omega}^m K)^q & & \xrightarrow{\quad i\times(id_n\wedge)^q \quad} & & J^{n+m}\pi_q \times_{\pi_q} (\bar{\Omega}^{n+m}(S^n\wedge K))^q \end{array}$$

Proof of this proposition is the same as the proof of Proposition 1–8.

We shall consider the case $K = S^m$ and passing to the limit, we obtain the following corollary.

COROLLARY 1–10. *The following diagram is homotopy commutative.*

$$(1\text{–}18)\qquad \begin{array}{ccccc} (J^\infty\pi_q/\pi_q)\times\bar{Q}_0 S^0 & \xrightarrow{h\times id} & \bar{Q}_q S^0 \times \bar{Q}_0 S^0 & \xrightarrow{\wedge} & \bar{Q}_0 S^0 \\ \big\downarrow{\scriptstyle id\times\triangle_q} & & & \nearrow{\scriptstyle\theta} & \\ J^\infty\pi_q \times_{\pi_q} (\bar{Q}_0 S^0)^q & & & & \end{array}$$

It is easy to prove the following proposition.

PROPOSITION 1–11. *We have the following commutative diagram.*

$$(1\text{–}19)\qquad \begin{array}{ccccc} (J^n\pi_q/\pi_q)\times\bar{\Omega}^m K & \xrightarrow{h_0\times id} & \bar{\Omega}_0^n S^n \times \bar{\Omega}^m K & \xrightarrow{\wedge} & \bar{\Omega}^{n+m}(S^n\wedge K) \\ \big\downarrow{\scriptstyle\triangle_{q+1}} & & & & \big\uparrow{\scriptstyle\vee} \\ (J^n\pi_q/\pi_q\times\bar{\Omega}^m K)^{q+1} & & & & \bar{\Omega}^{n+m}(S^n\wedge K)\times(\bar{\Omega}^{n+m}(S^n\wedge K))^q \\ \big\downarrow & & & & \big\uparrow{\scriptstyle id\times((-id_n)\wedge)^q} \\ (J^n\pi_q/\pi_q\times\bar{\Omega}^m K)\times(J^n\pi_q/\pi_q\times\bar{\Omega}^m K)^q & & \xrightarrow{(h\wedge id)\times(\pi_2)^q} & & \bar{\Omega}^{n+m}(S^n\wedge K)\times(\bar{\Omega}^m K)^q \end{array}$$

§ 2. Filtration on $H_*(Q_0S^0; Z_p)$.

2–1. In this chapter, p denotes an odd prime number unless otherwise stated. Let C denote $H_*(Q_0S^0; Z_p)$ as a Hopf algebra over Z_p. It is well known that $H_i(J^\infty\pi_p/\pi : Z_p) = Z_p$, $i = 0, 1, 2, \cdots$. We shall serect generators $e_i \in H_i(J^\infty\pi_p/\pi_p : Z_p)$ with the following properties.

(2–1) i) $e_0 = 1$ ii) $\triangle(e_j) = \sum_{i=0}^{j} e_i \otimes e_{j-i}$ iii) $\beta_p e_{2j} = e_{2j-1}$.

where β_p is Bockstein operation.

Dyer-Lashof [4] defined on $H_*(X : Z_p)$, the extended p-th power operations $Q_j^{(p)} = Q_j$, $j = 1, \cdots n$, with the following properties, where X is a H_p^n space in their sense.

(2–2)

1) $Q_j : H_k(X, Z_p) \longrightarrow H_{pk+j}(X, Z_p)$,
2) Q_j is a homomorphism for $j \leq n-1$,
3) Q_0 is the Pontrjagin p-th power,
4) $Q_{2j-1} = \beta_p Q_{2j}$, $2j \leq n-1$, β_p is Bockstein operation,
5) $x \in H_r(X, Z_p)$, $Q_{2j}(x) = 0$ unless the change in dimension, $2j+pr-r$ is an even multiple of $p-1$,
6) Cartan formula:

$X, Y : H_p^n$-space, $x \in H_r(X, Z_p)$, $y \in H_s(Y, Z_p)$, $2j < n$ then

$$Q_{2j}(x \otimes y) = (-1)^{rs(p-1)/2} \sum_{i=0}^{j} Q_{2i}(x) \otimes Q_{2j-2i}(y).$$

For $J = (\varepsilon_1, j_1, \cdots, \varepsilon_r, j_r)$, $\varepsilon_i = 0$ or 1 and $j_k \geq 0$, we denote $Q_J = \beta_p^{\varepsilon_1} Q_{j_1} \cdots \beta_p^{\varepsilon_r} Q_{j_r}$.

We shall now formulate the Adem relations for $Q_j's$. At first we shall comment on the homology of symmetric group.

Let X be a connected finite CW-complex and $x_1, x_2, \cdots \in H_*(X, Z_p)$ be a basis of Z_p-module consisting of homogenous elements. Then $e_i \underset{\pi}{\otimes} x_j^p$, $i \geq 0$, $j \geq 1$, and $e_0 \underset{\pi}{\otimes} x_{j_1} \otimes \cdots \otimes x_{j_p}$, is a basis of $H_*(J^\infty\pi_p \times_{\pi_p} X^p, Z_p)$, where not all the $j_1, \cdots, j_p$ are equal and $(j_1, \cdots, j_p)$ runs through all representative classes obtained by cyclic permutations of the indices. As the chapter VIII of Steenrod [13], we can obtain the following lemma.

LEMMA 2-1. *X is as above. Let $d : J^\infty\pi_p/\pi_p \times X \rightarrow J^\infty\pi_p \times_{\pi_p} X^p$ be the twisted diagonal map. Then the image of $d_* : H_*(J^\infty\pi_p/\pi_p \times X, Z_p) \rightarrow H_*(J^\infty \times_{\pi_p} X^p, Z_p)$ coincides with the sub-module generated by $e_j \otimes_\pi x_j^p$, $i \geq 0$, $j \geq 1$.*

LEMMA 2-2.

Let $\mu : J^\infty\pi_p \times_{\pi_p} (J^\infty\pi_p/\pi_p)^p \to J^\infty\left(\pi_p \int \pi_p\right) \Big/ \pi_p \int \pi_p \to J^\infty(\Sigma_{p^2})/\Sigma_{p^2}$ *be the natural inclusion. Then the following relations holds on*

$$\mu_* : H_*(J^\infty\pi_p \times_{\pi_p} (J^\infty\pi_p/\pi_p)^p : Z_p) \to H_*(J^\infty(\Sigma_{p^2})/\Sigma_{p^2} : Z_p).$$

(2-3) a) $\mu_*(e_i \otimes_\pi (e_j)^p) = 0$ *unless* (i, j) *is of the form* $(2s(p-1)-\varepsilon,\ 2t(p-1))$; $s \geq 0$, $t \geq 0$, $\varepsilon = 0$ *or* 1, *or* $((2s+1)(p-1)-\varepsilon,\ 2t(p-1)-1)$; $s \geq 0$, $t \geq 1$, $\varepsilon = 0$ *or* 1.

b) $t > s(p+1)$, $s \geq 0$

$$\mu_*(e_{(2t-2sp)(p-1)} \otimes_\pi (e_{2s(p-1)})^p)$$
$$= \sum_{k=[(t-s)/p]}^{[t/p]} (-1)^{k+s+t} \binom{(k-s)(p-1)-1}{kp+s-t} \mu_*(e_{(2t-2kp)(p-1)} \otimes_\pi (e_{2k(p-1)})^p).$$

c) $t \geq s(p+1)$, $s \geq 0$, $m = (p-1)/2$.

$$-m!\mu_*(e_{(2t+1-2sp)(p-1)} \otimes_\pi (e_{2s(p-1)-1})^p)$$
$$= \sum_{k=[(t-s)/p]}^{[t/p]} (-1)^{k+s+t} \binom{(k-s)(p-1)}{kp+s-t} \mu_*(e_{(2t-2kp)(p-1)-1} \otimes_\pi (e_{2k(p-1)})^p)$$
$$+ \sum_{k=[(t-s+1)/p]}^{[t/p]} (-1)^{k+s+t} \binom{(k-s)(p-1)-1}{kp+s-t} m!\mu_*(e_{(2t+1-2kp)(p-1)} \otimes_\pi (e_{2k(p-1)-1})^p)$$

Now the Adem relations are formulated as follows.

PROPOSITION 2-3. *Let X be an H^∞-space. Then we have the following relations.*

1) $x \in H_*(X, Z_p)$, $deg\ x = even \geq 0$,

a) $t > s(p+1)$, $s \geq 0$,

(2-4) $$Q_{(2t-2sp)(p-1)}Q_{2s(p-1)}(x)$$
$$= \sum_{k=[(t-s)/p]}^{[t/p]} (-1)^{k+s+t} \binom{(k-s)(p-1)-1}{kp+s-t} Q_{(2t-2kp)(p-1)}Q_{2k(p-1)}(x)$$

b) $t \geq s(p+1)$, $s > 0$, $m = (p-1)/2$,

$$-m!Q_{(2t+1-2sp)(p-1)}\beta_p Q_{2s(p-1)}(x)$$
$$= \sum_{k=[(t-s)/p]}^{[t/p]} (-1)^{k+s+t} \binom{(k-s)(p-1)}{kp+s-t} \beta_p Q_{(2t-2kp)(p-1)}Q_{2k(p-1)}(x)$$
$$+ \sum_{k=[(t-s+1)/p]}^{[t/p]} (-1)^{k+s+t} \binom{(k-s)(p-1)-1}{kp+s-t} m!Q_{(2t+1-2kp)(p-1)}\beta_p Q_{2k(p-1)}(x)$$

2) $x \in H_*(X, Z_p)$ $deg\, x = odd > 0$,

c) $t > s(p+1) + m + 1$, $s \geqq 0$, $m = (p-1)/2$,

$$Q_{(2t-(2s+1)p)(p-1)} Q_{(2s+1)(p-1)}(x)$$
$$= \sum_{k=[(t-s+1)/p]}^{[t/p-1/2]} (-1)^{m+k+s+t+1} \binom{(k-s)(p-1)-1}{kp+s-t+m-1} Q_{(2t-(2k+1)p)(p-1)} Q_{(2k+1)(p-1)}(x),$$

d) $t \geqq s(p+1) + m + 1$, $s \geqq 0$, $m = (p-1)/2$,

$$-m!Q_{(2t+1-(2s+1)p)(p-1)} \beta_p Q_{(2s+1)(p-1)}(x)$$
$$= \sum_{k=[(t-s-m-1)/p]}^{[t/p-1/2]} (-1)^{m+k+s+t+1} \binom{(k-s)(p-1)}{kp+s-t+m+1} \beta_p Q_{(2t-(2k+1)p)(p-1)} Q_{(2k+1)(p-1)}(x)$$
$$+ \sum_{k=[(t-s-m)/p]}^{[t/p-1/2]} (-1)^{m+k+s+t+1} \binom{(k-s)(p-1)-1}{kp+s-t+m} m! Q_{(2t+1-(2k+1)p)(p-1)} \beta_p Q_{(2k+1)(p-1)}(x)$$

On S^{2n+1}, cyclic group π_p acts freely in standard way, and S^{2n+1} has the CW-complex structure with p-cells in each dimension, and π_p acts cellularly. We denote this π_p CW-complex by $W^{(2n+1)}$, and put $W = \lim W^{(2n+1)}$. We fix a π_p equivariant homotopy equivalence $W \to J^\infty \pi_p$, and we identify these spaces, and hence identify $L_p = W/\pi_p$ and $J^\infty \pi_p/\pi_p$. In § 1 we define a continuous map $h_0 : L_p = J^\infty \pi_p/\pi_p \to \bar{Q}_0 S^0$. As in § 0, we define $x_j \in H_{2j(p-1)}(Q_0 S^0 : Z_p)$ by $x_j = h_{0*}(e_{2j(p-1)})$, $j = 1, 2, \cdots$, and $x_J = \beta_p^{\varepsilon_1} Q_{j_1} \cdots \beta_p^{\varepsilon_{r-1}} Q_{j_{r-1}} \beta_p^{\varepsilon_r} x_{j_r/2(p-1)}$ for $J \in H$, $J = (\varepsilon_1, j_1, \cdots, \varepsilon_r, j_r)$.

In $H_*(Q_0 S^0 : Z_p)$, the Adem relations between x_j and Q_j are following.

Proposition 2–4. *In $H_*(Q_0 S^0 : Z_p)$, the following relations hold.*

a) $t > s(p+1)$, $s > 0$.

(2–5)
$$Q_{(2t-2sp)(p-1)}(x_s)$$
$$= \sum_{k=[(t-s)/p]}^{[t/p]} (-1)^{k+s+t} \binom{(k-s)(p-1)-1}{kp+s-t} Q_{(2t-2kp)(p-1)}(x_k)$$
$$+ \sum_{r>0} (x_r)^p y_r, \quad y_r \in H_*(Q_0 S^0 : Z_p).$$

b) $t \geqq s(p+1)$, $s > 0$, $m = (p-1)/2$.

$$-m!\, Q_{(2t+1-2sp)(p-1)}(\beta_p x_s)$$
$$= \sum_{k=[(t-s)/p]}^{[t/p]} (-1)^{k+s+t} \binom{(k-s)(p-1)}{kp+s-t} \beta_p Q_{(2t-2kp)(p-1)}(x_k)$$
$$+ \sum_{k=[(t-s+1)/p]}^{[t/p]} (-1)^{k+s+t} \binom{(k-s)(p-1)-1}{kp+s-t} m! Q_{(2t+1-2sp)(p-1)}(\beta_p x_k)$$
$$+ \sum_{r>0} x_r^p y_r, \qquad y_r \in H_*(Q_0 S^0 : Z_p).$$

2-2. We shall define a filtration in C as follows;

(2-6) 1) $C = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots$

2) $G_1 = \ker \varepsilon$ where $\varepsilon : C \to Z_p$ is the augmentation.

3) $\omega(x_J) = p^r$, where $J \in H$, $J = (\varepsilon_1, j_1, \cdots, \varepsilon_{r+1}, j_{r+1})$ and $\omega(x) = \inf_q \{q \,;\, x \in G_q\}$ for $x \in C$.

4) $\omega(x_{J_1}{}^{k_1} \cdots x_{J_r}{}^{k_r}) = \sum_{i=1}^{r} k_i \omega(x_{J_i})$, $J_i \in H$, $k_i \geqq 1$.

if $\deg x_{J_s} =$ odd then $k_s = 1$.

Then C become a filtered algebra, i.e. $\omega(x \cdot y) \geqq \omega(x) + \omega(y)$. And E_0C denotes the associated graded algebra. Then we have easily obtain the following proposition.

PROPOSITION 2-5. *E_0C is a free commutative algebra generated by $\{x_J\}, J \in H$.*

By the definition of the filtration on C and by Proposition 2-3 and 2-4 we obtain the following proposition

PROPOSITION 2-6. *If $x \in C$ belongs to G_q, and $J = (\varepsilon_1, j_1, \cdots, \varepsilon_r, j_r)$, $\varepsilon_i = 0$ or 1, $j_i \geqq 0$, then $Q_J(x)$ belongs to $G_{p^r q}$.*

COROLLARY 2-7. *For $j \geqq 1$, and J as above, the element $Q_J(\beta_p^\varepsilon x_j)$ belongs to G_{p^r}.*

We shall define the Z_p module homomorphism $\wedge : C \otimes C \to C$ as follows;

(2-7) $$\wedge : H_*(Q_0S^0 : Z_p) \otimes H_*(Q_0S^0 : Z_p) \to H_*(Q_0S^0 \times Q_0S^0 : Z_p) \xrightarrow{\wedge_*} H_*(Q_0S^0 : Z_p),$$

Then we have the following proposition.

PROPOSITION 2-8. *The following relations hold. Let $a, b, c \in C$.*

(2-8) i) $\wedge((a+b)) \otimes c) = \wedge(a \otimes c) + \wedge(b \otimes c)$,

ii) $\wedge(a \otimes (b+c)) = \wedge(a \otimes b) + \wedge(a \otimes c)$,

iii) $\wedge(1 \otimes a) = \wedge(a \otimes 1) = 0$ *if* $\deg a > 0$,

$\wedge(1 \otimes 1) = 1$,

iv) $\wedge((a\ b) \otimes c) = \sum (-1)^{\deg b \deg c'} (a \wedge c) \cdot (b \wedge c'')$,

where $\triangle(c) = \sum c' \otimes c''$,

v) $\wedge(a \otimes (b\ c)) = \sum (-1)^{\deg a'' \deg b} (a' \wedge b)(a'' \wedge c)$.

where $\triangle(a) = \sum a' \otimes a''$.

Proof. i) and ii) are trivial. iii) follows from the result that if $0 \in Q_0(S^0)$ is the trivial element, then the image of $0 \times Q_0(S^0) \to Q_0(S^0)$ is 0. iv) and v) follows from Proposition 1-7.

Next we shall introduce a filtration on $C \otimes C$ as follows;

$$G_j(C \otimes C) = \sum_{j_1+j_2=j} G_{j_1}(C) \otimes G_{j_2}(C). \tag{2-9}$$

PROPOSITION 2-9. *If $x \in C$ belongs to G_q, then $\triangle(x) \in C \otimes C$ belongs to G_q.*

This follows easily from Cartan formula, and Proposition 2-6.

Our final object in this chapter is the following.

PROPOSITION 2-10. *If $x = (Q_J \beta_p^{\varepsilon} x_j) \otimes (Q_{J'} \beta_p^{\varepsilon'} x_{j'})$, where $J = (\varepsilon_1, j_1, \cdots, \varepsilon_r, j_r)$, $J' = (\varepsilon_1', j_1', \cdots, \varepsilon_s', j_s')$, and $j, j' > 0$, then $\wedge(x) \in C$ belongs to $G_{p^{r+s}-1}$.*

We shall prove this proposition in the last of this chapter.

COROLLARY 2-11. *If $x \in C \otimes C$ belongs to G_q, and $q > 0$, then $\wedge(x)$ belongs to G_{q+1}.*

This corollary follows from Proposition 2-8 and Proposition 2-10, by tedious calculation.

We shall define $\xi_r : L_p^r \to Q_0(S^0)$, $r = 1, 2, \cdots$, in the following way, where $L_p^r = L_p \times \cdots L_p$, r-fold product.

$$\xi_r : L_p^r = L_p^{r-1} \times L_p \xrightarrow{h^{r-1} \times h_0} (Q_p(S^0))^{r-1} \times Q_0(S^0) \xrightarrow{\wedge} Q_0(S^0).$$

LEMMA 2-11. *The image of $(\xi_r)_* : H_*(L_p^r) \to H_*(Q_0(S^0))$ coincides with the submodule generated by $Q_J \beta_p^{\varepsilon} x_j$, $J = (\varepsilon_1, j_1, \cdots, \varepsilon_{r-1}, j_{r-1})$, $\varepsilon_i = 0$ or 1, $j_i \geq 0, j \geq 1$, $\varepsilon = 0$ or 1, in positive degree.*

Proof. This follows easily, using induction on r, from lemma 2-1, and the commutativety of the following diagram:

$$\begin{tikzcd}
H_*(L_p^r) \arrow[r, "(\xi_r)_*"] \arrow[d] & H_*(Q_0(S^0)) & H_*(J^\infty \pi_p \times_{\pi_p} (Q_0 S^0)^p) \arrow[l, "\theta_*"'] \\
H_*(L_p^1 \times L_p^{r-1}) \arrow[rr, "(id \times_{\pi_p} \triangle_p)_*"] & & H_*(J^\infty \pi_p \times_{\pi_p} (L_p^{r-1})^p). \arrow[u, "(id \times_{\pi_p} (\xi_{r-1})^p)_*"']
\end{tikzcd}$$

LEMMA 2-12. *The following diagram is homotopy commutative.*

$$\begin{tikzcd}
L_p^r \times L_p^s \arrow[r, "\xi_r \times \xi_s"] \arrow[d, "\triangle_{p+1}"] & Q_0S^0 \times Q_0S^0 \arrow[r, "\wedge"] & Q_0S^0 \\
(L_p^r \times L_p^s) \times (L_p^r \times L_p^s)^p \arrow[d, "id \times (\pi_{r-1} \times id)^p"] & & Q_0S^0 \times (Q_0S^0)^p \arrow[u, "\vee"'] \\
(L_p^{r+s}) \times (L_p^{r-1} \times L_p^s)^p \arrow[rr, "\xi_{r+s} \times (\xi_{r+s-1})^p"] & & Q_0S^0 \times (Q_0S^0)^p \arrow[u, "id \times (-1)^p"']
\end{tikzcd}$$

where $\pi_{r-1} : L_p^r = L_p^{r-1} \times L_p \to L_p^{r-1}$ *is the projection to the first part. This lemma follows easily from the results that* h_0 *is equal to* $h \vee (\text{-}p\, id)$ *and the distributive law of Proposition* 1–7.

LEMMA 2-13. $c = (-1)_* : H_*(Q_0S^0) \to H_*(Q_0S^0)$ *is filtration preserving.*

Proof. The following two diagrams are homotopy commutative.

a)

$$\begin{tikzcd}
Q_0S^0 \arrow[r, "\triangle"] \arrow[d] & Q_0S^0 \times Q_0S^0 \arrow[r, "id \times (-1)"] & Q_0S^0 \times Q_0S^0 \arrow[d, "\vee"] \\
* \arrow[rr] & & Q_0S^0
\end{tikzcd}$$

b)

$$\begin{tikzcd}
Q_0S^0 \times Q_0S^0 \arrow[r, "(-1) \times (-1)"] \arrow[d, "\vee"] & Q_0S^0 \times Q_0S^0 \arrow[d, "\vee"] \\
Q_0S^0 \arrow[r, "(-1)"] & Q_0S^0
\end{tikzcd}$$

b) shows that c is algebra homomorphism, and $y \in H_*(Q_0S^0)$, $\triangle(y) = y \otimes 1 + 1 \otimes y + \sum y' \otimes y''$. Then $\varepsilon(y) = c(y) + y + \sum y' c(y'')$, where $\varepsilon : C \to Z_p$ is argumentation. Since c is algebra homomorphism, it is sufficient to prove $c(Q_J \beta_p^\varepsilon x_j) \in G_{p^r}$ if $|J| = r$. This follows by induction argument from Corollary 2–7. and Cartan formula.

Proof of Proposition 2–10. From lemma 2–11, it is sufficient to prove that the image of $\wedge_* \cdot (\xi_r \wedge \xi_s)_*$ belongs to $G_{p^{r+s-1}}$, $r, s \geq 1$, for positive dimension. If $y \in H_*(L_p^r \times L_p^s)$, and $\deg y > 0$, then $\triangle_{p+1}(y) = y \otimes 1 \otimes \cdots \otimes 1 + \sum y_1 \otimes y_2 \otimes \cdots \otimes y_2 + \sum y_1 \otimes y_2 \otimes \cdots \otimes y_{p+1}$, where in the third term, $(y_2, \cdots, y_{p+1})$ is not of the form $(y_2, \cdots, y_2)$. Then lemma 2–12 shows

$$\begin{aligned}
\wedge_*(\xi_r \times \xi_s)_*(y) = (\xi_{r+s})_*(y) &+ \sum [(\xi_{r+s})_*(y_1)] \circ [(-1)_*(\xi_{r+s-1})_*((\pi_{r-1} \times id)_*(y_2))]^p \\
&+ \sum [(\xi_{r+s})_*(y_1)] \circ [(-1)_*(\xi_{r+s-1})_*((\pi_{r-1} \times id)_*(y_2))] \cdots \\
&\quad [(-1)_*(\xi_{r+s-1})_*((\pi_{r-1} \times id)_*(y_{p+1}))].
\end{aligned}$$

But in the third term, since $(y_2, \cdots, y_{p+1})$ is not of the form $(y_2, \cdots, y_2)$ if $(y_2, \cdots, y_{p+1})$ appears then its cyclic permutation $(y_{\sigma(2)}, \cdots, y_{\sigma(p+1)})$ appears for $\sigma \in \pi_p$. So that the third term vanishes. By lemma 2-13, $(-1)_*(\xi_{r+s-1})_*(\pi_{r-1} \times id)_*(y_2)$ belongs to $G_{p^{r+s-2}}$, so that the second term belongs to $G_{p^{r+s-1}}$. The first term belongs to $G_{p^{r+s-1}}$ by lemma 2-11 and Corollary 2-7. This proves proposition

§ 3. Pontrjagin ring $H_*(SF, Z_p)$

3-1. In this chapter, p denotes an odd prime number. We shall consider $H_*(Q_0(S^0), Z_p)$ as a Hopf-algebra with product $\overline{\wedge}_* : H_*(Q_0(S^0), Z_p) \otimes H_*(Q_0(S^0), Z_p) \to H_*(Q_0S^0 \times Q_0S^0, Z_p) \to H_*(Q_0S^0, Z_p)$, and with standard diagonal. We shall denote this Hopf-algebra by $\bar{C}$. Then C and $\bar{C}$ are naturaly isomorphic as coalgebras. Since SF is an H-space, $H_*(SF, Z_p)$ is a Hopf-algebra over Z_p. Let $i : Q_0S^0 \to SF$ be the inclusion defined in (1-6). Then $i_* : \bar{C} = H_*(Q_0S^0) \to H_*(SF)$ is a Hopf-algebra isomorphism because of definition of $\overline{\wedge}$, c.f. (1-7). So to determine the structure of Pontrjagin ring $H_*(SF, Z_p)$, it is sufficient determine the ring $\bar{C}$.

PROPOSITION 3-1. *If $u, v \in C$, and $u \in G_i$, $v \in G_j$, then $\overline{\wedge}_*(u \otimes v)$ belongs to G_{i+j}, and $\overline{\wedge}_*(u \otimes v)$ and $u \cdot v$ are equal mod G_{i+j+1}.*

Proof. If $\triangle(u) = u \otimes 1 + 1 \otimes u + \sum u' \otimes u''$, and $\triangle(v) = v \otimes 1 + 1 \otimes v + \sum v' \otimes v''$, then by Proposition 2-9, $u' \otimes u''$ belong to G_i, and $v' \otimes v''$ belong to G_j. By Corollary 1-6.

$$\begin{aligned}\overline{\wedge}_*(u \otimes v) = uv &+ \wedge_*(u \otimes v)\\ &+ \sum(-1)^{\deg u'' \deg v'}(u'v')\wedge_*(u'' \otimes v'')\\ &+ \sum(-1)^{\deg u \deg v'} v' \wedge_*(u \otimes v')\\ &+ \sum u' \wedge_*(u'' \otimes v).\end{aligned}$$

The term uv belongs to G_{i+j}, and by Corollary 2-11, other terms belong to G_{i+j+1}. This proves the proposition.

We shall introduce a filtration in $\bar{C}$ by that of C. Then Proposition 3-1 shows the product in $\bar{C}$ is filtration preserving.

THEOREM 1. *As an algebra $H_*(SF, Z_p)$ is a free commutative algebra generated by $\tilde{x}_J = i_*(x_J)$, $J \in H$.*

Proof. Let E_0C, and $E_0\bar{C}$ denote associated graded algebras with respect to the filtrations. Then Proposition 3–1 shows that E_0C and $E_0\bar{C}$ are isomorphic as algebras by $E_0 i_*$. On the other hand C and E_0C are isomorphic, and these are free commutative algebras generated by x_J and $\{x_J\}$, $J \in H$, respectively. This proves the Theorem.

§ 4. H_p^∞ structure on BSF

4–1. If $\pi_1 : \xi \to X$ and $\pi_2 : \eta \to Y$ are two spherical fiberings, then we shall define the exterior Whitney join product as follows.

(4–1) $$\pi_1 \hat{*} \pi_2 : \xi \hat{*} \eta \to X \times Y.$$

where $$\xi \hat{*} \eta = \{(t_1(e_1 \times y) \oplus t_2(x \times e_2) \in (\xi \times X) * (X \times \eta)$$
$$; \pi_1(e_1) = x \text{ and } \pi_2(e_2) = y \text{ if } t_1, t_2 > 0\}.$$

and $$(\pi_1 \hat{*} \pi_2)(t_1(e_1 \times y) \oplus t_2(x \times e_2))$$
$$= \begin{cases} (\pi_1(e_1), y) & \text{if } t_1 \neq 0 \\ (x, \pi_2(e_2)) & \text{if } t_2 \neq 0. \end{cases}$$

And if $X = Y$, then we shall define the interior Whitney join $\xi * \eta \to X$ as fiber product.

(4–2) $$\begin{array}{ccc} \xi * \eta & \longrightarrow & \xi \hat{*} \eta \\ \downarrow & \triangle & \downarrow \\ X & \longrightarrow & X \times X \end{array}$$

By the same method as in Hall [5], it is easy to prove that Whitney join is a spherical fibering.

We can interpret the iterated exterior Whitney join of $\pi_i : \xi_i \to X_i$, $i = 1, \cdots, q$, by the following.

$$\pi_1 \hat{*} \cdots \hat{*} \pi_q : \xi_1 \hat{*} \cdots \hat{*} \xi_q \to X_1 \times \cdots \times X_q.$$
$$\xi_1 \hat{*} \cdots \hat{*} \xi_q$$
$$= \{(t_1(e_1 \times x_{1,2} \times \cdots x_{1,q}) \oplus \cdots \oplus (t_q(x_{q,1} \times \cdots \times x_{q,q-1} \times e_q)$$
$$\in (\xi_1 \times X_2 \times \cdots \times X_q) * \cdots * (X_1 \times \cdots \times X_{q-1} \times \xi_q).$$

with $$\pi_1(e_1) = x_{2,1} = \cdots = x_{q,1}$$
$$\cdots\cdots\cdots$$
$$x_{1,q} = \cdots = x_{q-1,q} = \pi_q(e_q).$$

if $t_j=0$ then we omit the condition on $\pi_j(e_j)$ and $x_{k,j}\cdot\}$

Let $\xi^q\to X^q$ denote the exterior q-th join of $\xi\to X$ with itself. Symmetric group $\sum_q$ acts on X^q as permutation, and on ξ^q as follows. For $\sigma\in\sum_q$.

$$\begin{aligned}&\sigma(t_1(e_1\times x_{1,2}\times\cdots\times x_{1,q})\oplus\cdots\oplus t_q(x_{q,1}\times\cdots\times x_{q,q-1}\times e_q))\\&=(t_{\sigma(1)}(e_{\sigma(1)}\times x_{\sigma(1),\sigma(2)}\times\cdots\times x_{\sigma(1),\sigma(q)})\oplus\cdots\oplus t_{\sigma(q)}(x_{\sigma(q),\sigma(1)}\times\\&\qquad\times\cdots\times x_{\sigma(q),\sigma(q-1)}\times e_{\sigma(q)}).\end{aligned}$$

Then the operation σ commutes with projection $\xi^q\to X^q$, and define a fiber map.

Let $\pi_2: J^\infty\sum_q\times X^q\to X^q$ be projection on the second factor. If $\pi_2^*(\xi^q)=J^\infty\sum_q\times\xi^q$ is the induced fibering of ξ^q by π_2, and $\sum_q$ operates on $\pi_2^*(\xi^q)$ by $\sigma(\omega,e)=(\sigma(\omega),\sigma(e))$, $\omega\in J^\infty\sum_q$, $e\in\xi^q$, $\sigma\in\sum_q$, then σ is a fiber map covering the operation $\sigma: J^\infty\sum_q\times X^q\to J^\infty\sum_q\times X^q$.

PROPOSITION 4-1. *There exists a spherical fiber space $P(\xi)\to J^\infty\sum_q\times_{\Sigma_q}X^q$ and a bundle map $\pi_2^*(\xi^q)\to P(\xi)$ such that the following diagram is commutative for any $\sigma\in\sum_q$.*

(4-3)
$$\begin{tikzcd}
\pi_2^*(\xi^q) \arrow[rr, "\sigma"] \arrow[dr] \arrow[dd] & & \pi_2^*(\xi^q) \arrow[dl] \arrow[dd] \\
& P(\xi) \arrow[dd] & \\
J^\infty\sum_q\times X^q \arrow[rr, "\sigma"] \arrow[dr] & & J^\infty\sum_q\times X^q \arrow[dl] \\
& J^\infty\sum_q\times_{\Sigma_q}X^q &
\end{tikzcd}$$

It is easy to prove this proposition so we omit it.

We shall call this fibering $P(\xi)\to J^\infty\sum_q\underset{\Sigma_q}{\times}X^q$ by the extended p-th join of ξ.

PROPOSITION 4-2. *Let $\pi_1;\xi\to X$ and $\pi_2;\eta\to Y$ be two spherical fiber spaces, then.*

a) *There is a natural fiber map as follows.*

(4-4)
$$\begin{tikzcd}
P(\xi\hat{*}\eta) \arrow[r] \arrow[d] & P(\xi)\hat{*}P(\eta) \arrow[d] \\
J^\infty\sum_q\times_{\Sigma_q}(X\times Y)^q \arrow[r] & (J^\infty\sum_q\times_{\Sigma_q}X^q)\times(J^\infty\sum_q\times_{\Sigma_q}Y^q)
\end{tikzcd}$$

b) *If $X=Y$, then the following two spherical fibering are naturally isomorphic.*

(4-5)
$$\begin{array}{ccc} P(\xi*\eta) & \longrightarrow & P(\xi)*P(\eta) \\ \downarrow & & \downarrow \\ J^{\infty}\sum_q \times_{\Sigma_q} X^q & \longrightarrow & J^{\infty}\sum_q \times_{\Sigma_q} X^q. \end{array}$$

COROLLARY 4-3. *The following isomorphism holds.*

$$\begin{array}{ccc} P(\xi*1) & \longrightarrow & P(\xi)*P(1) \\ \downarrow & & \downarrow \\ J^{\infty}\sum_q \times_{\Sigma_q} X^q & \longrightarrow & J^{\infty}\sum_q \times_{\Sigma_q} X^q \end{array}$$

where $1\to X$ denotes the trivial bundle with fiber S^0.

Let $BSG(n)$ be the classifying space of $SG(n)$, and $\gamma_n\to BSG(n)$ denote the universal oriented spherical fibering with fiber S^{n-1}. Consider $P(\gamma_n)\to J^{\infty}\sum_q \times_{\Sigma_q}(BSG(n))^q$, then if n is even, then $P(\gamma_n)$ has the natural orientation, since $\sigma: S^{n-1}*\cdots*S^{n-1}\to S^{n-1}*\cdots*S^{n-1}$, $\sigma\in\sum_q$ is orientation preserving. Define

(4-7)
$$\bar\theta=\bar\theta^q_n : J^{\infty}\sum_q \times_{\Sigma_q}(BSG(n))^p \to BSG(qn)$$

as the classifying map of $P(\gamma_n)$. We shall also consider

(4-8)
$$\bar\theta=\bar\theta^p_n \,;\, J^{\infty}\pi_q \times_{\pi_q}(BSG(n))^q \to BSG(qn)$$

as the restriction of $\bar\theta^q_n$ of (4-7).

4-2. Consider regular representation $N=N_q$

(4-9)
$$N=N_q : \textstyle\sum_q \to 0(q)\to G(q).$$

Then it is easy to see that the bundle $P(1)\to J^{\infty}\sum_q \times_{\Sigma_q} X^q$ is the associated spherical fiber space to the principal $\sum_q$ bundle $J^{\infty}\sum_q \times X^q \to J^{\infty}\sum_q \times_{\Sigma_q} X^q$ with $N:\sum_q\to G(q)$.

Consider the following map f_n

(4-10)
$$\begin{aligned} f_n : L_p^{(2m+1)} = W^{(2m+1)}/\pi_p &\to W^{(2m+1)}\times_{\pi_p}(x_0)^p \\ &\to J^{\infty}\pi_p \times_{\pi_q}(BSG(n))^p \xrightarrow{\bar\theta} BSG(pn). \end{aligned}$$

where p is odd prime number and $x_0\in BSG(n)$. Then f_n is the classifying map of the associated spherical fibering with π_p principal fibering $W^{(2m+1)}\to L_p^{(2m+1)}$ by n-times regular representation: $\pi_p\to SO(pn)\to SG(pn)$. By Kambe

[6], the order of regular representation in $K\tilde{O}(L_p^{(2m+1)})$ is a factor of p^s, where $s=[(2m+1)/(p-1)]+1$. So if n is divisible by p^s and greater than $2m+1$, then we can assume that $\bar{\theta}(W^{(2m+1)}\times_{\pi_p}(x_0)^p=y_0\in BSG(pn)$.

REMARK 4–4. *Since the order of the regular representation N in $KO(J^t\sum_q)$ is finite, if t is finite, the above consideration holds when we consider $J^t\sum_q$ instead of $W^{(2m+1)}$ for some t and n.*

Let $\pi: ESG(n)\to BSG(n)$ be the associated principal fibering with $\gamma_n\to BSG(n)$,

$$ESG(n)=\{f: \begin{array}{ccc} S^{n-1} & \to & \gamma_n \\ \downarrow & & \downarrow \\ * & \to & BSG(n) \end{array} \quad f: \text{orientation preserving fiber map}\} \tag{4–11}$$

$$\pi(f)=f(*).$$

Fix an element $g_n\in ESG(n)$ with $\pi(g_n)=x_0$, and define $\bar{g}_n: SG(n)\to ESG(n)$ by $\bar{g}_n(f)=g_n\cdot f$. Then we can identify the image of g_n with the fiber $\pi^{-1}(x_0)$. Define $\bar{g}_{pn}: SG(pn)\to ESG(pn)$ by putting $g_{pn}: S^{pn-1}\to\gamma_{pn}$, $\pi(g_{pn})=y_0$

$$g_{pn}: S^{pn-1}\xrightarrow{g_n*\cdots*g_n}\gamma_n*\cdots*\gamma_n\longrightarrow\gamma_{pn}$$

and $\bar{g}_{pn}(f)=f\circ g_{pn}$. And identify $\pi^{-1}(y_0)\subseteq ESG(pn)$, with $SG(pn)$ by this map $\bar{g}_{pn}$.

Define a map $\bar{\rho}_n: W^{(2m+1)}\to SG(pn)$ by

$$\begin{array}{ccccccc} \bar{\rho}_n(\omega): S^{pn-1} & \xrightarrow{(\omega,\, g_n*\cdots*g_n)} & W^{(2m+1)}\times\gamma_n^p & \longrightarrow & p_0(\gamma_n) & \xrightarrow{\bar{\theta}} & \gamma_{pn} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & W^{(2m+1)}\times(BSG(n))^p & \longrightarrow & W^{(2m+1)}\times_{\pi_p}BSG(n)^p & \longrightarrow & BSG(pn). \end{array} \tag{4–12}$$

$$\omega\in W^{2m+1}.$$

Define a homomorphism $\rho_n: \pi_p\to SG(pn)$ by

$$\rho(\sigma)(t_1x_1\oplus\cdots\oplus t_px_p)=(t_{\sigma(1)}x_{\sigma(1)}\oplus\cdots\oplus t_{\sigma(p)}x_{\sigma(p)})$$
$$(t_1x_1\oplus\cdots\oplus t_px_p)\in S^{n-1}*\cdots*S^{n-1}=S^{pn-1}.$$

Then we have,

PROPOSITION 4–5. *The following formula holds.*

$$\bar{\rho}_n(\sigma\omega)\rho_n(\sigma)=\bar{\rho}_n(\omega),\quad \sigma\in\pi_p,\quad \omega\in W^{(2m+1)}. \tag{4–13}$$

Proof. This follows from the commutativety of the following diagram.

$$\begin{tikzcd}
S^{pn-1} \arrow[rr, "{(\omega,\, g_n*\cdots*g_n)}"] \arrow[dd] \arrow[dr, "\rho(\sigma)"] & & W^{(2m+1)}\times\gamma_n^p \arrow[rr] \arrow[dd] \arrow[dr, "\sigma"] & & P_0(\gamma_n) \arrow[dd] \\
& S^{pn-1} \arrow[rr, "{(\sigma\omega,\, g_n*\cdots*g_n)}"] \arrow[dd] & & W^{(2m+1)}\times\gamma_n^p \arrow[ur] \arrow[dd] & \\
* \arrow[rr] \arrow[dr] & & W^{(2m+1)}\times(BSG(n))^p \arrow[rr] \arrow[dr] & & W^{(2m+1)}\times_{\pi_p}(BSG(n))^p \\
& * \arrow[rr] & & W^{(2m+1)}\times(BSG(n))^p \arrow[ur] &
\end{tikzcd}$$

PROPOSITION 4–6. *Let* $\bar{\rho}_{n_i}: W^{(2m+1)}\to SG(n_i)$, $i=1,2$, *be the map of* (4–12). *Then* $\bar{\rho}_{n_1}*\bar{\rho}_{n_2}$ *and* $\bar{\rho}_{n_1+n_2}$ *are* π_p *equivariantly homotopic as maps*, $W^{(2m+1)}\to SG(n_1+n_2)$.

Proof is easily follows from proposition 4–2.

Define a map $\bar{\theta}_n' : W^{(2m+1)}\times(ESG(n))^p\to ESG(pn)$ as follows. $\omega\in W^{(2m+1)}$, $f_1,\cdots,f_p\in ESG(n)$.

(4–14) $\bar{\theta}_n'(\omega;\, f_1,\cdots,f_p)$:

$$\begin{tikzcd}
S^{pn-1} \arrow[r, "\bar{\rho}_n(\omega)^{-1}"] \arrow[d] & S^{pn-1} \arrow[r, "{(\omega,\, f_1*\cdots*f_p)}"] \arrow[d] & W^{(2m+1)}\times\gamma_n^p \arrow[r] \arrow[d] & P_0(\gamma_n) \arrow[r] \arrow[d] & \gamma_{pn} \arrow[d] \\
* \arrow[r] & * \arrow[r] & W^{(2m+1)}\times(BSG(n))^p \arrow[r] & W^{(2m+1)}\times_{\pi_p}(BSG(n))^p \arrow[r] & BSG(pn)
\end{tikzcd}$$

PROPOSITION 4–7. $\bar{\theta}_n'$ *is a* π_p *equivariant map, where* π_p *operates on* $ESG(pn)$ *trivially.*

This follows easily from definition as that of proposition 4–5.

By proposition 4–7, we can define the following fiber wisemap.

$$\text{(4–15)}\qquad \begin{tikzcd}
& W^{(2m+1)}\times_{\pi_p}(SG(n))^p \arrow[r] \arrow[dl] & SG(pn) \arrow[dl] \\
\bar{\theta}_n : W^{(2m+1)}\times_{\pi_p}(ESG(n))^p \arrow[r] \arrow[d] & ESG(pn) \arrow[d] & \\
W^{(2m+1)}\times_{\pi_p}(BSG(n))^p \arrow[r] & BSG(pn) &
\end{tikzcd}$$

PROPOSITION 4–8. $\bar{\theta}_n : W^{(2m+1)}\times_{\pi_p}(SG(n))^p\to SG(pn)$ *is expressed as follows.* $\omega\in W^{(2m+1)}$

$$\text{(4–16)}\qquad \bar{\theta}(\omega;\, f_1,\cdots,f_p) = \bar{\rho}(\omega)\circ(f_1*\cdots*f_p)\circ\bar{\rho}(\omega)^{-1}.$$

PROPOSITION 4–9. *The following diagram is homotopy commutative.*

$$
\text{(4–17)}\qquad
\begin{tikzcd}
W^{(2m+1)}\times_{\pi_p}(SG(n)\times SG(n))^p \arrow[r, "id\times(\circ)^p"] \arrow[d, "\triangle\times id"] & W^{(2m+1)}\times_{\pi_p}(SG(n))^p \arrow[d, "\bar{\theta}_n"] \\
W^{(2m+1)}\times W^{(2m+1)}\times_{\pi_p\times\pi_p}(SG(n)\times SG(n))^p \arrow[d] & SG(n) \\
W^{(2m+1)}\times_{\pi_p}(SG(n))^p\times W^{(2m+1)}\times_{\pi_p}(SG(n))^p \arrow[r, "\bar{\theta}_n\times\bar{\theta}_n"] & SG(n)\times SG(n) \arrow[u, "\circ"']
\end{tikzcd}
$$

REMARK 4–10. *By remark* 4–4, *the above construction* $\bar{\theta}_n$ *can be extended as follows*

$$
\text{(4–18)}\qquad
\begin{tikzcd}
 & & B\sum_p^{(t)}\times_{\Sigma_p}(SG(n))^p \arrow[rr] \arrow[dl] & & SG(pn) \arrow[dl] \\
\bar{\theta}_n : & B\sum_p^{(t)}\times_{\Sigma_p}(ESG(n))^p \arrow[rr] \arrow[d] & & ESG(pn) \arrow[d] & \\
 & B\sum_p^{(t)}\times_{\Sigma_p}(BSG(n))^p \arrow[rr] & & BSG(pn) &
\end{tikzcd}
$$

At the last we shall consider the relationship between $\bar{\theta}_n$ and the suspension homomorphism.

PROPOSITION 4–11. *The following diagram is homotopy commutative, where* $s=[(2m+1)/(p-1)]+1$.

$$
\text{(4–19)}\qquad
\begin{tikzcd}
W^{(2m+1)}\times_{\pi_p}(BSG(n))^p \arrow[r] \arrow[d, "\bar{\theta}"] & W^{(2m+1)}\times_{\pi_p}(BSG(n+p^s))^p \arrow[d, "\bar{\theta}"] \\
BSG(pn) \arrow[r] & BSG(p(n+p^s))
\end{tikzcd}
$$

Proof. By proposition 4–2, the fiber space $P_0(\gamma_n*(p^s))$ is equivalent to $P_0(\gamma_n)*(p^sN)$. And the fibering $(p^sN)\to W^{(2m+1)}\times_{\pi_p}(BSG(n))^p$ is equivalent to the trivial fiber space. So proposition follows.

PROPOSITION 4–12. *The following diagram is homotopy commutative,* $s=[(2m+1)/(p-1]+1)$.

$$
\text{(4–20)}\qquad
\begin{tikzcd}
W^{(2m+1)}\times_{\pi_p}(SG(n))^p \arrow[r] \arrow[d, "\bar{\theta}"] & W^{(2m+1)}\times_{\pi_p}(SG(n+p^s))^p \arrow[d, "\bar{\theta}"] \\
SG(pn) \arrow[r] & SG(pn+p^{p+1}).
\end{tikzcd}
$$

Proof is analog as that of proposition 4–11.

§5. Decomposition of $\bar{\theta}$.

5–1. In this chapter we shall study the map $\bar{\theta} : W\times_{\pi_p}SG^p \to SG$. p is always an odd prime number. For topological spaces X, Y, we denote by $G(X,Y)$, the space of all continuous maps from X to Y with compact open topology. And if X and Y are endowed with base points, we denote by $F(X,Y)$, the space of all base preserving continuous maps. We denote by $G(n)$, the space $G(S^{n-1},S^{n-1})$, and denote $G_q(n)$, the subspace of $G(n)$ consisting of the maps of degree q, $q\in Z$.

We denote $\mathscr{E}=\{E=(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_p);\varepsilon_i=0 \text{ or } 1\}$. And for $E\in\mathscr{E}$, $|E|$ is the number of elements of the set $\{\varepsilon_i,\varepsilon_i=1; E=(\varepsilon_1,\cdots,\varepsilon_p)\}$. The cyclic group π_p operates on $\mathscr{E}$ by $\sigma(\varepsilon_1,\cdots,\varepsilon_p)=(\varepsilon_{\sigma(1)},\cdots,\varepsilon_{\sigma(p)})$. Introduce a total ordering in $\mathscr{E}$ by

$$E<E' \quad \varepsilon_1=\varepsilon_1',\cdots,\varepsilon_{j-1}=\varepsilon_{j-1}',\ \varepsilon_j<\varepsilon_j',$$

where $E=(\varepsilon_1,\cdots,\varepsilon_p)$ and $E'=(\varepsilon_1',\cdots,\varepsilon_p')$.

$\bar{\mathscr{E}}$ is by definition $\mathscr{E}/\pi_p$, and $\pi:\mathscr{E}\to\bar{\mathscr{E}}$ denotes the projection. Define a cross section $s:\bar{\mathscr{E}}\to\mathscr{E}$ by $S(\{E\})=$ the first element in $\{E\}$ by the total ordering, and $\mathscr{E}_0$ denotes the image $s(\bar{\mathscr{E}})$.

Define a map $\varphi_2 : S^{n-1}\to S_0^{n-1}\vee S_1^{n-1}$, by the following way, where $S_0^{n-1}\vee S_1^{n-1}$ denotes the one point union of two spheres S_0^{n-1} and S_1^{n-1}.

$$\varphi_2(\phi_{n-1}(t_1,\cdots,t_{n-1}))=\begin{cases}\phi_{n-1}(2t_1,t_2,\cdots,t_{n-1})\in S_0^{n-1}, & 0\leq t_1\leq 1/2.\\ \phi_{n-1}(2t_1-1,\ t_2,\cdots,t_{n-1})\in S_1^{n-1}, & 1/2\leq t_1\leq 1.\end{cases} \tag{5–1}$$

where $\phi_{n-1}:(I^{n-1},\partial I^{n-1})\to(S^{n-1},*)$ is relative homeomorphism.

For $E_0\in\mathscr{E}_0$, define a continuous map $\eta_{E_0}:(\Omega_0^{n-1}S^{n-1})^p\to G(pn)=G(S^{pn-1}, S^{pn-1})$ by the following diagram. $l_1,\cdots,l_p\in\Omega_0^{n-1}S^{n-1}$

$$\begin{tikzcd}[column sep=huge]
\eta_{E_0}(l_1,\cdots,l_p): S^{n-1}*\cdots*S^{n-1} \arrow[r, "\varphi_2*\cdots*\varphi_2"] \arrow[d] & (S_0^{n-1}\vee S_1^{n-1})*\cdots*(S_0^{n-1}\vee S_1^{n-1}) \arrow[d] \\
S^{pn-1} & \bigvee_{\triangle^{p-1},E\in\mathscr{E}} S_E^{pn-1} \arrow[l, "l_{E_0}(l_1,\cdots,l_p)"']
\end{tikzcd} \tag{5–2}$$

where $S_E^{pn-1}=S_{\varepsilon_1}^{n-1}*\cdots*S_{\varepsilon_p}^{n-1}$ for $E=(\varepsilon_1,\cdots,\varepsilon_p)$ and $l_{E_0}(l_1,\cdots,l_p)$ represents the following map.

$$l_{E_0}(l_1,\cdots,l_p)|_{S_E^{pn-1}} : S_E^{pn-1}\to S^{pn-1}$$

(5-3) a) if $E \neq \sigma(E_0)$ for any $\sigma \in \pi_p$, then $l_{E_0}(l_1, \cdots, l_p)$ is $0 * \cdots * 0$, where $0 : S^{n-1} \to * \to S^{n-1}$.

b) if $E = \sigma(E_0)$ for some $\sigma \in \pi_p$, then $l_{E_0}(l_1, \cdots, l_p)$ is $l_1^{\varepsilon_1} * \cdots * l_p^{\varepsilon_p}$, where $l_i^0 = id_{n-1}$, and $l_j^1 = l_j$, $E = (\varepsilon_1, \cdots, \varepsilon_p)$.

LEMMA 5-1. *The following formula holds for any* $\sigma \in \pi$, *and* $l_1, \cdots, l_p \in \Omega_0^{n-1} S^{n-1}$.

$$\eta_{E_0}(l_{\sigma(1)}, \cdots, l_{\sigma(p)}) = \rho(\sigma)\eta_{E_0}(l_1, \cdots, l_p)\rho(\sigma)^{-1}. \tag{5-4}$$

Proof. This follows from the commutativety of the following diagram.

$$\begin{array}{ccccc}
S^{n-1} * \cdots * S^{n-1} & \xrightarrow{\varphi_2 * \cdots * \varphi_2} & \bigvee_{\Delta^{p-1}, E \in \mathscr{E}} S_E^{pn-1} & \xrightarrow{l_{E_0}(l_1, \cdots, l_p)} & S^{pn-1} \\
\uparrow \rho(\sigma) & & \uparrow \rho(\sigma) & & \uparrow \rho(\sigma) \\
S^{n-1} * \cdots * S^{n-1} & \xrightarrow{\varphi_2 * \cdots * \varphi_2} & \bigvee_{\Delta^{p-1}, E \in \mathscr{E}} S_E^{pn-1} & \xrightarrow{l_{E_0}(l_1, \cdots, l_p)} & S^{pn-1}
\end{array}$$

where $\rho(\sigma)|_{S_E^{pn-1}} : S_E^{pn-1} \to S_{\sigma(E)}^{pn-1}$ is defined by $\rho(\sigma)(t_1 x_1 \oplus \cdots \oplus t_p x_p) = (t_{\sigma(1)} x_{\sigma(1)} \oplus \cdots \oplus t_{\sigma(p)} x_{\sigma(p)})$.

Next define a map $\bar{\theta}'_{E_0} : W^{(2m+1)} \times (\Omega_0^{n-1} S^{n-1})^p \to G(pn)$, by the following for $E_0 \in \mathscr{E}_0$. $\omega \in W^{(2m+1)}$, $l_1, \cdots, l_p \in \Omega_0^{n-1} S^{n-1}$.

$$\bar{\theta}'_{E_0}(\omega : l_1, \cdots, l_p) = \bar{\rho}(\omega)\eta_{E_0}(l_1, \cdots, l_p)\bar{\rho}(\omega)^{-1}. \tag{5-5}$$

PROPOSITION 5-2. $\bar{\theta}'_{E_0} : W^{(2m+1)} \times (\Omega_0^{n-1} S^{n-1})^p \to G(pn)$ *is a* π_p *equivariant map. So we can obtain*

$$\bar{\theta}_{E_0} : W^{(2m+1)} \times_{\pi_p} (\Omega_0^{n-1} S^{n-1})^p \to G(pn). \tag{5-6}$$

This follows from the formula (4-13); $\bar{\rho}(\sigma\omega)\rho(\sigma) = \bar{\rho}(\omega)$, and lemma 5-1.

5-2. Denote $\bigvee_{\Delta^{p-1}, E \in \mathscr{E}} S_E^{pn-1}$ by X, $\bigvee_{\Delta^{p-1}, E_0 \in \mathscr{E}_0} S_{E_0}^{pn-1}$ by X_0, and $\bigvee_{\Delta^{p-1}, \sigma \in \pi_p} S_{\sigma(E_0)}^{pn-1}$ by X_{E_0} for $E_0 \in \mathscr{E}_0$. Let $i_{E_0} : X_{E_0} \to X$, $i_{E_0} : S_0^{pn-1} \to X_0$ be natural inclusion, for $E_0 \in \mathscr{E}_0$. Define continuous maps, $\pi : X \to X_0$, $\pi_0 : X_0 \to S^{pn-1}$, $\pi_{E_0} : X \to X_{E_0}$, $\bar{\pi}_{E_0} : X_0 \to S_{E_0}^{pn-1} = S^{pn-1}$, for $E_0 \in \mathscr{E}_0$ as follows.

i) $$\pi|_{S_E^{pn-1}} : S_E^{pn-1} = S^{pn-1} \xrightarrow{id} S^{pn-1} = S_{s\pi(E)}^{pn-1}.$$

(5-7) ii) $$\pi_0|_{S_{E_0}^{pn-1}} : S_{E_0}^{pn-1} = S^{pn-1} \xrightarrow{id} S^{pn-1}.$$

iii) $$\pi_{E_0}|_{S_E^{pn-1}}\begin{cases} S_E^{pn-1} = S^{pn-1} \xrightarrow{id} S^{pn-1} = S_E^{pn-1}, & \text{if } E = \sigma(E_0) \text{ for some } \sigma\in\pi_p. \\ S_E^{pn-1} = S^{pn-1} \xrightarrow{0*\cdots*0} (*)*\cdots*(*) = \triangle^{p-1}\subseteq X_{E_0}, & \text{if } E \neq \sigma(E_0) \text{ for any } \sigma\in\pi_p. \end{cases}$$

iv) $$\bar{\pi}_{E_0}|_{S_E^{pn-1}}\begin{cases} id & \text{if } E = E_0 \\ 0*\cdots*0 & \text{if } E \neq E_0. \end{cases}$$

Define the maps $\tilde{\eta}, \tilde{\eta}_{E_0} : (\Omega_0^{n-1}S^{n-1})^p \to G(S^{pn-1}, X_0)$, $E_0\in\mathscr{E}_0$, by the following way. $E_0 = (\varepsilon_1, \cdots, \varepsilon_p)$, $l_1, \cdots, l_p\in\Omega_0^{n-1}S^{n-1}$.

(5–8)
$$\tilde{\eta}(l_1, \cdots, l_p) = \pi_0((id\vee l_1)*\cdots*(id\vee l_p))\circ(\varphi_2*\cdots*\varphi_2) : S^{pn-1}\to X\to X\to X_0.$$
$$\tilde{\eta}_{E_0}(l_1, \cdots, l_p) = i_{E_0}\cdot\bar{\pi}_{E_0}\cdot\tilde{\eta}(l_1, \cdots, l_p) : S^{pn-1}\to X_0\to S_{E_0}^{pn-1}\to X_0$$

For $\omega\in W^{(2m+1)}$, define $\bar{\rho}'(\omega) : X_0\to X_0$ as follows.

(5–9) $$\bar{\rho}'(\omega)|_{S_{E_0}^{pn-1}} : S_{E_0}^{pn-1} = S^{pn-1}\xrightarrow{\bar{\rho}(\omega)} S^{pn-1} = S_{E_0}^{pn-1}.$$

For $\sigma\in\pi_p$, define $\rho'(\sigma) : X_0\to X_0$ as follows.

(5–10) $$\rho'(\sigma)|_{S_{E_0}^{pn-1}} : S_{E_0}^{pn-1} = S^{pn-1}\xrightarrow{\rho(\sigma)} S^{pn-1} = S_{E_0}^{pn-1}.$$

Then it is easy to show the following formula.

(5–11) $$\bar{\rho}'(\sigma\omega)\rho'(\sigma) = \bar{\rho}'(\omega),\quad \omega\in W^{(2m+1)},\quad \sigma\in\pi_p.$$

Define continuous maps $\tilde{\theta}', \tilde{\theta}'_{E_0} : W^{(2m+1)}\times(\Omega_0^{n-1}S^{n-1})^p\to G(S^{n-1}, X_0)$, $E_0\in\mathscr{E}_0$, by the following.

(5–12)
i) $$\tilde{\theta}'(\omega; l_1, \cdots, l_p) = \bar{\rho}'(\omega)\cdot\tilde{\eta}(l_1, \cdots, l_p)\bar{\rho}(\omega)^{-1}$$
ii) $$\tilde{\theta}'_{E_0}(\omega; l_1, \cdots, l_p) = \bar{\rho}'(\omega)\cdot\tilde{\eta}_{E_0}(l_1, \cdots, l_p)\bar{\rho}'(\omega)^{-1} = i_{E_0}\cdot\bar{\pi}_{E_0}\cdot\tilde{\theta}'(l, \cdots, l_p).$$

Then it is easy to show that $\tilde{\theta}'$, and $\tilde{\theta}'_{E_0}$ are π_p equivariant, and we obtain the following maps.

(5–13)
i) $$\tilde{\theta} : W^{(2m+1)}\times_{\pi_p}(\Omega_0^{n-1}S^{n-1})^p\to G(S^{pn-1}, X_0)$$
ii) $$\tilde{\theta}_{E_0} : W^{(2m+1)}\times_{\pi_p}(\Omega_0^{n-1}S^{pn-1})^p\to G(S^{pn-1}, X_0)$$

5-3. We shall consider the relations between $\tilde{\theta}$ and $\tilde{\theta}_{E_0}$, and between $\bar{\theta}$ and $\bar{\theta}_{E_0}$. Let A be a finite CW complex, (not pointed), and EA denotes the (not reduced) suspension of A, i.e. $EA = A \times I/\sim$. We endow the base point on EA by $\{(A, 0)\}$. And $\sum^2 A$ denote $S(EA) = (EA) \wedge S^1$. Define a map $\varphi : \sum^2 A \to \sum^2 A$ by,

$$\varphi((a, t_1, t_2)) = \begin{cases} (a, t_1, 2t_2) & 0 \leq t_2 \leq 1/2 \\ (a, t_1, 2t_2 - 1), & 1/2 \leq t_2 \leq 1. \end{cases}$$

Then $\sum^2 \tilde{\theta}$ and $\sum^2 \tilde{\theta}_{E_0}$ are defined as follows.

(5-14) $$\sum{}^2 \tilde{\theta} : W^{(2m+1)} \times_{\pi_p} (\Omega_0^{n-1} S^{n-1})^p \to G(S^{pn-1}, X_0) \to F(\sum{}^2 S^{pn-1}, \sum{}^2 X_0)$$

$$\sum{}^2 \tilde{\theta}_{E_0} : W^{(2m+1)} \times_{\pi_p} (\Omega_0^{n-1} S^{n-1})^p \to G(S^{pn-1}, X_0) \to F(\sum{}^2 S^{pn-1}, \sum{}^2 X_0).$$

Introduce a product in $F(\sum^2 S^{pn-1}, \sum^2 X_0)$ by the following.

$$(f \vee g) : \sum{}^2 S^{pn-1} \xrightarrow{\varphi} \sum{}^2 S^{pn-1} \vee \sum{}^2 S^{pn-1} \xrightarrow{(f \vee g)} (\sum{}^2 X_0) \vee (\sum{}^2 X_0) \longrightarrow \sum{}^2 X_0.$$

Then define the map $\bigvee_{E_0 \in \mathscr{E}_0} \sum^2 \tilde{\theta}_{E_0}$ by the following

(5-15) $$\begin{array}{ccc} \bigvee_{E_0 \in \mathscr{E}_0} \sum^2 \tilde{\theta}_{E_0} : W^{(2m+1)} \times_{\pi_p} (\Omega_0^{n-1} S^{n-1})^p & \longrightarrow & F(\sum^2 S^{pn-1}, \sum^2 X_0) \\ \downarrow \Delta & & \uparrow \vee \\ \prod_{E_0 \in \mathscr{E}_0} (W^{(2m+1)} \times_{\pi_p} (\Omega_0^{n-1} S^{n-1})^p)_{E_0} & \xrightarrow[\prod \sum^2 \tilde{\theta}_{E_0}]{} & \prod_{E_0 \in \mathscr{E}_0} F(\sum^2 S^{pn-1}, \sum^2 X_0)_{E_0} \end{array}$$

PROPOSITION 5-3. *$\sum^2 \tilde{\theta}$ and $\bigvee_{E_0 \in \mathscr{E}_0} \sum^2 \tilde{\theta}_{E_0}$ are homotopic on $(pn-5)$ skeleton of $W^{(2m+1)} \times_{\pi_p} (\Omega_0^{n-1} S^{n-1})^p$.*

Proof. By definition $\sum^2 \tilde{\theta}_{E_0} = (\sum^2 i_{E_0}) \circ (\sum^2 \bar{\pi}_{E_0}) \cdot (\sum^2 \tilde{\theta})$ so that proposition follows easily from the following lemma.

LEMMA 5-4. *Let $X_1, \cdots, X_r$ be connected finite CW complex with base points, and X_i is $(n + m_i)$ connected, $n > 0$, $m_i > 1$. Then $\Omega^n(X_1 \vee \cdots \vee X_r) \xrightarrow{i} \Omega^n(X_1 \times \cdots \times X_1) = \Omega^n(X_1) \times \cdots \times \Omega^n(X_r) \xrightarrow{\vee} \Omega^n(X_1 \vee \cdots \vee X_r)$ is homotopy equivalence on $(m-2)$ skeleton, where $m = \min(m_1, \cdots, m_r)$.*

Continuous maps $\pi_0 : X_0 \to S^{pn-1}$, and $\bar{\pi}_{E_0} : X_0 \to S^{pn-1}$, c.f. (5-7), define maps $\pi_0, \bar{\pi}_{E_0} : G(S^{pn-1}, X_0) \to G(S^{pn-1}, S^{pn-1}) = G(pn)$. In §4 we introduce a continuous map $\bar{\theta} : W^{(2m+1)} \times_{\pi_p} (SG(n))^p \to SG(pn)$. We also denote by $\bar{\theta}$ the following map : $W^{(2m+1)} \times_{\pi_p} (\Omega_0^{n-1}S^{n-1}) \xrightarrow{id\times(id_{n-1}\vee)^p} W^{(2m+1)} \times_{\pi_p} (SG(n))^p \xrightarrow{\bar\theta} SG(pn)$. Then we have

PROPOSITION 5-5. *$\bar\theta = \pi_0 \cdot \tilde\theta$ and $\bar\theta_{E_0} = \bar\pi_{E_0} \cdot \tilde\theta_{E_0}$, $E_0 \in \mathscr{E}_0$, as maps $W^{(2m+1)} \times_{\pi_p} (\Omega_0^{n-1}S^{n-1})^p \to G(pn)$.*

From this proposition and proposition 5-3 we have.

PROPOSITION 5-6. *$\sum^2\bar\theta$ and $\bigvee_{E_0\in\mathscr{E}_0} \sum^2\bar\theta_{E_0}$ are homotopic on $(pn-5)$ skeleton as the maps : $W^{(2m+1)} \times_{\pi_p} (\Omega_0^{n-1}S^{n-1})^p \to F(\sum^2 S^{pn-1}, \sum^2 S^{pn-1}) = \Omega^{pn+1}S^{pn+1}$.*

It is easy to show that $\bar\theta_{(0,\dots,0)} : W^{(2m+1)} \times_{\pi_p} (\Omega_0^{n-1}S^{n-1})^p \to G_1(pn)$ is constant map, so we obtain.

PROPOSITION 5-7. *The following diagram is homotopy commutative on $(pn-5)$ skeletons.*

$$\begin{array}{ccc} W^{(2m+1)}\times_{\pi_p}(\Omega_0^{n-1}S^{n-1})^p \xrightarrow{\bar\theta} G(pn) & \xrightarrow{\sum^2} & \Omega_1^{pn+1}S^{pn+1} \\ & & \uparrow (\vee\, id_{pn+1}) \\ \Big\downarrow \triangle & & \Omega_0^{pn+1}S^{pn+1} \\ & & \uparrow \vee \\ \prod_{E_0\in\mathscr{E}_0,\,|E_0|\neq 0} (W^{(2m+1)}\times_{\pi_p}(\Omega_0^{n-1}S^{n-1})^p)_{E_0} & \xrightarrow{\Pi\sum^2\bar\theta_{E_0}} & \prod_{E_0\in\mathscr{E}_0,\,|E_0|\neq 0} (\Omega_1^{pn+1}S^{pn+1})E_0 \end{array} \tag{5-16}$$

5-4. For $E_0 \in \mathscr{E}_0$, define continuous maps $p_{E_0} : X \to S^{pn-1}$, and $\bar p_{E_0} : X_{E_0} \to S^{pn-1}$ by

i) $p_{E_0}|_{S_E^{pn-1}}$ $\begin{cases} S_E^{pn-1} = S^{pn-1} \xrightarrow{id} S^{pn-1} & \text{if } E = \sigma(E) \text{ for some } \sigma\in\pi_p \\ S_E^{pn-1} = S^{pn-1} \xrightarrow{0*\cdots*0} S^{pn-1} & \text{if } E \neq \sigma(E) \text{ for any } \sigma\in\pi_p \end{cases}$

ii) $\bar p_{E_0} = p_{E_0}|_{X_{E_0}}$.

Introduce continuous maps $\bar h_{E_0} : L_p^{(2m+1)} = W^{(2m+1)}/\pi_p \to G(pn)$, for $E_0 \in \mathscr{E}_0$, as follows. $\bar h_{E_0}(\omega)$, $\omega \in W^{(2m+1)}$ represents the following map.

$$\bar h_{E_0}(\omega) : S^{pn-1} \xrightarrow{\bar\rho(\omega)^{-1}} S^{pn-1} \xrightarrow{\varphi_2*\cdots*\varphi_2} X \xrightarrow{p_{E_0}} S^{pn-1} \xrightarrow{\bar\rho(\omega)} S^{pn-1}. \tag{5-17}$$

PROPOSITION 5-8. *The following diagram is homotopy commutative for $E_0 \in \mathscr{E}_0$, $0 \lneq |E_0| \lneq p$.*

$$
(5\text{-}18)\qquad
\begin{array}{ccccc}
W^{(2m+1)}/\pi_p\times\Omega_0^{n-1}S^{n-1} & \xrightarrow{id\times\triangle_p} & W^{(2m+1)}\times_{\pi_p}(\Omega_0^{n-1}S^{n-1})^p & \xrightarrow{\bar\theta_{E_0}} & G_0(pn)\\
\downarrow{\scriptstyle \bar h_{E_0}\times\bar l_{E_0}} & & * & & \downarrow{\scriptstyle *id_{pn-1}}\\
G(pn)\times G_0(pn) & & \xrightarrow{\qquad\qquad} & & G_0(2pn)
\end{array}
$$

where $l_{E_0}(l)=l^{\varepsilon_1}*\cdots*l^{\varepsilon_p}$, $E_0=(\varepsilon_1,\cdots,\varepsilon_p)$, $l_j^0=id$, $l_j^1=l_j$.

Proof. At first, choose a homotopy $F_{E_0,t}:\Omega_0^{n-1}S^{n-1}\to G_0(2pn)$ with the properties, a) $F_{E_0,0}(l)=(l^{\varepsilon_1}*\cdots*l^{\varepsilon_p})*id_{pn-1}$.
b) $F_{E_0,1}(l)=id_{pn-1}*(l^{\varepsilon_1}*\cdots*l^{\varepsilon_p})$. And then define $\varphi_{E_0,t}:\Omega_0^{n-1}S^{n-1}\to G(\bar X_{E_0}*S^{pn-1},\bar X_{E_0}*S^{pn-1})$ as follows, where $\bar X_{E_0}=X_{E_0}/\triangle^{p-1}=\bigvee_{\sigma\in\pi_p}\bar S^{pn-1}_{\sigma(E_0)}$, $\bar S^{pn-1}_{\sigma(E)}=S^{pn-1}_{\sigma(E)}/\triangle^{p-1}$.

$$\varphi_{E_0,t}(l)\,|_{\bar S^{pn-1}_{\sigma(E_0)}*S^{pn-1}}=(\rho(\sigma)*id_{pn-1})\circ F_{E_0,t}(l)\circ(\rho(\sigma)^{-1}*id_{pn-1}).$$

And define $\eta_{E_0,t}:\Omega_0^{n-1}S^{n-1}\to G(2pn)$, as follows, $l\in{}_0^{n-1}S^{n-1}$.

$$
\begin{array}{ccc}
\eta_{E_0,t}(l):(S^{n-1}*\cdots*S^{n-1})*S^{pn-1} & \xrightarrow{(\varphi_2*\cdots*\varphi_2)*id_{pn-1}} & X*S^{pn-1}\\
\downarrow & & \downarrow{\scriptstyle \pi_{E_0}*id_{pn-1}}\\
S^{2pn-1} & & X_{E_0}*S^{pn-1}\\
\uparrow{\scriptstyle \bar p_{E_0}*id_{pn-1}} & & \downarrow\\
\bar X_{E_0}*S^{pn-1} & \xleftarrow{\varphi_{E_0,t}(l)} & \bar X_{E_0}*S^{pn-1}
\end{array}
$$

And define $\bar\theta_{E_0,t}(\omega,l)=(\bar\rho(\omega)*id_{pn-1})\circ(\eta_{E_0,t}(l))\circ(\bar\rho(\omega)^{-1}*id_{pn-1})$. Then it is easy to show that $\bar\theta_{E_0,0}$ and $(*id_{pn-1})(id\times\triangle_p)$ is homotopic, and $\bar\theta_{E_0,1}$ and $(*)(\bar hE_0\times\bar l_{E_0})$ is homotopic. This gives the proof.

Now introduce the following map $\bar\theta_p:W^{(2m+1)}\times(\Omega_0^{n-1}S^{n-1})^p\to G_0(pn)$ as follows.

$$(5\text{-}19)\qquad \bar\theta_p(\omega\,;\,l_1,\cdots,l_p)=\bar\rho(\omega)\cdot(l_1*\cdots*l_p)\cdot\bar\rho(\omega)^{-1}.$$

PROPOSITION 5-9. *$\bar\theta_p$ and $\bar\theta_{(1,\dots,1)}:W^{(2m+1)}\times_{\pi_p}(\Omega_0^{n-1}S^{n-1})^p\to G_0(pn)$ are homotopic.*

This proposition is proved by the same idea of two proof of proposition 5-8, so we omit the proof.

The following is the easy consequence of proposition 5-9.

PROPOSITION 5-10. *The following diagram is commutative.*

(5-20)

$$
\begin{array}{ccc}
W^{(2m+1)}\times_{\pi_p}(\Omega_0^{n-1}S^{n-1}\times\Omega_0^{n-1}S^{n-1})^p & \xrightarrow{id\times\pi_p(\circ)^p} & W^{(2m+1)}\times_{\pi_p}(\Omega_0^{n-1}S^{n-1})^p \\
\downarrow{\scriptstyle \triangle\times id} & & \downarrow{\scriptstyle \bar\theta_{(1,\dots,1)}} \\
(W^{(2m+1)}\times W^{(2m+1)})\underset{(\pi_p\times\pi_p)}{\times}(\Omega_0^{n-1}S^{n-1}\times\Omega_0^{n-1}S^{n-1})^p & & G_0(pn) \\
\downarrow & & \downarrow{\scriptstyle \circ} \\
W^{(2m+1)}\times_{\pi_p}(\Omega_0^{n-1}S^{n-1})\times W^{(2m+1)}\times_{\pi_p}(\Omega_0^{n-1}S^{n-1})^p & \xrightarrow{\bar\theta_{(1,\dots,1)}\times\bar\theta_{(1,\dots,1)}} & G_0(pn)\times G_0(pn)
\end{array}
$$

Next define $\bar\theta_q^{(q)}: W^{(2m+1)}\times_{\pi_p}(\Omega_q^{n-1}S^{n-1})^p \longrightarrow G_{q^p}(pn)$ as $\bar\theta_p^{(q)}(\omega\ ;\ l_1,\cdots,l_p) = \bar\rho(\omega)\cdot(l_1*\cdots*l_p)\cdot\bar\rho(\omega)^{-1}$. Then we obtain following proposition easily.

PROPOSITION 5-11. *The following diagram is commutative.*

(5-21)

$$
\begin{array}{ccc}
W^{(2m+1)}\times_{\pi_p}(\Omega_q^{n-1}S^{n-1}\times\Omega_0^{n-1}S^{n-1}) & \xrightarrow{id\times\pi_p(\circ)^p} & W^{(2m+1)}\times_{\pi_p}(\Omega_0^{n-1}S^{n-1})^p \\
\downarrow{\scriptstyle \triangle\times id} & & \downarrow{\scriptstyle \bar\theta_p} \\
(W^{(2m+1)}\times W^{(2m+1)})\underset{\pi_p\times\pi_p}{\times}(\Omega_q^{n-1}S^{n-1}\times\Omega_0^{n-1}S^{n-1})^p & & G_0(pn) \\
\downarrow & & \uparrow{\scriptstyle \circ} \\
W^{(2m+1)}\times_{\pi_p}(\Omega_0^{n-1}S^{n-1})\times W^{(2m-1)}\times_{\pi_p}(\Omega_0^{n-1}S^{n-1})^p & \xrightarrow{\bar\theta_p^{(q)}\times\bar\theta_p} & G_{q^p}(pn)\times G_0(pn)
\end{array}
$$

Remark 5-12. *By remark* 4-4, $\bar\theta_p$, *and* $\bar\theta_p^{(q)}$ *can be extended on* $J^{(t)}\sum_p\times_{\Sigma_p}(\Omega^{n-1}S^{n-1})^p\to G(pn)$.

§ 6. Computation of the spectral sequence.

6-1. We shall introduce the extended p-th power operations $\bar Q_j$, $j=0, 1,2,\cdots$ on $H_*(BSF, Z_p)$ and $H_*(SF, Z_p)$, where p is an odd prime number. For an element $x\in H_*(BSF, Z_p)$ and $j\geq 0$, we shall pick up a large number n divisible by p^s for large s, and represent x as an element of $H_*(BSG(n), Z_p)$, and then define $\bar Q_j(x)$ as the element $\bar\theta_*(e_j\otimes x^p)$. Then by Proposition 4-11, $\bar Q_j(x)$ does not depend on the choice of n. For $x\in H_*(SF, Z_p)$ we shall define $\bar Q_j(x)$ similarly.

These operations $\bar Q_j$ have the similar properties as the extended p-th power operation Q_j defined by Dyer-Lashof [4].

(6-1) a) $\bar Q_i$ is Z_p-module homomorphism. $j=0,1,2,\cdots$

b) $\bar Q_0$ is the Pontrjagin p-th power.

c) $\bar{Q}_{2j-1} = \beta_p \bar{Q}_{2j}$, where β_p is the Bockstein operation.

d) For $x \in H_r(SF, Z_p)$ or $x \in H_r(BSF, Z_p)$, $\bar{Q}_{2j}(x) = 0$ unless the change of dimension $2j + pr - r$ is even multiple of $p-1$.

e) Cartan-formula holds, i.e. for $x \in H_r(BSF, Z_p)$, $y \in H_s(BSF, Z_p)$ or $x \in H_r(SF, Z_p)$, $y \in H_s(SF, Z_p)$, following formula holds:

$$\bar{Q}_{2j}(xy) = (-1)^{rs(p-1)/2} \sum_{i=0}^{j} \bar{Q}_{2i}(x) \bar{Q}_{2j-2i}(y).$$

Now we shall consider the following principal fibering $SF \to ESF \to BSF$. And then consider the Serre spectral sequence associated with this fibering. Then we obtain the following proposition.

PROPOSITION 6-1. (*transgression theorem*) *In the spectral sequence* $E^2_{**} \cong H_*(BSF, Z_p) \otimes H_*(SF, S_p)$, $E^\infty_{**} \cong Z_p$. *We obtain the following relation.*

Suppose $x \in E^2_{2n,0}$ *is a transgressive element, and* $y \in E^2_{0,2n-1}$ *is an element such that,* $\tau(x) = y$ *in* $E^{2n}_{0,2n-1}$. *Then*

(6-2) a) $\tau(\bar{Q}_0(x)) = \tau(x^p) = c\bar{Q}_{p-1}(y)$ in $E^{2np}_{0,2np-1}$, $c \neq 0$,

b) $\tau(x^{p-1} \otimes y) = c\bar{Q}_{p-2}(y)$ in $E^{2n(p-1)}_{0,2np-2}$, $c \neq 0$.

This proposition can be proved by the same method as Theorem 4-7 of Dyer-Lashof [4], so we omit the proof.

We will compute this spectral sequence using this proposition. So we must compute $\bar{Q}_{p-2}(x)$ and $\bar{Q}_{p-1}(x)$ in $H_*(SF, Z_p)$. The answer of this problem is the following proposition.

PROPOSITION 6-2. *For any* $J = (\varepsilon_1, j_1, \cdots, \varepsilon_r, j_r)$, $r \geq 1$, $\varepsilon_i = 0$ *or* 1, $j_i \geq 0$, *and* $\varepsilon = 0$ *or* 1, *and* $j > 0$. $\bar{Q}_{p-2}(Q_J \beta_p^\varepsilon x_j)$ *and* $\bar{Q}_{p-1}(Q_J \beta_p^\varepsilon x_j)$ *belong to* $G_{p^{r+1}}$, *and as elements of* $G_{p^{r+1}}/(G_{p^{r+1}+1} + \textit{decomp.})$, *they coincide with* $c(Q_{p-2} Q_J \beta^\varepsilon x_j)$ *and* $c(Q_{p-1} Q_J \beta^\varepsilon x_j)$ *respectively, where c is a non-zero constant. And decomp. means subspace of* $G_{p^{r+1}}$ *consisting of decomposable elements in* $H_*(SF)$.

Let $q_j \in H^{2j(p-1)}(BSF, Z_p)$ denote the j-th Wu-class $j = 1, 2, \cdots$, and Δq_j denotes its Bockstein image.

LEMMA 6-3. *For any* $x \in H_*(SF, Z_p)$, $x \in G_2$, $\langle x, \sigma(\triangle q_j) \rangle = 0$ *and* $\langle x, \sigma(q) \rangle = 0$, *where* σ *denotes the suspension homomorphism and* $\langle \tilde{x}_j, \sigma(\triangle q_j) \rangle \neq 0$ *and* $\langle \beta_p \tilde{x}_j, \sigma(q_j) \rangle \neq 0$.

LEMMA 6-4. *For* $J = (\varepsilon_1, j_1, \cdots, \varepsilon_r, j_r)$, $r \geq 0$, $\varepsilon_i = 0$ *or* 1, $j_i \geq 0$, $i \geq 0$, $\varepsilon = 0$ *or* 1, $j > 0$. $\bar{\theta}_{E_0 *}(e_i \underset{\pi}{\otimes} (Q_J \beta_p^\varepsilon x_j)^p)$ *belongs to* $G_{p^{r+1}+1}$, *if* $|E_0| \neq 0, 1, p$.

LEMMA 6–5. *If* J, i, ε *and* j *are the same as lemma* 6–4. *Then* $\bar{\theta}_{(1,0\ldots 0)*}(e_i\otimes(Q_J\beta_p^{\varepsilon}x_j)^p)$ *belongs to* $G_{p^{r+1}}$, *and as an element of* $G_{p^{r+1}}/(G_{p^{r+1}+1}+\text{decomp.})$ *it coincides with* $c(Q_iQ_J\beta_p^{\varepsilon}x_j)$, $c\neq 0$, *decomposable means in* $H_*(Q_0S^0:Z_p)$.

LEMMA 6–6. *If* $J=(\varepsilon_1, j_1, \cdots, \varepsilon_r, j_r)$, $r\geq 1$ *and* $i\leqq p-1$. *Then* $\bar{\theta}_{(1,\ldots,1)*}(e_i\otimes(Q_J\beta_p^{\varepsilon}x_j)^p)$ *belongs to* $G_{p^{r+1}+1}$.

These lemmas will be proved in §7.

Proof of Proposition 6–2. From the proposition 5–7, the following diagram is homotopy commutative:

$$\begin{tikzcd}
W\times_{\pi_p}Q_0(S^0)^p \arrow[r, "\bar{\theta}"] \arrow[d, "\triangle"] & SG \arrow[r] & \Omega_1^{\infty}S^{\infty} \\
\prod\limits_{\substack{E_0\in\mathscr{E}\\ |E_0|\neq 0}}(W\times_{\pi_p}Q_0(S^0)^p)_{E_0} \arrow[d, "\Pi\Sigma^2\bar{\theta}_{E_0}"] & & \\
\Pi Q_0(S^0)_{E_0} \arrow[rr, "\vee"] & & Q_0S^0 \arrow[uu, "(\vee(id))"']
\end{tikzcd}$$

On the other hand, we have

$$\begin{aligned}
&\triangle_*(e_i\otimes(Q_J\beta_p^{\varepsilon}x_j)^p)\\
&=\sum(-1)^*(e_{i_1}\otimes(Q_{J_1}\beta_p^{\varepsilon_1}x_{j_1})^p)\otimes\cdots\otimes(e_{i_q}\otimes(Q_{J_q}\beta_p^{\varepsilon_q}x_{j_q})^p).
\end{aligned}$$

So above homotopy commutative diagram, and Lemma 6–4, 6–5 and 6–6 show that $(\vee_*)(\Pi\Sigma^2\bar{\theta}_{E_0})_*\triangle_*(e_i\otimes(Q_J\beta_p^{\varepsilon}x_j)^p)$ belongs to $G_{p^{r+1}}$, and as an element of $H_*(Q_0S^0\,;Z_p)$, it is of the form $cQ_iQ_J\beta_p^{\varepsilon}x_j+x+y$, $x\in G_{p^{r+1}}$, $y\in G_{p^{r+1}+1}$, and x is decomposable as an element of $H_*(Q_0S^0)$. Since $i_*(x)\in H_*(SF\,;Z_p)$ can be expressed as x_1+x_2, $x_1\in G_{p^{r+1}}$, $x_2\in G_{p^{r+1}}$, and x_1 is decomposable as an element of $H_*(SF)$, this proves proposition 6–2.

It is well known the following results.

(6–3) a) $H_*(SO, Z_p)\cong\Lambda(u_1, u_2, \cdots)$ as an algebra, where deg $u_i=4i-1$.

b) $H_*(BSO, Z_p)\cong Z_p[v_1, v_2, \cdots]$ as an algebra, where deg $v_i=4i$, and $\triangle(v_j)=\sum\limits_{j_1+j_2=j}v_{j_1}\otimes v_{j_2}$. $v_0=1$.

c) In the homology spectral sequence associated to the universal fibering $SO\to ESO\to BSO$, $E^2_{**}\cong H_*(BSO, Z_p)\otimes H_*(SO, Z_p)$, $E^{\infty}_{**}\cong Z_p$.

i) $d_{4jp^k}(v_j^{p^k})=y_{p^kj}$ if $(j,p)=1$, $k\geq 0$

ii) $d_{4jp^{k-1}(p-1)}(v_{j\,p^k})=(v_j)^{p^{k-1}(p-1)}\otimes u_{j\,p^{k-1}}$, $(j,p)=1$, $k\geq 1$.

We shall denote the inclusion $SO \to SF$, $BSO \to BSF$ by j. Then by Peterson-Toda [12], $Im\ j_*\ H_*(BSF, Z_p)$ is the polynomial ring generated by $j_*(v_{\frac{p-1}{2}i})$, $i = 1, 2, \cdots$, and by dimensional reason, $j_*(v_j) = 0$, if $j \not\equiv 0$, $\frac{(p-1)}{2}$. We shall denote $\tilde{z}_j = j_*(v_{\frac{p-1}{2}j})$, $j = 0, 1, 2, \cdots$, then $\triangle(\tilde{z}_j) = \sum_{j_1+j_2=j} \tilde{z}_j \otimes \tilde{z}_{j_2}$. We consider $j_*(u_{\frac{p-1}{2}i}) = \tilde{y}_i$, $i = 1, 2, \cdots$. Then we obtain the following lemma.

LEMMA 6-7. $\tilde{y}_j = c\widetilde{\beta_p x_j} + x$, $x \in G_2$, $c \neq 0$, *in* $H_*(SF, Z_p)$.

Proof. Because $\langle \tilde{y}_j, \sigma(q_j) \rangle \neq 0$, so this follows from Lemma 6-3.

PROPOSITION 6-8. *As the algebraic generators of* $H_*(SF, Z_p)$, *we can choose the following elements:*

(6-4) i) $\tilde{x}_j, \tilde{y}_j$, $j = 1, 2, \cdots$

ii) $\tilde{x}_I$, $I \in H_i^+$, $i = 1, 2$.

iii) $\bar{Q}_{p-1} \cdots \bar{Q}_{p-1}(\tilde{x}_I)$, $I \in H_i^-$, $i = 1, 2$. $\bar{Q}_{p-1}$ operates on $\tilde{x}_I$, k-times $k \geq 0$.

iv) $\bar{Q}_{p-2}\bar{Q}_{p-1} \cdots \bar{Q}_{p-1}(\tilde{x}_I)$, $I \in \bar{H}_i$, $i=1, 2$. $\bar{Q}_{p-1}$ operates on $\tilde{x}_I$, k-times $k \geq 0$.

Proof. This follows trivially from Proposition 6-2 and Lemma 6-7.

We can now formulate the main Theorem and prove it.

THEOREM 2. i) $H_*(BSF, Z_p) = Z_p[\tilde{z}_1, \tilde{z}_2, \cdots] \otimes \Lambda(\sigma\tilde{x}_1, \sigma\tilde{x}_2, \cdots) \otimes C_*$, *where* C_* *is the free commutative algebra generated by* $\sigma\tilde{x}_J$, $J \in H_1 \cup H_2$, $\sigma\tilde{x}_j$ *and* $\sigma\tilde{x}_J$ *are primitive elements and* $\triangle(\tilde{z}_j) = \sum_{j_1+j_2=j} \tilde{z}_{j_1} \otimes \tilde{z}_{j_2}$.

ii) $H^*(BSF, Z_p) \cong Z_p[q_1, q_2, \cdots] \otimes \Lambda(\triangle q_1, \triangle q_2, \cdots) \otimes C$, $C = \Lambda_{I \in H_1^+ \cup H_2^+}((\sigma\tilde{x}_1)^*) \otimes \Gamma_{p\ J \in H_1^- \cup H_2^-}[(\sigma(\tilde{x}_J))^*]$, *where* $(\ \ )^*$ *denote the dual elements.*

Proof. ii) follows easily from i) and the following facts

a) $\langle x, q_j \rangle = 0$, $\langle x, \triangle q_j \rangle = 0$, if $x \in C_*$;

b) $\langle \sigma\tilde{x}_j, \triangle q_j \rangle \neq 0$, $j = 1, 2, \cdots$;

c) $\tilde{z}_i$ is in the image of $j_* : H_*(BSO) \to H_*(BSF)$.

So it is sufficient to prove i).

We shall consider the following formal spectral algebra:

$$'E^2_{**} \cong (Z_p[\tilde{z}_j] \otimes \wedge(\sigma\tilde{x}_j) \otimes C_*) \otimes H_*(SF, Z_p),$$

with differential d_r:

a) $d_r(xy) = d_r(x)y + (-1)^{\deg x} x d_r(y)$,

b) $d_{2(p-1)j\, p^k}((\tilde{z}_j)^{p^k}) = \tilde{y}_{p^k j}$, if $(j, p) = 1$, $k \geq 0$,

c) $d_{2(p-1)j\, p^{k-1}}(\tilde{z}_{j\, p^k}) = (\tilde{z}_j)^{p^{k-1}(p-1)} \otimes y_{j\, p^{k-1}}$, if $(j, p) = 1$, $k \geq 1$.

d) $d_{2j(p-1)}(\sigma(\tilde{x}_j)) = \tilde{x}_j$, $j = 1, 2, \cdots$,

e) $d_{p^k q}(\sigma\tilde{x}_1)^{p^k}) = \bar{Q}^k_{p-1}(\tilde{x}_I)$, $I \in H^-_i$, $i = 1, 2$ and $q = \deg\ (\sigma(x_I))$, $\bar{Q}^k_{p-1} = \bar{Q}_{p-1} \cdots \bar{Q}_{p-1}$, k-times, $k \geq 0$,

f) $d_{p^k(p-1)}(\sigma(\tilde{x}_I)^{p^k(p-1)} \otimes \bar{Q}^k_{p-1}(\tilde{x}_I)) = \bar{Q}_{p-2}\bar{Q}^k_{p-1}(\tilde{x}_I)$,
$q = \deg \sigma(\tilde{x}_I)$, $I \in H^-_i$, $i = 1, 2$, $k \geq 0$,

g) $d_q(\sigma(\tilde{x}_I)) = \tilde{x}_I$, $I \in H^+_i$, $i = 1, 2$, $q = \deg(\sigma(x_I))$.

Then d_r is determined uniquely and $'E^\infty_{*,*} \cong Z_p$. Then we shall difine the spectral algebra homomorphism $f^r : {'E^r_{*,*}} \to E^q_{*,*}$ with $f^2(z_j) = z_j$, $f^2(\sigma(x)) = \sigma(x)$, $x = \tilde{x}_j$ or $\tilde{x}_J$. By Proposition 6-1, and the properties of d_r in the homology spectral sequence associated to $SO \to ESO \to BSO$, f^r extsts. Then the comparision theorem for spectral sequence shows that f^r is an isomorphism for $r \geq 2$. So we obtain $H_*(BSF, Z_p) \cong Z_p[\tilde{z}_j] \otimes \Lambda(\sigma(\tilde{x}_j)) \otimes C_*$. So we obtain the theorem.

§7. Proof of Lemma 6-3, 6-4, 6-5, and 6-6

7-1. The object of this section is to prove Lemma 6-3, 6-4, 6-5 and 6-6. p is always an odd prime number.

Let X be a finite connected CW complex with base point, and $f : X \to SG(N)$ be a continuous map. Let $\xi = \xi_f \to SX$ be the spherical fiber space of fiber S^{N-1} over SX associated to f. Let $f : X \times S^{N-1} \to S^{N-1}$ be the representative of f, and $G(f) : X * S^{N-1} \to S^N$ be the Hopf construction of f.

LEMMA 7-1. *Let $T(\xi)$ be the Thom complex of $\xi = \xi_f$. Then $T(\xi)$ is homotopy equivalent to $S^N \underset{G(f)}{\cup} C(X * S^{N-1})$, the mapping cone of $G(f)$.*

Let $g : X \to \Omega^{N-1}_0 S^{N-1}$ be a continuous map, and consider $\bar{g} = (g \vee id_{N-1}) : X \to \Omega^{N-1}_1 S^{N-1} \to SG(N)$. Let $x_0 \in X$, $s_0 \in S^{N-1}$ be the base points, then $X * S^{N-1}/(X * s_0) \vee (x_0 * S^{N-1})$ is equal to $X \wedge S^1 \wedge S^{N-1}$, and this gives the homotopy equivalence between $X * S^{N-1}$ and $X \wedge S^1 \wedge S^{N-1}$, and we identify $X * S^{N-1}$ with $X \wedge S^1 \wedge S^{N-1}$ by this map.

LEMMA 7-2. *$G(\bar{g}) : X * S^{N-1} \to S^N$ is homotopic to $(\widetilde{id_1 \wedge g}) : X \wedge S^N \to S^N$, where $\widetilde{id_1 \wedge g}$ is adjoint map of $id_1 \wedge g : X \to \Omega_0^N S^N$.*

LEMMA 7-3. *Let X_1, X_2 be finite connected CW complexes with base points. And f_i; $X_i \to S^{n_i}$ are continuous maps preserving base points, $i = 1, 2$, $n_i > 0$. And assume $f_i^* : \tilde{H}^*(S^{n_i} : Z_p) \to \tilde{H}^*(X_i : Z_p)$ are zero maps, $i = 1, 2$. Consider $f_1 \wedge f_2 : X_1 \wedge X_2 \to S^{n_1} \wedge S^{n_2} = S^{n_1+n_2}$. Then in $H^*(S^{n_1+n_2} \underset{f_1 \wedge f_2}{\cup} C(X_1 \wedge X_2) : Z_p)$, $P^j(s) = 0$, $j \geqq 1$, where P^j is Steenrod reduced power, and $s \in H^{n_1+n_2}(S^{n_1+n_2} \cup C(X_1 \wedge X_2) : Z_p)$ is the generator representing $S^{n_1+n_2}$.*

7-2. *Proof of Lemma* 6-3. If $x \in H_*(SF, Z_p)$ is a decomposable element, it is well known that $\langle x, \sigma(\triangle q_j) \rangle = \langle x, \sigma(q_j) \rangle = 0$. By the result of §2 and §3, the algebraic generators of Pontrjagin ring $H_*(SF)$ are in the image of $i_*(\xi_1 \wedge \xi_r)_* : H_*(L_p \wedge L_p^r) \to H_*(Q_0 S^0) \to H_*(SF)$, $r \geq 0$. So to prove the result that for $x \in G_2$, $\langle x, \sigma(\triangle q_j) \rangle = \langle x, \sigma(q_j) \rangle = 0$, we can assume that x is in the image of $i_*(\xi_1 \wedge \xi_r)$, $r \geq 1$. Let $g : (L_p^{(m)})^{r+1} \to \Omega_0^{N-1} S^{N-1}$ be the representative of $\xi_1 \wedge \xi_r$, $r \geq 1$. And consider $\bar{g} = g \vee id_{N-1} : (L_p^{(m)})^{r+1} \to \Omega_0^{N-1} S^{N-1} \to SG(N)$. Then by lemma 7-1 and 7-2, Thom complex of $\xi_{\bar{g}}$ is of the form $S^N \underset{g}{\cup} C((L_p^{(m)})^{r+1} \wedge S^N)$. By lemma 7-3, in $H^*(S^N \cup C((L_p^{(m)})^{r+1} \wedge S^N) : Z_p)$, $P^j(s_N)$ and $\triangle P^j(s_N)$ is equal to zero, $j \geq 1$. This proves the results that $\langle x, \sigma(\triangle q_j) \rangle = \langle x, \sigma(q_j) \rangle = 0$. To prove the results that $\langle \tilde{x}_j, \sigma(\triangle q_j) \rangle \neq 0$, $\langle \widetilde{\beta_p x_j}, \sigma(q_j) \rangle \neq 0$, $j \geq 1$, it is sufficient to prove that $\sigma(\triangle q_j) \neq 0$ in $H^*(SF : Z_p)$. This is the result of Peterson-Toda [12], indeed they proved that there is a continuous map $h : SL_p \to BSF$ such that $h^*(\triangle q_j) \neq 0$.

7-3. At first we shall prove the following lemma.

LEMMA 7-4. *Let $\xi = (\vee)_* \circ (\triangle_p)_* : H_*(\Omega_0^{n-1} S^{n-1} : Z_p) \to H_*(\Omega_0^{n-1} S^{n-1} \times \cdots \times \Omega_0^{n-1} S^{n-1} : Z_p) \to H_*(\Omega_0^{n-1} S^{n-1} : Z_p)$. If $x \in H_r(\Omega_0^{n-1} S^{n-1})$ belongs to G_q, $r > 0$. then $\xi(x)$ is of the form $\sum y^p$, $y \in G_q$.*

Proof. Since ξ is an algebra homomorphism, it is sufficient to assume $x = Q_J \beta_p^\varepsilon x_j$. Then Cartan formula shows the lemma.

7-4. *Proof of lemma* 6-4. By proposition 5-8, the following diagram is commutative.

$$\begin{array}{ccc} H_*(L_p^{(2m+1)}\times\Omega_0^{n-1}S^{n-1}) & \xrightarrow{id\times\triangle_p} H_*(W^{(2m+1)}\times_{\pi_p}(\Omega_0^{n-1}S^{n-1})^p) \xrightarrow{\bar\theta_{E_0}} & H_*(G_0(pn)) \\ \Big\downarrow \bar h_{E_0}\times\bar l_{E_0} & (*)_* & \Big\downarrow *id_{pn-1} \\ H_*(G_p(pn)\times G_0(pn)) & \xrightarrow{\qquad\qquad\qquad} & H_*(G_0(2pn)) \end{array}$$

By lemma 2-11, and its proof, the element $e_i\otimes(Q_I(\beta_p^\varepsilon x_j)^p\in H_*(W^{(2m+1)}\times_{\pi_p}(\Omega_0^{n-1}S^{n-1})^p)$ is in the image of $(id\times\triangle_p)_*(A)$, where A is the submodule of $H_*(L_p^{(2m+1)}\times\Omega_0^{n-1}S^{n-1})$ generated by $e_k\otimes Q_I\beta_p^\varepsilon x_l$, $r=|I|=|J|$, $l=0,1,\cdots$, $k=0,1,2,\cdots$, $\varepsilon=0$ or 1. So that it is sufficient to prove that $(*)_*(\bar h_{E_0}\times\bar l_{H_0})_*(e_k\otimes Q_I\beta_p^\varepsilon x_l)$ belongs to $G_{p^{r+1}+1}$, where $e_k\otimes Q_I\beta_p^\varepsilon x_l\in A$. If $\deg(Q_I\beta_p^\varepsilon x_l)=0$, then $\deg e_k>0$, so that $(*)_*(\bar h_{E_0}(e_k)\otimes\bar l_{E_0}(Q_I\beta_p^\varepsilon x_l))=0$. So that we can assume $\deg(Q_I\beta_p^\varepsilon x_l)>0$. On the other hand $\bar l_{E_0}:\Omega_0^{n-1}S^{n-1}\to G_0(pn)$ is homotopic to the map $:\Omega_0^{n-1}S^{n-1}\xrightarrow{\triangle_{k_0}}\Omega_0^{n-1}S^{n-1}\times\cdots\times\Omega_0^{n-1}S^{n-1}\xrightarrow{\wedge}\Omega_0^{k_0(n-1)}S^{k_0(n-1)}\xrightarrow{i}G_0(pn)$, $1\lneqq k_0=|E_0|\lneqq p$. So $\bar l_{E_0*}(Q_I\beta_p^\varepsilon x_l)$ belongs to $G_{p^{k_0r+k_0-1}}$ by Cartan formula for Q_I, proposition 2-8, iii), and proposition 2-10. Let $\bar h_{E_0,0}$ denote the following map: $L_p^{(2m+1)}\xrightarrow{\bar h_{E_0}}G_0(pn)\xrightarrow{i}\Omega_p^{pn}S^{pn}\xrightarrow{(\vee(-pid))}\Omega_0^{pn}S^{pn}$. Then $L^{(2m+1)}\times G_0(pn)\xrightarrow{\bar h_{E_0}\times id}G_p(pn)\times G_0(pn)\xrightarrow{*}G_0(2pn)\xrightarrow{i}\Omega_0^{2pn}S^{2pn}$ is homotopic to the map, $L_p^{(2m+1)}\times G_0(pn)\xrightarrow{\bar h_{E_0,0}\times\triangle_2}\Omega_0^{pn}S^{pn}\times G_0(pn)\times G_0(pn)\xrightarrow{id\times i\times\triangle_p}\Omega_0^{pn}S^{pn}\times\Omega_0^{pn}S^{pn}\times(G_0(pn))^p\xrightarrow{\wedge\times(i)^p}\Omega_0^{2pn}S^{2pn}\times(\Omega_0^{2pn}S^{2pn})^p\xrightarrow{id\times\vee}\Omega_0^{2pn}S^{2pn}\times\Omega_0^{2pn}S^{2pn}\xrightarrow{\vee}\Omega_0^{2pn}S^{2pn}$. So the above homomorphism maps A in $G_{p^{k_0r+k_0-1}+1}$ by using lemma 7-4 and the result of §2. On the other hand $k_0r+k_0-1\geqq r+1$, since $k_0\geqq 2$, $r\geqq 0$. This proves the lemma.

7-5. We shall consider $\bar h_1\equiv\bar h_{(1,0\ldots 0)};\ L_p^{(2m+1)}\to G_p(pn)$ defined in §5. Let $\bar h_1:L_p^{(2m+1)}\times S^{pn-1}\to S^{pn-1}$ be the representative of $\bar h_1$. And consider the mapping cone $C_{\bar h_1}$ of $\bar h_1$.

LEMMA 7-5. *In* $H^*(C_{\bar h_1}:Z_p)$, $P^j(s)\neq 0$, $\triangle P^j(s)\neq 0$, $j=1,2,\cdots[2m+1/p-1]$, *where* $s\in H^{pn-1}(C_{\bar h}:Z_p)$ *be the generator representing* S^{pn-1} *of* $C_{\bar h_1}=S^{pn-1}\cup C(L_p^{(2m+1)}\times S^{pn-1})$.

This lemma is proved by tediously long caluculation acording to the result of Nakaoka [10], so we omit the proof.

Next define $\bar h_{1,0}$ as follows, $\bar h_{1,0}:L_p^{(2m+1)}\xrightarrow{\bar h_1}G_p(pn)\xrightarrow{i}\Omega_p^{pn}S^{pn}\xrightarrow{(\vee\text{-}pidn)}\Omega_0^{pn}S^{pn}$.

COROLLARY 7-6. *In* $H^*(C_{\bar{h}_{1,0}} : Z_p)$, $P^j(S) \neq 0$, $\triangle P^j(S) \neq 0$, $j = 1, \cdots [(2m+1)/(p-1)]$, *for* $s \in H^{pn-1}(C_{\bar{h}_{1,0}} : Z_p)$ *generator.*

LEMMA 7-7. *In* $H_*(L_p : Z_p)$ *for any* $i_0 > 1$, *there is a number* $r > 0$, such that $P^r_*(e_{2i_0(p-1)}) \neq 0$. *or* $P^r_*(e_{2i_0(p-1)-1}) \neq 0$.

LEMMA 7-8. *Consider* $\bar{h}_{1,0*} : H_*(L_p^{(2m+1)} : Z_p) \to H_*(\Omega_0^{n-1} S^{n-1} : Z_p)$, *then we have*

$$\bar{h}_{1,0*}(e_{2i(p-1)}) = cx_i + x$$

$$\bar{h}_{1,0*}(e_{2i(p-1)}) = c\beta_p x_i + y, \quad i = 1, 2, \cdots [(2m+1)/(p-1)]$$

$$\bar{h}_{1,0*}(e_j) = 0 \quad \textit{if} \quad j \neq 2i(p-1) \quad \textit{or} \quad 2i(p-1)-1.$$

where $x, y \in G_2$, *and* $c \in Z_p$ *is non zero constant not dependent on* i.

Proof. By lemma 7-6 and lemma 6-3, $\bar{h}_{1,0*}(e_{2i(p-1)}) = c_i x_i + x$, $c_i \neq 0$, $x \in G_2$. We shall prove that c_i is indepent on i by induction. Assume $c = c_1 = \cdots = c_{i_0-1}$ for $i_0 > 1$. By lemma 7-7, there exists $r > 0$, such that $P^r_*(e_{2i_0(p-1)}) = ae_{2(i_0-r)(p-1)}$, or $P^r_*(\beta_p e_{2i_0(p-1)}) = a\beta_p e_{2(i_0-r)(p-1)}$, for some $0 \neq a \in Z_p$. And since $x_i = h_{0*}(e_{2i(p-1)})$ for $h_0 : L_p^{(2m+1)} \to \Omega_0^{n-1} S^{n-1}$, $P^r_*(x_{i_0}) = ax_{i_0-r}$ or $P^r_*(\beta_p x_{i_0}) = a\beta_p x_{i_0-r}$. So that $\bar{h}_{1,0*}(P^r_* e_{2i_0(p-1)}) = \bar{h}_{1,0*}(ae_{(i_0-r)(p-1)}) = acx_{i_0-r} + x'$ or $\bar{h}_{1,0*}(P^r_*\beta_p e_{2i(p-1)}) = ac\beta_p x_{i_0-r} + y'$ for some, x' or $y' \in G_2$. On the other hand by naturality of P^r_* or $P^r\beta_*$, $\bar{h}_{1,0*}(P^r_* e_{2i_0(p-1)}) = P^r_*(\bar{h}_{1,0*}(e_{2i_0(p-1)})) = P^r_*(c_{i_0} x_{i_0} + x)$ $= ac_{i_0} x_{i_0-r} + P^r_*(x)$ or $\bar{h}_{1,0*}(P^r\beta_p e_{2i_0(p-1)}) = P^r_*\beta_p(\bar{h}_{1,0*}(e_{2i_0(p-1)})) = P^r_*\beta_p(c_{i_0} x_{i_0} + x)$ $= aci_0\beta_p x_{i_0-r} + P^r_*\beta_p x$. On the other hand by the result of Nishida [11], $P^r_*(x)$, $P^r_*\beta_p x \in G_2$. This shows $c_{i_0} = c$. The results that $\bar{h}_{1,0*}(e_j) = 0$ for $j \neq 2i(p-1)$, $2i(p-1)-1$, follows from the Remark in §5 that $\bar{h}_1$ factors through $B\sum_p(t)$ as follows, $\bar{h}_1 : L_p^{(2m+1)} \to B\sum_p^{(t)} \to G_p(pn)$.

7-6. *Proof of Lemma* 6-5. We are given an element $e_i \otimes_{\pi_p} (Q_I \beta_p^{\varepsilon} x_j)^p \in H_*(W^{(2m+1)} \times_{\pi_p} (\Omega_0^{n-1} S^{n-1})^p : Z_p)$ where $J = (\varepsilon_1, j_1, \cdots, \varepsilon_r, j_r)$, $r \geq 0$. By proposition 1-9 the following diagram is commutative.

$$\begin{array}{ccc} H_*(L_p \times Q_0 S^0) & \xrightarrow{id \times \triangle_p} & H_*(W \times_{\pi_p} (Q_0 S^0)^p) \\ \downarrow h \times id & \wedge & \downarrow \theta \\ H_*(Q_p S^0 \times Q_0 S^0) & \longrightarrow & H_*(Q_0 S^0) \end{array}$$

On the other hand by proposition 5-8, the following diagram is commutative.

$$\begin{array}{ccc} H_*(L_p\times Q_0S^0) & \xrightarrow{id\times\triangle_p} & H_*(W\times_{\pi_p}(Q_0S^0)^p) \\ \downarrow \bar{h}_{(1,0\ldots0)}\times\bar{l}_{(1,0\ldots0)} & \wedge & \downarrow \bar{\theta}_{(1,0\ldots0)} \\ H_*(QS_0\times Q_0S^0) & \longrightarrow & H_*(Q_0S^0) \end{array}$$

We can choose an element $x\in H_*(L_p\times Q_0S^0)$ such that $(id\times\triangle_p)_*(x)=e_i\otimes(Q_J\beta_p^\varepsilon x_j)^p$. Then by lemma 2-11, x is of the form $\sum c(k,I,\varepsilon',l)(e_k\otimes Q_I\beta_p^\varepsilon x_l)$, where $j\geq 0$, $I=(\varepsilon_1,i_1,\cdots,\varepsilon_r,j_r)$, $\varepsilon'=0$ or 1, and $l=0,1,2,\cdots$. So $Q_iQ_J\beta_p^\varepsilon x_j=\sum c(k,I,\varepsilon',l)(h_*(e_k)\wedge(Q_I\beta_p^{\varepsilon'}x_l))$. On the other hand $\bar{\theta}_{(1,0\ldots0)*}(e_i\otimes Q_J\beta_p^\varepsilon x_j)^p)=\sum c(k,I,\varepsilon',l)(\bar{h}_{(1,0\ldots0)*}(e_k)\wedge(Q_I\beta_p^{\varepsilon'}x_l))=\sum c(0,I,\varepsilon',l)(\bar{h}_{(1,0\ldots0)*}(e_0)\wedge(Q_I\beta_p^{\varepsilon'}x_l))+\sum_{j\neq 0}c(k,I,\varepsilon',l)(\bar{h}_{(1,0\ldots0)*}(e_k)\wedge(Q_I\beta_p^{\varepsilon'}x_l))$. By lemma 7-8, $\bar{h}_{(1,0\ldots0)*}(e_k)=c\cdot h_*(e_k)+x$, if $k\neq 0$, and $x\in G_2$, $c\neq 0$. And by extension of proposition 2-8, iv), Cartan formula, and extension of proposition 2-10 shows that $x\wedge(Q_I\beta_p^{\varepsilon'}x_l)\in G_{p^{r+1}+1}$, and by lemma 7-4, $h_*(e_0)\wedge(Q_I\beta_p^{\varepsilon'}x_l)=\bar{h}_{(1,0\ldots0)*}(e_0)\wedge(Q_I\beta_p^{\varepsilon'}x_l)$ belongs to $G_{p^{r+1}}$, and decomposable. So $\bar{\theta}_{(1,0\ldots0)*}(e_i\otimes(Q_J\beta_p^\varepsilon x_j)^p)=cQ_iQ_J\beta_p^\varepsilon x_j+x+y$, for $x\in G_{p^{r+1}}$, x: decomposable, and $y\in G_{p^{r+1}+1}$. This shows the lemma.

7.7. *Proof of Lemma* 6-6. By proposition 5-9, $\bar{\theta}_{(1,\ldots,1)*}=\bar{\theta}_{p*}$. If $i=0$, then $\bar{\theta}_{p*}(e_0\otimes(Q_J\beta_p^\varepsilon x_j)^p)=\wedge_*((Q_J\beta_p^\varepsilon x_j)^p)=\wedge_*((Q_J\beta_p^\varepsilon x_j)\otimes\cdots\otimes(Q_J\beta_p^\varepsilon x_j))$, where $\wedge:\Omega_0^{n-1}S^{n-1}\times\cdots\times\Omega_0^{n-1}S^{n-1}\to\Omega_0^{(n-1)p}S^{(n-1)p}\xrightarrow{i}G_0(pn)$. So this element belongs to $G_{p^{pr+p-1}}$. So lemma is valid for this case. By Remark 5-12, $\bar{\theta}_{p*}(e_i\otimes x^p)=0$, if $i\not\equiv 0 \bmod (p-1)$ or $(p-2)$ or $2(p-1)-1$. So we can assume $i=p-2$ or $p-1$. And we shall prove in the case $i=p-2$, when $i=p-1$ the proof is similar. By proposition 5-11, the following diagram is commutative:

$$\begin{array}{ccc} H_*(W^{(2m+1)}\times_{\pi_p}(\Omega_p^{n-1}S^{n-1}\times\Omega_0^{n-1}S^{n-1})^p) & \xrightarrow{(id\times(\circ)^p)_*} & H_*(W^{(2m+1)}\times_{\pi_p}(\Omega_0^{n-1}S^{n-1})^p) \\ \downarrow(\triangle\times id)_* & & \downarrow\bar{\theta}_{p*} \\ H_*((W^{(2m+1)}\times W^{(2m+1)})\underset{\pi_p\times\pi_p}{\times}(\Omega_p^{n-1}S^{n-1}\times\Omega_0^{n-1}S^{n-1})^p) & & H_*(G_0(pn)) \\ \downarrow & & \uparrow(\circ)_* \\ H_*(W^{(2m+1)}\times_{\pi_p}(\Omega_p^{n-1}S^{n-1})^p\times W^{(2m+1)}\times_{\pi_p}(\Omega_0^{n-1}S^{n-1})^p) & \xrightarrow{\bar{\theta}_*^{(p)}\times\bar{\theta}_{p*}} & H_*(G_{p^p}(pn)\times G_0(pn)) \end{array}$$

On the other hand $Q_J\beta_p^\varepsilon x_j\in H_*(\Omega_0^{n-1}S^{n-1})$ belongs to the image of $B_r\supseteq H_*(\Omega_p^{n-1}S^{n-1}\times\Omega_0^{n-1}S^{n-1})$ by $(\circ)_*$, where B_r is the submodule of $H_*(\Omega_p^{n-1}S^{n-1})\otimes H_*(\Omega_0^{n-1}S^{n-1})$, generated by $(\beta_p^\varepsilon x_k)\otimes(Q_I\beta_p^{\varepsilon'}x_l)$, $k=0,1,\cdots,\varepsilon$, $\varepsilon'=0$ or 1, $l=0,1,2,\cdots$, $|I|=r-1$, $r\geq 1$. We shall prove this lemma by induction or r.

i) $r=1$. It is sufficient to prove $(\circ)_*(\bar{\theta}_p\times\bar{\theta}_p)_*(\triangle\times id)_*(e_{p-2}\otimes(\beta_p^{\varepsilon}x_k\otimes\beta_p^{\varepsilon'}x_l))$ belongs to G_{p^2+1}. $(\triangle\times id)_*((e_{p-2})\otimes(\beta_p^{\varepsilon}x_k\otimes\beta_p^{\varepsilon'}x_l)^p)=\sum_{i_1+i_2=p-2}(-1)^*(e_{i_1}\otimes(\beta_p^{\varepsilon}x_k)^p)\otimes(e_{i_2}\otimes(\beta_p^{\varepsilon'}x_l)^p)$. On the other hand $\bar{\theta}_p^{(p)}{}_*(e_{i_1}\otimes\beta_p^{\varepsilon}x_k)^p)=0$ except the case $i_1=0$, $p-2$, and so on $\bar{\theta}_{p*}(e_{i_2}\otimes(\beta_p^{\varepsilon'}x_l)^p)=0$. So that $(\bar{\theta}_p^{(p)}\otimes\bar{\theta}_p)_*(\triangle\times id)_*(e_{p-2}\otimes(\beta_p^{\varepsilon}x_k\otimes\beta_p^{\varepsilon'}x_l)^p)=\bar{\theta}_p^{(p)}{}_*(e_{p-2}\otimes(\beta_p^{\varepsilon}x_k)^p)\otimes\bar{\theta}_{p*}(e_0\otimes(\beta_p^{\varepsilon'}x_l)^p)+(-1)^*\bar{\theta}_p^{(p)}{}_*(e_0\otimes(\beta_p^{\varepsilon}x_k)^p)\otimes\bar{\theta}_{p*}(e_{p-2}\otimes(\beta_p^{\varepsilon'}x_l)^p)$. And $\bar{\theta}_p^{(p)}{}_*(e_0\otimes(\beta_p^{\varepsilon}x_k)^p)=\underset{i}{\wedge}{}_{p*}((\beta_p^{\varepsilon}x_k)^p)\in G_{p^{p-1}}$, if $\deg\beta_p^{\varepsilon}x_k>0$, where $\wedge_p:\Omega_p^{n-1}S^{n-1}\times\cdots\times\Omega_p^{n-1}S^{n-1}\to\Omega_{pp}^{p(n-1)}S^{p(n-1)}\to G_{p^{p(pn)}}$. And $\bar{\theta}_{p*}(e_0\otimes(\beta_p^{\varepsilon'}x_l)^p)=\wedge_*((\beta_p^{\varepsilon'}x_l)^p)\in G_{p^{p-1}}$ if $\deg\beta_p^{\varepsilon'}x_l>0$, where $\wedge:\Omega_0^{n-1}S^{n-1}\times\cdots\times\Omega_0^{n-1}S^{n-1}\to\Omega_0^{p(n-1)}S^{p(n-1)}\to G_0(pn)$. So lemma in this case is proved by dividing three cases a) $\deg(\beta_p^{\varepsilon}x_k)>0$, $\deg(\beta_p^{\varepsilon'}x_l)>0$, b) $\deg(\beta_p^{\varepsilon}x_k)>0$, $\deg(\beta_p^{\varepsilon'}x_l)=0$ c) $\deg(\beta_p^{\varepsilon}x_k)=0$, $\deg(\beta_p^{\varepsilon'}x_l)>0$.

ii) *We assume that lemma holds when* $r\leq r_0$, $r_0\geq 1$. *We shall prove* $(\circ)_*(\bar{\theta}_p^{(p)}\times\bar{\theta}_p)_*(\triangle\times id)_*(e_{p-2}\otimes(\beta_p^{\varepsilon}x_k\otimes Q_I\beta_p^{\varepsilon'}x_l)^p)$ belongs to $G_{p^{r_0+2}+1}$, where $I=(\varepsilon_1,j_1,\cdots,\varepsilon_{r_0},j_{r_0})$. $(\bar{\theta}_p^{(p)}{}_*\times\bar{\theta}_p)_*\bar{\theta}_p)_*(\triangle\times id)_*(e_{p-2}\otimes(\beta_p^{\varepsilon}x_k\otimes Q_I\beta_p^{\varepsilon'}x_l)^p)=\bar{\theta}_p^{(p)}{}_*(e_{p-2}\otimes(\beta_p^{\varepsilon}x_k)^p)\otimes\bar{\theta}_{p*}(e_0\otimes(Q_I\beta_p^{\varepsilon'}x_l)^p)+(-1)^*\bar{\theta}_p^{(p)}{}_*(e_0\otimes(\beta_p^{\varepsilon}x_k)^p)\otimes\bar{\theta}_{p*}(e_{p-2}\otimes(Q_I\beta_p^{\varepsilon'}x_l)^p)$. Then lemma in this case is proved by using induction dividing two cases a) $\deg\beta_p^{\varepsilon}x_k>0$, $\deg(Q_I\beta_p^{\varepsilon'}x_l)>0$, b) $\deg\beta_p^{\varepsilon}x_k=0$, $\deg(Q_I\beta_p^{\varepsilon'}x_l)>0$. And these proves the lemma.

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