SEQUENTIAL GAUSSIAN MARKOV INTEGRALS*

JOHN A. BEEKMAN

I. Introduction

In [6] R.H. Cameron defined and studied a sequential Wiener integral. This was motivated by the function space integral R.P. Feynman used in [12] to give a solution to the Schröedinger equation. In [5] the present author studied sequential Gaussian Markov integrals with a positive parameter. This paper gives sufficient conditions on the integrand for such integrals to exist, when the parameter is complex. These sequential integrals are related to ordianry Gaussian Markov integrals through a Fourier transform type formula extended from [5]. We shall show that such integrals are equal to conditional Wiener integrals of suitably modified functionals.

As is well-known, function space itegrals are used in many fields. We will use the sequential integrals in certain applications in physics, but it is believed they will prove useful in other areas. Specifically, we shall show that sequential integrals of appropriate functionals satisfy generalized Schröedinger equations and Dirac delta function conditions. We shall also prove that certain sequential integrals solve integral equations formally analogous to the differential equations of [5]. Our use of the word potential is the quantum mechanics use—see [5], for example.

For completeness, several references will be mentioned. References [1] through [5] consider the connection between Gaussian Markov stochastic processes and generalized Schröedinger equations. Reference [20] discusses the same connection, with heavy emphasis on the physics involved. R.H. Cameron's papers [6] through [9] have contributed much to this area.

II. Sequential Integrals

Let $\{X(\tau),\ s \leq \tau \leq t\}$ be a Gaussian Markov process with transition density function

$$(2.1) \qquad p(x, s; y, t) = \frac{\partial}{\partial y} P[X(t) \le y \,|\, X(s) = x]$$

$$= \{2\pi A(s, t)\}^{-1/2} \exp\left\{-\frac{[y - (v(t)/v(s))x]^2}{2A(s, t)}\right\}$$

where

$(2.2) \quad A(s, t) = [u(t)v(t) - u(s)v^2(t)/v(s)], \quad s \le t$

$(2.3) \quad u(\tau) \ge 0, \quad v(\tau) > 0, \quad s \le \tau \le t$

$(2.4) \quad u''(\tau), \ v''(\tau)$ are continuous, $s \le \tau \le t$

$(2.5) \quad [v(\tau)u'(\tau) - u(\tau)v'(\tau)] > 0, \quad s \le \tau \le t.$

The u and v functions for the Wiener, Doob-Kac, and Ornstein-Uhlenbeck processes are given in [3]. The transition density function determines a stochastic process with $X(s) = x, X(t) = y$ with probability one since

$$\lim_{t \to s+} P[X(t) \le y \,|\, X(s) = x] = \begin{cases} 1, & y > x \\ 0, & y < x. \end{cases}$$

Almost all of the sample functions of the process are continuous, since there is a transformation of the process to the Weiner process—see Lemma 2 of [2] and its references. Denote the expected value of a functional $F[X]$ for this process by

$$E\{F[X]\,|\,X(s) = x, X(t) = y\}.$$

Expectations not tied down at t can be obtained thruogh the equation

$$(2.6) \qquad E\{G[X]\,|\,X(s) = x\} = \int_{-\infty}^{\infty} E\{G[X]\,|\,X(s) = x, X(t) = y\} p(x, s; y, t) dy.$$

In [6], a complex variance parameter λ was used to consider an analytic Feynman integral. We will now introduce a parameter λ which serves essentially the same purpose. Let $p^*(x, s; y, t)$ equal $p(x, s; y, t)$ with $A(s, t)$ replaced by $A(s, t)/\lambda$, where $Re \, \lambda \ge 0, \ \lambda \ne 0.$

Let $\tau \equiv [\tau_1, \cdots, \tau_n]$ be a variable vector of a variable number of dimensions whose components form a subdivision of $[s, t]$ so that $\tau_0 \equiv s < \tau_1 < \tau_2 < \cdots < \tau_n \equiv t.$ Let $\|\tau\| = \max_{j=1,\ldots,n}(\tau_j - \tau_{j-1}).$ Let $\xi \equiv [\xi_1, \cdots, \xi_{n-1}]$ denote an unrestricted real vector, where n is determined by τ, and let $\xi_0 \equiv x, \ \xi_n \equiv y.$ Let $\psi_{\tau,\xi}(\tau_i) = \xi_i, \ i = 0, 1, \cdots, n$ and $\psi_{\tau,\xi}$ be linear on $[\tau_{i-1}, \tau_i].$

Then we define the sequential Gasusian Markov integral

$$(2.7) \qquad E_\lambda^s\{F[X] \,|\, X(s) = x, X(t) = y\} = \lim_{||\tau||\to 0} \int_{R^{n-1}} G_\lambda(\tau, \xi) F(\psi_{\tau,\xi}) d\xi$$

where

$$(2.8) \qquad G_\lambda(\tau, \xi) = \frac{1}{p^*(x, s\,;\, y, t)} \prod_{i=1}^{n} p^*(\xi_{i-1}, \tau_{i-1}\,;\, \xi_i, \tau_i).$$

Assume that ξ_n is unrestricted. Then we make the definition

$$(2.9) \qquad E_\lambda^s\{F[X] \,|\, X(s) = x\} = \lim_{||\tau||\to 0} \int_{R^n} \prod_{i=1}^{n} p^*(\xi_{i-1}, \tau_{i-1}\,;\, \xi_i, \tau_i) \, F(\psi_{\tau,\xi}) d\xi.$$

Clearly we have

$$(2.10) \qquad E_\lambda^s\{F[X] \,|\, X(s) = x\} = \int_{-\infty}^{\infty} E_\lambda^s\{F[X] \,|\, X(s) = x, \ X(t) = y\} p^*(x, s\,;\, y, t) dy.$$

Let $C[x, s\,;\, y, t]$ denote the space of continuous functions with x and y endpoints. Let $C[s, t]$ denote the space of continuous functions with $X(s) = 0$. For $X \in C[x, s\,;\, y, t]$ or $X \in C[s, t]$, let $\|X\| = \sup_{s \le \tau \le t} |X(\tau)|$.

A subset S of $C[x, s\,;\, y, t]$ or $C[s, t]$ is a Borel set if it is a member of the smallest σ-ring containing the quasi-intervals

$$\{X \in C[x, s\,;\, y, t] \text{ or } C[s, t] : \alpha_i < X(\tau_i) < \beta_i, \ i = 1, 2, \cdots, n\}$$

where τ ranges over all subdivision vectors of $[s, t]$ and α_i, β_i range over the extended reals. $F[X]$ is a Borel functional if it is measurable with respect to the σ-ring of Borel measurable subsets of $C[x, s\,;\, y, t]$ or $C[s, t]$.

THEOREM 1. *Let Λ be any open set of complex numbers $\lambda \ni \mathrm{Re}\, \lambda > 0$, $\lambda \ne 0$. Let Λ^* denote the closure of Λ with $\lambda = 0$ omitted. Let $p(x, s\,;\, y, t)$, $p^*(x, s\,;\, y, t)$ and their related integrals be as specified earlier. Let $F[X]$ be a Borel functional for $X \in C[x, s\,;\, y, t]$. Assume that F also satisfies the following two conditions.*

(2.11) *$F[X]$ is a continuous function of X in the uniform topology almost everywhere in $C[x, s\,;\, y, t]$.*

(2.12) *For all X in $C[x, s\,;\, y, t]$, $|F[X]| \le A \exp(M\|X\|^r)$ where A and M are given positive integers, and $0 < r < 2$.*

Then for $\lambda \in \Lambda^*$, and functionals for which the right side exists and is analytic in Λ, and continuous in Λ^*, we have

$$(2.13) \qquad E_\lambda^s\{F[X] \,|\, X(s) = x, \ X(t) = y\} p^*(x, s\,;\, y, t) =$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\mu\left[y-x\frac{v(t)}{v(s)}\right]} E\left[F\left[\lambda^{-1/2}X(\cdot) + x\frac{v(\cdot)}{v(s)}\right]\exp[i\mu X(t)/\sqrt{\lambda}]\mid X(s) = 0\right]d\mu.$$

Proof. By obvious changes in Theorem 1 of [5],

$$E_\lambda^s\{F[X]\mid X(s) = x, \ X(t) = y\} = E^*\{F[X]\mid X(s) = x, \ X(t) = y\}$$

for $\lambda > 0$, where the second integral uses p^* as the transition density function. Hence from (3.2) of [5], (2.13) holds for $\lambda > 0$.

Now we will prove that the left hand side of (2.13) is analytic in $\varLambda$. First we show that for each subdivision vector

(2.14) $\int_{R_{n-1}} G_\lambda(\tau, \xi)F(\phi_{\tau,\xi})d\xi$ is an analytic function in $\varLambda$.

(2.15) $|p^*(x, s; y, t)||G_\lambda(\tau, \xi)F(\phi_{\tau,\xi})| \, |\lambda|^{-n/2}[(2\pi)^n A(s, t_1)\cdots A(t_{n-1}, t_n)]^{1/2}$

$$\leq A \exp\left(M\max_{i=1,\cdots,n}|\xi_i|^r\right) \exp\left\{-\operatorname{Re}\lambda\sum_{i=1}^{n}\frac{[\xi_i - v(t_i)\xi_{i-1}/v(t_{i-1})]^2}{2A(t_{i-1}, t_i)}\right\}.$$

This is integrable over R_{n-1} for each λ in $\varLambda$.

Now we can integrate (2.14) around a contour in $\varLambda$, and exchange order of integration by the Fubini Theorem. Because of the analyticity of the integrand of (2.14) in λ, the repeated integral vanishes, and by Morera's Theorem, (2.14) is an analytic function in $\varLambda$.

As in Cameron, [6], Theorem 2, the limit of the finite dimensional integrals is also analytic in $\varLambda$ and continuous in $\varLambda^*$.

Then the two sides of (2.13) are equal for $\operatorname{Re}\lambda > 0$ by analytic continuation. We get equality for $\operatorname{Re}\lambda \geq 0$, $\lambda \neq 0$, by continuity in λ of the two sides.

EXAMPLE 1 OF THEOREM 1.   Let $F[X] \equiv 1$, corresponding to a potential $V \equiv 0$.   Assume $\operatorname{Re}\lambda > 0$ as the only requirement for $\lambda \in \varLambda$.

$$E_\lambda^s\{1\mid X(s) = x, \ X(t) = y\} \, p^*(x, s; y, t)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\mu[y - xv(t)/v(s)]} E[e^{i\mu X(t)/\sqrt{\lambda}}\mid X(s) = 0]d\mu$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\mu[y - xv(t)/v(s)]}\int_{-\infty}^{\infty} e^{i\mu y/\sqrt{\lambda}} p(0, s; y, t)dyd\mu$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\mu[y - xv(t)/v(s)]} \exp\left\{-\frac{1}{2}A(s, t)\mu^2/\lambda\right\}d\mu$$

(by the Lemma of [5]).

$$= p^*(x, s; y, t)$$

This is analytic in λ in Λ. For $Re\,\lambda = 0$, one also obtains $p^*(x, s; y, t)$ by using Fresnel integrals. (See [10], pages 114-115, for example.) Clearly $p^*(x, s; y, t)$ is continuous in λ in Λ^*. For $\lambda = -i$,

$$p^*(x, s; y, t) = [-i/(2\pi A(s, t))]^{1/2} \exp\left\{\frac{i[y - v(t)x/v(s)]^2}{2A(s, t)}\right\}$$

satisfies a pair of generalized Schröedinger equations:

$$(2.16) \qquad iA(t)\frac{\partial^2 p^*}{\partial y^2} - B(t)\frac{\partial}{\partial y}[yp^*] - iV(y, t)p^* = \frac{\partial p^*}{\partial t}$$

$$(2.17) \qquad iA(s)\frac{\partial^2 p^*}{\partial x^2} + xB(s)\frac{\partial p^*}{\partial x} - iV(x, s)p^* = \frac{\partial p^*}{\partial s}$$

where

$$(2.18) \qquad A(t) = [v(t)u'(t) - u(t)v'(t)]/2, \ 0 \leq t \leq T$$

$$(2.19) \qquad B(t) = v'(t)/v(t), \ 0 \leq t \leq T.$$

The $A(t)$ and $B(t)$ functions for the Wiener, Doob-Kac, and Ornstein-Uhlenbeck processes are given in [3].

It would be desirable to establish a class of functions $g(y)$ such that the following Dirac delta function property holds:

$$(2.20) \qquad \lim_{t \to s+}\int_{-\infty}^{\infty} g(x)[-i/(2\pi A(s, t))]^{1/2}\exp\{i[y - xv(t)/v(s)]^2/(2A(s, t))\}dx$$

$$= g(y), \ -\infty < y < \infty.$$

This does not seem possible. However, to indicate that the class is rather broad, two examples will be considered.

Let $g(y) \equiv 1, \ -\infty < y < \infty$. Using Fresnel integrals, and the continuity of $v(t)$,

$$\lim_{t \to s+}\int_{-\infty}^{\infty} [-i/(2\pi A(s, t))]^{1/2}\exp\{i[y - xv(t)/v(s)]^2/(2A(s, t))\}dx$$

$$= \lim_{t \to s+}\int_{-\infty}^{\infty} [-i/\pi]^{1/2}e^{iw^2}v(t)/v(s)dw = 1.$$

Let $g(y) = e^{-y^2/2}, \ -\infty < y < \infty$. Then completing the square, and using the Lemma of [5],

$$\int_{-\infty}^{\infty} g(x)[-i/(2\pi A(s,t))]^{1/2} \exp\{i[y - xv(t)/v(s)]^2/(2A(s,t))\}\, dx$$

$$= \frac{(-i)^{1/2}}{[A(s,t) - iv^2(t)/v^2(s)]^{1/2}} \exp\left\{\frac{iy^2}{2[A(s,t) - iv^2(t)/v^2(s)]}\right\}.$$

Using the continuity of $v(t)$, and the fact that $\lim_{t \to s+} A(s,t) = 0$, the limit as $t \to s+$ of the previous quantity equals $e^{-y^2/2}$.

One can also prove that

$$(2.21) \qquad \lim_{s \to t-} \int_{-\infty}^{\infty} g(y)[-i/(2\pi A(s,t))]^{1/2} \exp\{i[y - xv(t)/v(s)]^2/(2A(s,t))\}\, dy$$

$$= g(x), \quad -\infty < x < \infty$$

for the above g's and other suitable functions.

EXAMPLE 2 OF THEOREM 1.   Let $F[X] = \exp\left\{-i\int_{s}^{t}[X^2(\tau) - f(\tau)X(\tau)]\, d\tau\right\}$ where $f(0) = 0$, $f(T) = 0$, and $f(\tau) \in L_2[0, T]$. The calculations of Section V of [5] can be carried out and the result is analytic for $Re\, \lambda > 0$, using the analyticity of $D_r(\mu)$ and $R_r(a, b; \mu)$. See [18], pages 215–216. It is also continuous for $Re\, \lambda \geq 0$ if at the point where we assume $Re\, \theta > 0$, one uses instead Fresnel integrals as in Example 1. This indicates that for this example the true solution to the generalized Schröedinger equation, subject to the initial conditions (2.20) and (2.21), is a complex sequential integral with $\lambda = -i$.

III. Approximation of Integrals.

The following theorem was motivated by a paper by L.D. Fosdick, [14], which derives an approximation for the conditioned Wiener integral which could be used in electronic computer work. One could split the right hand side functional into real and imaginary parts and approximate the resulting integrals.

Other papers on this subject of Monte Carlo approximations of Wiener integrals and their relations to differential equations are [13] by Fosdick, [15] by Fosdick and Jordan, and [17] by Tsuda, Ichida, and Kiyono.

THEOREM 2.   *Let Λ be any open set of complex numbers λ such that $Re\, \lambda > 0$, $|\lambda| \geq \lambda_0 > 0$. Let Λ^* denote the closure of Λ. Let $p(x, s; y, t)$, $p^*(x, s; y, t)$ and their related integrals be as specified earlier. Let $F[X]$ be a Borel functional for $X \in C[x, s; y, t]$ and let the integrand of the right side of (3.1) be Borel measurable*

for (λ, X, x, y) *in* $\Lambda \times C[0,0;0,1] \times (-\infty, \infty) \times (-\infty, \infty)$. *Assume that* F *satisfies* (2.11) *for both* $C[x,s;y,t]$ *and* $C[0,0;0,1]$. *Assume that for all* X *in* $C[x,s;y,t]$ *or in* $C[0,0;0,1]$, $|F[r_1\lambda^{-1/2}X(\cdot) + r_2]| \leq A(r_2)e^{M(r_2)||X||^\gamma}$ *for* $0 < \gamma < 2$, *where there exists a number* B *such that* $0 < r_1 \leq B$ *but* $-\infty < r_2 < \infty$. *Here* $A(r_2)$ *and* $M(r_2)$ *are positive nondecreasing functions of* r_2 *alone, so that for fixed* r_2 *they may be taken to be constants. Assume* F *is such that the right side of* (2.13) *exists for* $\lambda > 0$. *Assume that* $F[\lambda^{-1/2}X(\cdot)]$ *is analytic in* λ *throughout* Λ *for each* X *in* $C[x,s;y,t]$ *and continuous in* λ *throughout* Λ^* *for each* X *in* $C[x,s;y,t]$. *Let* $E^w\{G[X]|X(0) = 0, X(1) = 0\}$ *denote the expectation of* $G[X]$ *over the Wiener process conditioned by* $X(0) = 0$, $X(1) = 0$ *(i.e.* $u(\tau) = \tau$, $v(\tau) = 1 - \tau$ *in* (2.1) *and* (2.2)*); this is sometimes called the Doob-Kac process. See* [3]*).*

Then for $\lambda \in \Lambda^*$,

(3.1) $\quad E_\lambda^s\{F[X]|X(s) = x, X(t) = y\}$

$$= E^w\left\{ F\left[\frac{v(\cdot)}{v(t)} \frac{[A(s,t)]^{1/2}}{[\lambda]^{1/2}} X\left(\frac{v^2(t)}{v^2(\cdot)} \frac{A(s,\cdot)}{A(s,t)}\right) + x\,\frac{v(\cdot)}{v(s)} \frac{A(\cdot,t)}{A(s,t)} \right.\right.$$

$$\left.\left. + y\,\frac{v(t)}{v(\cdot)} \frac{A(s,\cdot)}{A(s,t)} \right] \,\right|\, X(0) = 0, X(1) = 0\right\}.$$

Proof. Equation (3.1) is true for $\lambda > 0$ by (4.1) of [5].

Now the hypotheses insure that the left hand side of (3.1) is analytic for $Re\,\lambda > 0$, continuous for $Re\,\lambda \geq 0$, $|\lambda| \geq \lambda_0$. This was proved in Theorem 1.

Let I be the right side of (3.1), and let its integrand be $H[X; \lambda]$. To apply the bound on $H[X; \lambda]$ we note that $0 < g \leq v(\tau) \leq G$, $s \leq \tau \leq t$, and $A(s,\theta) \leq G^2 A(s,\tau)/v^2(\tau)$, $\theta \leq \tau \leq t$, and $A(s,\tau) \geq A(\theta,\tau)$, $s \leq \theta$. Let $r_2 = |x|G/v(s) + |y|G^2/[gv(t)]$. Then by well-known Wiener integral results, (see [5], Theorem 1 for example),

$$|E^w\{H[X; \lambda]|X(0) = 0\}| \leq \frac{4}{\sqrt{2\pi}} \int_0^\infty A(r_2)e^{M(r_2)y^\gamma} e^{-y^2/2}dy < \infty.$$

But

$$E^w\{H[X; \lambda]|X(0) = 0\} = \int_{-\infty}^\infty E^w\{H[X; \lambda]|X(0) = 0, X(1) = x\} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx.$$

Hence the real and imaginary parts of the right integrand are finite almost everywhere. By continuity in x (this uses 2.7), we get finiteness for all x, including $x = 0$. Now we can integrate I around a contour in Λ and ex-

change order of integration by the Fubini Theorem. Because of the analyticity of the integrand of I in λ, the repeated integral vanishes, and by Morera's Theorem, I is an analytic function in λ for $|\lambda| \geq \lambda_0 > 0$. Using bounded convergence and the continuity of F in λ, we can show I is continuous in λ for $Re\,\lambda \geq 0$, $|\lambda| \geq \lambda_0$.

Hence (3.1) follows by analytic continuation for $Re\,\lambda > 0$, $|\lambda| \geq \lambda_0$, and by continuity for $Re\,\lambda \geq 0$, $|\lambda| \geq \lambda_0$.

IV. An Integral Equation.

The following integral equation (4.2) is formally analogous to the differential equations (3.3) and (3.4) of [5] for complex λ, $Re\,\lambda > 0$. Let us proceed in a formal fashion to show that.

$$\frac{A(t)}{\lambda}\frac{\partial^2 r^*}{\partial y^2} - B(t)\left[y\,\frac{\partial r^*}{\partial y} + r^*\right] - iVr^*$$

$$= \frac{A(t)}{\lambda}\frac{\partial^2 p^*}{\partial y^2} - B(t)y\frac{\partial p^*}{\partial y} - B(t)p^* - iVr^* - i\int_s^t\!\!\int_{-\infty}^{\infty} r^*V\left[\frac{A(t)}{\lambda}\frac{\partial^2 p^*}{\partial y^2}\right.$$

$$\left. - B(t)y\,\frac{\partial p^*}{\partial y} - B(t)p^*\right]d\alpha d\tau \text{ by (4.2) (which follows)}$$

$$= \frac{\partial p^*}{\partial t} - i\int_s^t\!\!\int_{-\infty}^{\infty} r^*(x,s;\alpha,\tau)V(\alpha,\tau)\frac{\partial p^*}{\partial t}(\alpha,\tau;y,t)d\alpha d\tau - iVr^* \text{ by } \qquad (3.3)$$

of [5] for its $V = 0$ ($r^* = p^*$). This holds for complex λ such that $Re\,\lambda > 0$.

From (4.2)

$$\frac{\partial r^*}{\partial t} = \frac{\partial p^*}{\partial t} - i\frac{\partial}{\partial t}\int_s^t\!\!\int_{-\infty}^{\infty} r^*(x,s;\alpha,\tau)V(\alpha,\tau)p^*(\alpha,\tau;y,t)d\alpha d\tau$$

$$= \frac{\partial p^*}{\partial t} - i\lim_{\theta\to t}\int_{-\infty}^{\infty} r^*(x,s;\alpha,\theta)V(\alpha,\theta)p^*(\alpha,\theta;y,t)d\alpha - i\int_s^t\!\!\int_{-\infty}^{\infty} r^*V\frac{\partial p^*}{\partial t}\,d\alpha d\tau$$

$$= \frac{\partial p^*}{\partial t} - ir^*(x,s;y,t)V(y,t) - i\int_s^t\!\!\int_{-\infty}^{\infty} r^*V\frac{\partial p^*}{\partial t}\,d\alpha d\tau$$

by the singular nature of p^* at $\alpha = y$, $\theta = t$.

This completes the formal derivation.

The backwards equation would proceed similarly, except one uses the assumed singular nature of r^* at $\alpha = x$, $\theta = s$ (See (3.8) of [5] for the case $\lambda > 0$).

THEOREM 3. *Let Λ be the open set of complex numbers $\ni Re\,\lambda > 0$, $|\lambda| \geq \lambda_0 > 0$. Let Λ^* denote the closure of Λ. Let $p(x,s;y,t)$ and $p^*(x,s;y,t)$ and their related*

integrals be as specified earlier. Assume that $V(z, \tau)$ is defined for $S: 0 \leq \tau \leq T < \infty$, z any complex number. Let $V(z, \tau)$ be an analytic function of z for each $\tau \in [0, T]$ and $z \notin F$, F a bounded subset of $(-\infty, \infty)$. Let V be continuous over S, $z \notin F$ and let $V(x, \tau)$ be continuous for $0 \leq \tau \leq T$, $x \in (-\infty, \infty) - E$, E a finite set of real points, and for $x \in E$, let V be continuous in τ.

Assume that for complex z, and any $\tau \in [0, T]$, $|V(z, \tau)| \leq A + N|z|$ for some positive integers A and N.

For $\lambda \in \Lambda^*$, let

$$(4.1) \qquad r^*(x, s; y, t) = E_\lambda^s \Big\{ \exp \Big[-i \int_s^t V[X(\tau), \ \tau] d\tau \Big] | X(s) = x, X(t) = y \Big\} p^*(x, s; y, t).$$

Then $r^*(x, s; y, t)$ satisfies the following integral equation

$$(4.2) \qquad r^*(x, s; y, t) = p^*(x, s; y, t) - i \int_s^t \int_{-\infty}^{\infty} r^*(x, s; \alpha, \tau) V(\alpha, \tau) p^*(\alpha, \tau; y, t) d\alpha d\tau$$

for $\lambda \in \Lambda$.

Proof of Theorem 3.

For $\lambda > 0$,

$$E_\lambda^s \{ F[X] | X(s) = x, X(t) = y \} = E^{r*} \{ F[X] | X(s) = x, X(t) = y \}.$$

For $\lambda > 0$, (4.2) is true by [11] provided we show that $\int_s^t E^{r*} \{ |V(X(\theta), \theta)| |X(s) = x, X(t) = y \} d\tau < \infty$, which is requirement (2.1) of reference [11].

Assume that $s < \theta < t$. Then by hypothesis,

$$E^{r*} \{ |V(X(\theta), \theta)| \ |X(s) = x, X(t) = y \} < A + N E^{r*} \{ |X(\theta)| \ |X(s) = x, X(t) = y \}$$

which by Theorem 3 of [5]

$$= A + \frac{N}{p^*(x, s; y, t)} \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\mu[y - xv(t)/v(s)]}$$

$$E^r \Big[\Big| \frac{1}{\sqrt{\lambda}} X(\theta) + x \frac{v(\theta)}{v(s)} \Big| e^{i\mu X(t)/\sqrt{\lambda}} \Big| X(s) = 0 \Big] d\mu$$

$$= A + \frac{N}{p^*(x, s; y, t)} \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\mu[y - xv(t)/v(s)]}$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Big| \frac{1}{\sqrt{\lambda}} w + x \frac{v(\theta)}{v(s)} \Big| e^{i\mu z/\sqrt{\lambda}} [2\pi A(s, \theta)]^{-1/2} e^{-w^2/[2A(s, \theta)]}$$

$$[2\pi A(\theta, t)]^{-1/2} \exp \{ -[z - wv(t)/v(\theta)]^2/[2A(\theta, t)] \} \, dw dz d\mu$$

$$= A + \frac{N}{p^*(x, s; y, t)} \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\mu[y - xv(t)/v(s)]}$$

$$\int_{-\infty}^{\infty} \left| \frac{1}{\sqrt{\lambda}} w + x \frac{v(\theta)}{v(s)} \right| [2\pi A(s,\theta)]^{-1/2} e^{-w^2/[2A(s,\theta)]}$$

$$\exp \{i \mu v(t)w/[\sqrt{\lambda}\, v(\theta)] - A(\theta,t)\mu^2 v^2(t)/[2\lambda v^2(\theta)]\}\, dw\, d\mu$$

by the Fubini Theorem and the Lemma of [5]

$$= A + \frac{N(2\pi)^{-1/2}}{p^*(x,s;y,t)} [\lambda v^2(\theta)/(v^2(t)A(\theta,t))]^{1/2}$$

$$\int_{-\infty}^{\infty} \exp \left\{ -\frac{1}{2} \frac{\lambda v^2(\theta)}{v^2(t)A(\theta,t)} \left[x \frac{v(t)}{v(s)} - y + \frac{v(t)}{\sqrt{\lambda}} \frac{w}{v(\theta)} \right]^2 \right\}$$

$$\left| \frac{w}{\sqrt{\lambda}} + x \frac{v(\theta)}{v(s)} \right| [2\pi A(s,\theta)]^{-1/2} \exp \{- w^2/[2A(s,\theta)]\}\, dw$$

by the Fubini Theorem, and the Lemma of [5]

$$\leq A + \frac{N(2\pi)^{-1/2}}{p^*(x,s;y,t)} [\lambda v^2(\theta)/(v^2(t)A(\theta,t))]^{1/2}$$

$$[(2/(\lambda\pi))^{1/2}(A(s,\theta))^{1/2} + |x|v(\theta)/v(s)]$$

using the fact that the first factor of the above integrand is ≤ 1.

The above bound is integrable in θ between s and t by the hypotheses on $v(\theta)$, $u(\theta)$ and inequality (4) of Lemma 6.1 of [3]. A bound is not needed for the two point set $\{\theta = s, t\}$ but could be furnished by $A + N \max(|x|, |y|)$.

Now $p^*(x,s;y,t)$ is analytic for $Re\, \lambda \geq 0$, $\lambda \neq 0$.

Let D be a closed and bounded subset of $\Lambda - PF$ where $PF = \{x : x \in F$ and $x > 0\}$.

Considering r^* expressed as in (2.7), it is analytic in D as pointed out in the proof of Theorem 1.

We now verify that $I = \int_{-\infty}^{\infty} r^*(x,s;\alpha,\tau)V(\alpha,\tau)p^*(\alpha,\tau;y,t)d\alpha$ is continuous for $s < \tau < t$, $\lambda \in D$. Clearly the integrand is measurable in α. For the class of V's allowed, one can verify the hypotheses of Theorem 2, including the existence of the right side of (2.13). So by (3.1)

$$|r^*(x,s;\alpha,\tau)| \leq E^w \left\{ e^{At} \exp \left\{ Nt \left\{ \frac{G^2}{gv(t)} \frac{\sqrt{A(s,t)}}{\sqrt{\lambda_0}} \|X\| + Nt |x| \frac{G}{v(s)} \right. \right. \right.$$

$$\left. \left. \left. + Nt|\alpha| \frac{G^3}{g^3} \right\} |X(0) = 0, X(1) = 0 \right\} |p^*(x,s;\alpha,\tau)| \right.$$

where we again recall that $0 < g \leq v(\tau) \leq G$, $s \leq \tau \leq t$, and $A(s,\theta) \leq G^2 A(s,\tau)/v^2(\tau)$, $\theta \leq \tau \leq t$, and $A(s,\tau) \geq A(\theta,\tau)$, $s \leq \theta$.

Now there exists a positive integer K such that

$$|V(\alpha, \tau)| \; |p^*(\alpha, \tau; y, t)| \leq |\lambda|^{1/2}K[2\pi A(\tau, t)]^{-1/2} \leq |\lambda|^{1/2}K[2\pi A(\tau_0, t)]^{-1/2}, \tau \leq \tau_0 < t.$$

Using Fubini's Theorem, as justified below,

$$\int_{-\infty}^{\infty} \exp[Nt|\alpha|G^3/g^3]|p^*(x, s; \alpha, \tau)|d\alpha \leq \frac{2\sqrt{|\lambda_1|}}{\sqrt{\lambda_2}} \exp\left\{Nt\frac{G^4|x|}{g^3v(s)} + \frac{N^2t^2G^8A(s, t)}{g^82\lambda_2}\right\}$$

where we have assumed that $\lambda \in D$ implies that $\lambda_0 \leq |\lambda| \leq \lambda_1$, $Re \lambda \geq \lambda_2 > 0$.

As in Theorem 2, the Wiener integral is finite, and its bound does not involve either τ or λ.

Hence I is bounded by a quantity which does not involve τ or λ. (For each τ such that $s < \tau < t$, we can find τ_0 such that $\tau \leq \tau_0 < t$.)

Using (2.7) we see that the integrand of I is continuous in τ and λ, and so by Lebesgue's dominated convergence theorem, I is continuous in τ and λ for $s < \tau < t$, $\lambda \in D$.

By minor changes in the previous argument, I converges uniformly for λ in D by the analogue of Weierstrass' M test. The integrand of I is analytic in λ for each α since $r^*(x, s; \alpha, \tau)$ and $p^*(\alpha, \tau; y, t)$ are and $V(\alpha, \tau)$ is independent of λ. Using (2.7) we see that the integrand of I is continuous in λ and α, $\alpha \notin E$. The exceptional set E can be worked into the proof on page 108 of Copson, [10], by preliminary steps. (For example if $E = \{\alpha_1, \alpha_2\}$,

$$\int_0^T F(\lambda, \alpha)d\alpha = \int_0^{\alpha_1} F(\lambda, \alpha)d\alpha + \int_{\alpha_1}^{\alpha_2} F(\lambda, \alpha)d\alpha + \int_{\alpha_2}^T F(\lambda, \alpha)d\alpha.$$

For appropriate F's, each of the three integrals on the right is analytic and hence their sum is analytic.) Hence I is analytic in D for $s < \tau < t$ by Copson, page 110.

Therefore we have verified the hypotheses needed for $\int_s^t Id\tau$ to be analytic in D.

Therefore (4.2) holds for D by analytic continuation. Since any $\lambda \in A - PF$ can be enclosed by such a D, we obtain (4.2) for $A - PF$.

Equation (4.2) is true for PF as stated in the opening of the proof.

EXAMPLES. Examples of appropriate potentials are

$$V[z, \tau] = A(\tau) + B(\tau)z$$

which corresponds to motion in a homogeneous field when $A(\tau)$ and $B(\tau)$ are constants, and

$$V(z) = \begin{cases} 0, & -a < z < a, \ z \ \text{real} \\ V_0, & \text{either } z \text{ not real or } |z| \geq a, \ \text{for } z \ \text{real} \end{cases}$$

corresponding to a square well potential.

Remark. The author conjectures that r^* satisfies a pair of generalized Schroedinger equations plus initial conditions; that is, it is believed that most of Theorem 4 of [5] holds for $\lambda = -i$. However, several attempted proofs failed.

This theorem was partially motivated by Theorem 9 of [9], by Cameron and Storvick, which considers the Wiener case.

Acknowledgments. The author wishes to thank Professor R.H. Cameron for his help with Theorem 1, and the referee for helpful suggestions.

FUTURE RESEARCH. R. Kallman and the author are currently extending Theorem 3 to the important case $\lambda = -i$. Such an integral equation would be more equivalent to the generalized Schröedinger equation and would extend the Cameron-Storvick paper, [9], from the Wiener process to Gaussian Markov processes.

BIBLIOGRAPHY

[1] Beekman, John A., "Solutions to Generalized Schröedinger Equations via Feynman Integrals Connected with Gaussian Markov Stochastic Processes," Thesis, Univ. of Minnesota, 1963.

[2] Beekman, John A., "Gaussian Processes and Generalized Schröedinger Equations," *J. Math. Mech.* **14** (1965), 789–806.

[3] Beekman, John A., "Gaussian Markov Processes and a Boundary Value Problem," *Trans. Amer. Math. Soc.* **126** (1967), 29–42.

[4] Beekman, John A., "Feynman-Cameron Integrals," *J. Math. and Phys.* **46** (1967), 253–266.

[5] Beekman, John A., "Green's Functions for Generalized Schröedinger Equations," *Nagoya Math. J.* **35** (1969), 133–150. Correction: *Nagoya Math. J.* **39** (1970), 199.

[6] Cameron, R.H., "A Family of Integrals Serving to Connect the Wiener and Feynman Integrals," *J. Math. and Phys.* **39** (1960), 126–140.

[7] Cameron, R.H., "The Ilstow and Feynman Integrals," *J. Analyse Math.* **10** (1962–3), 287–361.

[8] Cameron, R.H., "Approximations to Certain Feynman Integrals," *J. Analyse Math.* **21** (1968), 337–371.

[9] Cameron, R.H. and D.A. Storvick, "An Operator Valued Function Space Integral and a Related Integral Equation," *J. Math. Mech.* **18** (1968), 517–552.

[10] Copson, E.T., *Theory of Functions of a Complex Variable*, Oxford, 1935.

[11] Darling, D.A. and A.J.F. Siegert, "Integral Equations for the Characteristic Functions of Certain Functionals of Multi-dimensional Markoff Processes," Rand Report, P-429, Rand Corporation, Santa Monica, California, 1955.

[12] Feynman, R.P., "Space-Time Approach to Non-relativistic Quantum Mechanics," *Rev. Modern Phys.* **20** (1948), 367–387.

[13] Fosdick, L.D., "Numerical Estimation of the Partition Function in Quantum Statistics," *J. Mathematical Phys.* **3** (1962), 1251–1264.

[14] Fosdick, L.D., "Approximation of a Class of Wiener Integrals," *Math. Comp.* **19** (1965), 225–233.

[15] Fosdick, L.D. and H.F. Jordan, "Approximation of a Conditional Wiener Integral," *J. Computational Phys.* **3** (1968), 1–16.

[16] Peña-Auerbach, L. de la and L.S. Garcia-Colin, "Simple Generalization of Schrödinger's Equations," *J. Mathematical Phys.* **9** (1968), 922–927.

[17] Tsuda, Takao, Ichida, Kozo, and Takeshi Kiyono, "Monte Carlo Path-Integral Calculations for Two-Point Boundary-Value Problems," *Numer. Math.* **10** (1967), 110–116.

[18] Whittaker, E.T. and G.N. Watson, *A Course of Modern Analysis*, Cambridge, 1927.

Ball State University
Muncie, Indiana