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A NOTE ON HOLOMORPHIC VECTOR BUNDLES OVER COMPLEX TORI

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1. Let $\omega : \boldsymbol{Z}^{2r} \to \boldsymbol{C}^r$ be an isomorphism of the free additive group of rank $2r$ into the complex vector n-space such that the quotient group $\boldsymbol{T} = \boldsymbol{C}^r/\omega(\boldsymbol{Z}^{2r})$ is compact, i.e., $\boldsymbol{T}_\omega$ is a complex torus.

We mean by a matric multiplier of rank n with respect to ω a family of complex holomorphic $n\times n$-matrix functions $\{\mu_\alpha(z)\}_{\alpha\in \boldsymbol{Z}^{2r}}$ on $\boldsymbol{C}^r$ such that

1) $\qquad \det \mu_\alpha(z) \neq 0 \quad (z\in \boldsymbol{C}^r),$

2) $\qquad \mu_\alpha(z)\mu_\beta(z+\omega(\alpha)) = \mu_{\alpha+\beta}(z), \quad (\alpha,\beta\in \boldsymbol{Z}^{2r}).$

By virtue of the conditions 1) and 2) we may define an action of $\boldsymbol{Z}^{2r}$ on the product $\boldsymbol{C}^r\times \boldsymbol{C}^n$ as follows:

$$(z,u)\to(z+\omega(\alpha),\ v\mu_\alpha(z)), \quad (\alpha\in \boldsymbol{Z}^{2r}).$$

The quotient $\boldsymbol{V}_\mu$ of $\boldsymbol{C}^r\times \boldsymbol{C}^n$ by this action of $\boldsymbol{Z}^{2r}$ is a holomorphic vector n-bundle over the complex torus $\boldsymbol{T}_\omega$, and conversely every holomorphic vector bundle over $\boldsymbol{T}_\omega$ is constructed by this method with a matric multiplier, since holomorphic vector bundles over a vector space are always trivial.[1)]

2. We shall recall the definition of *finite Heiseuberg groups* and their canonical representations.

Let G be an additive group of order n and of exponent d, and ζ be a primitive d-th root of unity. Let $\hat{G}$ be the dual group of G defined by a pairing $(\hat{a},a)\to\langle\hat{a},a\rangle$ of $\hat{G}\times G$ into the multiplicative group $\{1,\zeta,\cdots,\zeta^{d-1}\}$. We mean by the finite Heisenberg group $H(G)$ associated with G the group consisting of triples $\{(\hat{a},a,\zeta^l)\,|\,\hat{a}\in\hat{G},\ a\in G,\ 0\leq l\leq d-1\}$ with the composition law

$$(\hat{a},a,\zeta^l)(\hat{b},b,\zeta^h) = (\hat{a}+\hat{b},\ a+b,\ \langle\hat{a},b\rangle\zeta^{l+h}).$$

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The Heisenberg group $H(G)$ has a faithful irreducible $n\times n$-matric representation $\{U_{(\hat{a},a,\zeta^l)}\}$ characterized by its character

$$trU_{(\hat{a},a,\zeta^l)}=\begin{cases}0 & \text{for } \hat{a}+a\neq 0,\\ n\zeta^l, & \text{for } \hat{a}+a=0,\end{cases}$$

We call such a representation U the canonical representation of $H(G)$. Actually U is given by

$$U_{(\hat{a},a,\zeta^l)}=(\zeta^l u_{b,c}(\hat{a}+a))_{b,c\in G},$$

where

$$u_{b,c}(\hat{a}+a)=\langle\hat{a},b\rangle\,\delta_{b,c+a},$$

$$\delta_{b,a}=\begin{cases}1 & \text{for } b=a\\ 0 & \text{otherwise,}\end{cases}$$

In these terminologies we shall show the following result:

THEOREM 1. *Let $\omega:\boldsymbol{Z}^{2r}\to\boldsymbol{C}^r$ be an isomorphism such that $\boldsymbol{T}_\omega=\boldsymbol{C}^r/\omega(\boldsymbol{Z}^{2r})$ is a complex torus. If a matric multiplier $\{\mu_\alpha(z)\}$ of rank n with respect to ω satisfies the conditions*

i) *$\mu_\alpha(z)=\mu_\alpha(0)\chi_\alpha(z)$ with scalar functions $\chi_\alpha(z)$ $(\alpha\in\boldsymbol{Z}^{2r})$,*

ii) *the commutors of $\{\mu_\alpha(0)\}_{\alpha\in\boldsymbol{Z}^{2r}}$ are scalor matrices,*

then there exist an additive group G of order n, a surjective homomorphism $\hat{\sigma}\oplus\sigma$ of $\boldsymbol{Z}^{2r}$ onto $\hat{G}\oplus G$ and a family of holomorphic functions $\{\xi_\alpha(z)\}_{\alpha\in\boldsymbol{Z}^{2r}}$ such that

$$\mu_\alpha(z)=U_{(\hat{\sigma}(\alpha),\sigma(\alpha),1)}\xi_\alpha(z),\quad(\alpha\in\boldsymbol{Z}^{2r}),$$

where U is the canonical representation of the Heisenberg group $H(G)$.

Proof. Putting $z=0$ in $\mu_\alpha(z)\mu_\beta(z+\omega(\alpha))=\mu_{\alpha+\beta}(z)$, we have

$$\mu_\alpha(0)\mu_\beta(0)\chi_\beta(\omega(\alpha))=\mu_{\alpha+\beta}(0),$$

$$\chi_\beta(\omega(\alpha))^n=\frac{\det\mu_{\alpha+\beta}(0)}{\det\mu_\alpha(0)\det\mu_\beta(0)},\qquad(\alpha,\beta\in\boldsymbol{Z}^{2r}).$$

Since the left hand side of the last equation is symmetric with respect to α and β, denoting

$$\zeta_{\alpha,\beta}=\chi_\beta(\omega(\alpha))^{-1}\chi_\alpha(\omega(\beta)),$$

we have

$$\zeta_{\alpha,\beta}^n = 1, \quad \zeta_{\alpha,\beta}\zeta_{\beta,\alpha} = 1,$$

and the commutation relation

$$\mu_\alpha(0)\mu_\beta(0) = \zeta_{\alpha,\beta}\mu_\beta(0)\mu_\alpha(0).$$

Moreover $\zeta_{\alpha,\beta}$ is bimultiplicative, namely

$$\zeta_{\alpha,\beta+\gamma} = \zeta_{\alpha,\beta}\zeta_{\alpha,\gamma}.$$

because

$$\begin{aligned}\mu_\alpha(0)\mu_\beta(0)\mu_\gamma(0) &= \mu_\alpha(0)\mu_{\beta+\gamma}(0)\chi_\gamma(\omega(\beta))^{-1}\\ &= \zeta_{\alpha,\beta+\gamma}\mu_{\beta+\gamma}(0)\mu_\alpha(0)\chi_\gamma(\omega(\beta))^{-1}\\ &= \zeta_{\alpha,\beta+\gamma}\mu_\beta(0)\mu_\gamma(0)\mu_\alpha(0)\\ &= \zeta_{\alpha,\beta+\gamma}\zeta_{\beta,\alpha}\zeta_{\gamma,\alpha}\mu_\alpha(0)\mu_\beta(0)\mu_\gamma(0)\\ &= \zeta_{\alpha,\beta+\gamma}\zeta_{\alpha,\beta}^{-1}\zeta_{\alpha,\gamma}^{-1}\mu_\alpha(0)\mu_\beta(0)\mu_\gamma(0).\end{aligned}$$

Therefore $\mu_\alpha(0)^n\mu_\beta(0) = \zeta_{\alpha,\beta}^n\mu_\beta(0)\mu_\alpha(0)^n = \mu_\beta(0)\mu_\alpha(0)^n$, $(\alpha, \beta \in \boldsymbol{Z}^{2r})$ and these $\mu_\alpha(0)^n(\alpha\in\boldsymbol{Z}^{2r})$ are scalar matrices by the condition (ii). On the other hand

$$\mu_{n\alpha}(0) = \chi_\alpha(\omega(\alpha))\chi_\alpha(\omega(2\alpha))\cdots\chi_\alpha(\omega(n-1)\alpha))\mu_\alpha(0)^n,$$

hence $\mu_{n\alpha}(0)$, $(\alpha\in\boldsymbol{Z}^{2r})$, are scalar matrices. Let N be the subgroup consisting of all the elements α such that $\mu_\alpha(0)$ is a scalar matrix. Then $N\supset n\boldsymbol{Z}^{2r}$ and the element α is characterized by $\zeta_{\alpha,\beta} = 1$ for every β in $\boldsymbol{Z}^{2r}$,
Since $\zeta_{\alpha,\beta}$ is bimultiplicative and $\zeta_{\alpha,\beta} = \zeta_{\beta,\alpha}^{-1}$, there exists a skew symmetric rational matrix A such that

$$\zeta_{\alpha,\beta} = e^{2\pi\sqrt{-1}\alpha A^t\beta}$$

and nA is an integral matrix. Therefore we can choose a base $\{a_1, \cdots, a_s, \hat{a}_1, \cdots, \hat{a}_s\}$ of the quotient additive group $\boldsymbol{Z}^{2r}/N$ and their representatives $\alpha_1, \cdots, \alpha_s, \hat{\alpha}_1, \cdots, \hat{\alpha}_r$ in $\boldsymbol{Z}^{2r}$ such that

$$\begin{aligned}&\zeta_{\alpha_i,\alpha_j} = \zeta_{\hat{\alpha}_i,\hat{\alpha}_j} = 1, && (1\leq i,\ j\leq s)\\ &\zeta_{\hat{\alpha}_l,\alpha_h} = 1, && (l\neq h)\\ &\zeta_{\hat{\alpha}_i,\alpha_i}^{d_i} = 1, && (1\leq i\leq s),\end{aligned}$$

where d_i is the common order of α_i and $\hat{\alpha}_i$. We denote by G and $\hat{G}$ the subgroups of $\boldsymbol{Z}^{2r}/N$ generated by $\{a_1, \cdots, a_s\}$ and $\{\hat{a}_1, \cdots, \hat{a}_s\}$, respectively. Then $\boldsymbol{Z}^{2r}/N = G\oplus\hat{G}$ and the map

$$(\sum_{i=1}^{s} l_i\hat{a}_i, \sum_{i=1}^{s} h_i a_i) \to \zeta_{\sum_{i=1}^{s} l_i\hat{\alpha}_i, \sum_{i=1}^{s} h_i\alpha_i} = \prod_{i=1}^{s} \zeta_{\hat{\alpha}_i,\alpha_i}^{l_i h_i}$$

is the pairing of the dual pair $(G, \hat{G})$. We denote by $\langle \sum_{i=1}^{s} l_i\hat{a}_i, \sum_{i=1}^{s} h_i a_i\rangle$ its value. Denote by G^* and $\hat{G}^*$ the inverse images of G and $\hat{G}$ in Z^{2r}, respectively. Then from the definition of G and $\hat{G}$ it follows

$$\mu_\alpha(0)\mu_\beta(0) = \mu_\beta(0)\mu_\alpha(0), \qquad (\alpha, \beta \in G^*)$$
$$\mu_{\hat{\alpha}}(0)\mu_{\hat{\beta}}(0) = \mu_{\hat{\beta}}(0)\mu_{\hat{\alpha}}(0), \qquad (\hat{\alpha}, \hat{\beta} \in \hat{G}^*),$$

Since $d_i\hat{\alpha}_i$ and $d_i\alpha_i$ are elements in N, the matrices $\mu_{d_i\hat{\alpha}_i}(0)$ and $\mu_{d_i\alpha_i}(0)$ are scalar matrices and we can choose scalar matrices $\nu_{\hat{\alpha}_i}$ and ν_{α_i} such that

$$\nu_{\hat{\alpha}_i}^{d_i} = \mu_{d_i\hat{\alpha}_i}(0) \quad \nu_{\alpha_i}^{d_i} = \mu_{d_i\alpha_i}(0)(1 \leq i \leq s).$$

Let us construct an irreducible $n\times n$-representation of the Heisenberg group $H(G)$. From the definition follows the commutation relations

$$(\nu_{\hat{\alpha}_i}^{-1}\mu_{\hat{\alpha}_i}(0))\,(\nu_{\hat{\alpha}_j}^{-1}\mu_{\hat{\alpha}_j}(0)) = (\nu_{\hat{\alpha}_j}^{-1}\mu_{\hat{\alpha}_j}(0))(\nu_{\hat{\alpha}_i}^{-1}\mu_{\hat{\alpha}_i}(0)),$$
$$(\nu_{\alpha_i}^{-1}\mu_{\alpha_i}(0))(\nu_{\alpha_j}^{-1}\mu_{\alpha_j}(0)) = (\nu_{\alpha_j}^{-1}\mu_{\alpha_j}(0))(\nu_{\alpha_i}^{-1}\mu_{\alpha_i}(0)),$$
$$(\nu_{\hat{\alpha}_l}^{-1}\mu_{\hat{\alpha}_l}(0))(\nu_{\alpha_h}^{-1}\mu_{\alpha_h}(0)) = (\nu_{\alpha_h}^{-1}\mu_{\alpha_h}(0))(\nu_{\hat{\alpha}_l}^{-1}\mu_{\hat{\alpha}_l}(0)), \qquad (l \neq h),$$
$$(\nu_{\hat{\alpha}_i}^{-1}\mu_{\hat{\alpha}_i}(0))(\nu_{\alpha_i}^{-1}\mu_{\alpha_i}(0)) = \langle \hat{\alpha}_i, \alpha_i\rangle (\nu_{\alpha_i}^{-1}\mu_{\alpha_i}(0)(\nu_{\hat{\alpha}_i}^{-1}\mu_{\hat{\alpha}_i}(0)).$$

This shows that we may define a map

$$\sum_{i=l}^{s} l_i a_i + \sum_{i=1}^{s} h_i\hat{a} \to M(\sum_{i=1}^{s} l_i a_i + \sum_{i=1}^{s} h_i\hat{a}_i) = \prod_{i=1}^{s} (\nu_{\alpha_i}^{-1}\mu_{\alpha_i}(0))^{l_i} \prod_{i=1}^{s} (\nu_{\hat{\alpha}_i}^{-1}\mu_{\hat{\alpha}_i}(0))^{h_i}$$

such that

$$M(\sum_{i=1}^{s} l_i a_i)M(\sum_{i=1}^{s} h_i\hat{a}_i) = M(\sum_{i=1}^{s} l_i a_i + \sum_{i=1}^{s} h_i\hat{a}_i),$$
$$M(\sum_{i=1}^{s} h_i\hat{a}_i)M(\sum_{i=1}^{s} l_i a_i) = \langle \sum_{i=1}^{s} h_i\hat{a}_i, \sum_{i=1}^{s} l_i a_i\rangle\, M(\sum_{i=1}^{s} l_i a_i + \sum_{i=1}^{s} h_i\hat{a}_i).$$

Let d be the exponent of G and ζ be the primitive d-th root of unity. Then the map

$$(\hat{a}, a, \zeta^l) \to U_{(a,\hat{a},\zeta^l)} = \zeta^l M(\hat{a} + a)$$

is a representation of the Heisenberg group of $H(G)$.

From the commutation relation

$$U_{(\hat{a},a,\zeta^l)}U_{(\hat{b},b,\zeta^h)} = \langle\hat{a}, b\rangle U_{(\hat{b},b,\zeta^h)}U_{(\hat{a},a,\zeta^l)}$$

follows

$$\mathrm{tr}\, U_{(\hat{a},a,\zeta^l)} = \langle\hat{a}, b\rangle\, \mathrm{tr}\, U_{(\hat{a},a,\zeta^l)},$$

$$\mathrm{tr}\, U_{(\hat{b},b,\zeta^h)} = \langle\hat{a}, b\rangle\, \mathrm{tr}\, U_{(\hat{b},b,\zeta^h)}.$$

This mean that

$$\mathrm{tr}\, U_{(\hat{a},a,\zeta^l)} = \begin{cases} 0 & \text{for } a+\hat{a} \neq 0 \\ n\zeta^l & \text{for } a+\hat{a} = 0, \end{cases}$$

Since the commutors of $\{U_{(\hat{a},a,\zeta^l)}\}$ are scalar matrices, U is the canonical representation of $H(G)$. Denote by $\sigma(\alpha)+\hat{\sigma}(\alpha)|$ the direct sum decomposition of the image of α in $\boldsymbol{Z}^{2r}/N$ with respect to the decomposition $\boldsymbol{Z}^{2r}/N = G\oplus\hat{G}$ and put

$$\rho(\alpha) = (\hat{\sigma}(\alpha),\ \sigma(\alpha), 1),$$

$$\xi_\alpha(z) = U^{-1}_{\rho(\alpha)}\mu_\alpha(z), \qquad (\alpha\in\boldsymbol{Z}^{2r}).$$

Then $\xi_\alpha(z)\,(\alpha\in\boldsymbol{Z}^{2r})$ are scalar functions satisfying

$$\mu_\alpha(z) = U_{\rho(\alpha)}\xi_\alpha(z), \qquad (\alpha\in\boldsymbol{Z}^{2r}).$$

This completes the proof of Theorem 1.

3. In the notations in the proof of Theorem 1, we denote by λ the natural isogeny of the complex tori.

$$\lambda : \boldsymbol{C}^r/\omega(\hat{G}^*) \to \boldsymbol{T}_\omega = \boldsymbol{C}^r/\omega(\boldsymbol{Z}^{2r}).$$

After changing the base, we may assume that

$$U_{(a,\hat{a},1)} = (\langle\hat{a}, b\rangle\delta_{b,c+a})_{b,c\in G}, \qquad (a\in G,\ \hat{a}\in\hat{G})$$

We shall define line bundles $L_{\eta(a)}\ (a\in G)$ over $\boldsymbol{C}^r/\omega(\hat{G}^*)$ as follows. Let $\{\eta^{(a)}_\beta(z)\}_{\beta\in\hat{G}^*}\,(a\in G)$ be families of functions defined by

$$\eta^{(a)}_\beta(z) = \langle\hat{a}(\beta), a\rangle\xi_\beta(z) \qquad (\beta\in\hat{G}^*;\ a\in G).$$

Then it follows

$$\eta^{(a)}_\alpha(z)\eta^{(a)}_\beta(z+\omega(\alpha)) = \eta^{(a)}_{\alpha+\beta}(z) \qquad (\alpha,\beta\in\hat{G}^*;\ a\in G),$$

and thus there exist line bundles $L_{\eta(a)}\,(a\in G)$ over $\boldsymbol{C}^r/\omega(\hat{G}^*)$ associated with multipliers $\{\eta_{\beta}^{(a)}(z)\}_{\beta\in\hat{G}^*}(a\in G)$, respectively. The direct sum

$$\bigoplus_{a\in G} L_{\eta(a)}$$

may be regarded as the pull back by λ of the vector bundle on $\boldsymbol{T}_\omega=\boldsymbol{C}^r/\omega(\boldsymbol{Z}^{2r})$ associated with the multiflier $\{\mu_\alpha(z)\}_{\alpha\in\boldsymbol{Z}^{2r}}$ such that

$$\mu_\alpha(z) = U_{\rho(\alpha)}\xi_\alpha(z), \qquad (\alpha\in\boldsymbol{Z}^{2r}),$$

This means that the vector bundle $\boldsymbol{V}_\mu$ is the direct image $\lambda_*(L_{\eta(a)})$ of any one of the line bundles $L_{\eta(a)}$ with respect to the isogeny λ.

Then next is the geometric expression of Theorem 1.

THEOREM 2. *Let $\boldsymbol{V}$ be a holomorphic vector n-bundle over a complex torus $\boldsymbol{T}$ such that*

i) $End_{\boldsymbol{C}}(V) = \boldsymbol{C}$,

ii) *the projective $(n-1)$-bundle $\boldsymbol{P}(\boldsymbol{V})$ associated with $\boldsymbol{V}$ has a family of constant transition functions. Then there exist an isogeny $\lambda : \boldsymbol{S} \to \boldsymbol{T}_\omega$ of degree n and a line bundle L over the complex torus $\boldsymbol{S}$ such that $\boldsymbol{V}$ is isomorphic to the direct image $\lambda_*(L)$ of L[2] with respect to λ.*

REFERENCES

[1] Gunning, Rossi; Analytic Functions of Several Complex Variables, Prentice-Hall, 1965.
[2] T. Oda; Vector bundles on an elliptic curve. (to appear in Nagoya Math. J.).

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1) See [1].

2) T. Oda dealed with such direct images $\lambda_*(L)$ systematically in his paper [2]; our result is nothing else than a characterization of simple direct images $\lambda_*(L)$.