B.G. EKE
Nagoya Math. J.
Vol. 41 (1971), 43–53

ON THE DIFFERENTIABILITY OF CONFORMAL MAPS AT THE BOUNDARY[1]

B.G. EKE

1. Introduction. Let S be a simply connected domain in the $w = u + iv$ plane and let ∂S denote its boundary which we assume passes through $w = \infty$. Suppose that the segment $L = \{u \geqq u_0;\ v = 0\}$ of the real axis lies in S and that w_∞ is the point of ∂S accessible along L. Let $z = z(w) = x(w) + iy(w)$ map S in a $(1-1)$ conformal way onto $\sum = \left\{z = x + iy:\ -\infty < x < +\infty;\ |y| < \frac{\pi}{2}\right\}$ so that $\lim\limits_{u \to +\infty} x(u) = +\infty$. The inverse map is $w = w(z) = u(z) + iv(z)$. S is said to possess a *finite angular derivative* at w_∞ if $z(w) - w$ approaches a finite limit (called the angular derivative) as $w \to w_\infty$ in certain substrips of S.[2]

The problem of determining necessary and sufficient conditions for S to have a finite angular derivative at w_∞ has long been studied. (see [4], pp. 140, 216–7, for historical background). For the special cases when

$$\text{(a)} \qquad S \subset \left\{ |\mathscr{I} w| < \frac{\pi}{2} \right\},$$

$$\text{(b)} \qquad \partial S \subset \left\{ \frac{\pi}{2} \leqq |\mathscr{I} w| \leqq \pi \right\},$$

Lelong-Ferrand ([4], pp. 215–6) has given a necessary and sufficient condition and we state the result for case (a).

THEOREM A. *For a domain* $S \subset \left\{ |\mathscr{I} w| < \frac{\pi}{2} \right\}$ *to have a finite angular derivative at* w_∞ *it is necessary and sufficient that for each increasing unbounded sequence* $\{\sigma_n\}_1^\infty$ *such that*

Received June 10, 1969.

1) Work supported partially by U.S.A.F. Contract AFOSR–68–1514, with the University of California, San Diego.

2) More precisely: if $z - w(z)$ tends to a finite limit as $z \to z(w_\infty)$ with $|\mathscr{I} z| < \frac{\pi}{2} - \delta\ (\delta > 0)$. This implies the above definition, and if, for each $\Psi > 0$, there is a $u(\Psi)$ such that $\left\{w:\ \Re w > u(\Psi);\ |\mathscr{I} w| < \frac{\pi}{2} - \Psi\right\} \subset S$, then the implication can be reversed.

$$\sum_{n=1}^{\infty}(\sigma_{n+1}-\sigma_n)^2<+\infty$$

we have the convergence of

$$\sum_{n=1}^{\infty}\left(\frac{\pi-\Psi_n}{\Psi_n}\right)(\sigma_{n+1}-\sigma_n),$$

where

$$\Psi_n=\inf_{\substack{u\in[\sigma_n,\sigma_{n+1}]\\ u+iv\in\partial S,\, v>0}} v-\sup_{\substack{u\in[\sigma_n,\sigma_{n+1}]\\ u+iv\in\partial S,\, v<0}} v$$

and σ_1 is large enough for Ψ_n to be positive for all n.

DEFINITION 1. $\mathscr{D}_1$ denotes the class of simply connected domains S lying in $\left\{|\mathscr{I}w|<\frac{\pi}{2}\right\}$ with $w_\infty\in\partial S$.

DEFINITION 2. $\mathscr{D}_2$ denotes the class of simply connected domains S with $w_\infty\in\partial S$ and for which we can find a $u_0=u_0(S)$ such that S assumes finite area in $\left\{\mathfrak{R}w>u_0;\ |\mathscr{I}w|>\frac{\pi}{2}\right\}$.

DEFINITION 3.

$$\mathscr{D}=\mathscr{D}_1\cup\mathscr{D}_2.$$

For $u>u_0$, we denote by Θ_u the segment of $\{\mathfrak{R}w=u\}\cap S$ which contains $w=u$. The length of Θ_u will be $\Theta(u)$. If $S\in\mathscr{D}$, then

$$\int_{u_0}^{\infty}\max(\Theta(u)-\pi,0)\,du<+\infty. \tag{1}$$

Remark. We may extend $\mathscr{D}_2$ by defining new crosscuts Φ_u in the following way (c.f. [4], p. 191). If $u+\frac{i\pi}{2}\notin\Theta_u$, take Φ_u to agree with Θ_u in $\mathscr{I}w\geqq 0$.

If $u+\frac{i\pi}{2}\in\Theta_u$, then in the upper half plane Φ_u coincides with Θ_u in $0\leqq\mathscr{I}w\leqq\frac{\pi}{2}$ and is completed by a circular arc γ_u centred on $\mathscr{I}w=\frac{\pi}{2}$, pssing through $u+\frac{i\pi}{2}$, lying initially in $\mathscr{I}w>\frac{\pi}{2}$ and of length $\gamma(u)$.

We define Φ_u analogously in $\mathscr{I}w\leqq 0$ where the circular arcs, if necessary, are denoted by γ_u' with length $\gamma'(u)$.

Suppose such circular arcs γ_u can be found which are mutually disjoint and such that the values of u for which γ_u is defined can be partitioned into disjoint intervals on which the γ_u are concentric. Similarly for γ'_u.

If $\int \gamma(u)du + \int \gamma'(u)du$ is finite, the integrals being taken over values of u in $[u_0, \infty)$ for which the integrand is defined, then we have broadened the class $\mathscr{D}_2$. Taking this larger class as $\mathscr{D}_2$ does not affect the validity of Theorems 1 and 2 (below) and this observation may be useful if, say, $\Theta(u) = +\infty$ on an unbounded sequence of intervals that are quite short. We present the proofs however for the simpler case.

We shall prove

THEOREM 1. *A necessary and sufficient condition for* $S \in \mathscr{D}$ *to have a finite angular derivative at* w_∞ *is that given* $\varepsilon > 0$ *we can find a non-negative function* $\beta(u)$ *(defined for* $u \geqq u'_0$, u'_0 *independent of* ε*) such that*

(i) $\left\{w: u = \Re w \geqq u'_0;\ |\mathscr{I} w| < \frac{\pi}{2} - \beta(u)\right\} \subset S,$

(ii) $\int_{u'_0}^{\infty} \beta(u)du < +\infty,$

(iii) $|\beta(u_2) - \beta(u_1)| \leqq \varepsilon |u_2 - u_1|$ *for all* u_1, u_2 *greater than* u'_0.

Theorem 1 shows that if $S \subset \left\{|\mathscr{I} w| < \frac{\pi}{2}\right\}$ then a necessary and sufficient condition for S to have a finite angular derivative at w_∞ is that a large subdomain of $\left\{|\mathscr{I} w| < \frac{\pi}{2}\right\}$ having a *smooth* boundary is contained in S. This necessary and sufficient condition is of a different nature to that given in Theorem A.

DEFINITION 4. $\mathscr{D}'$ is the class of simply connected domains S with $w_\infty \in \partial S$ and such that

$$\int_{u_0}^{\infty} \max(\Theta(u) - \pi, 0)du < +\infty.$$

THEOREM B. (*Warschawski* [5] *pp.* 96–7, 100). *If* $S \in \mathscr{D}'$, *then a sufficient condition for* S *to have a finite angular derivative at* w_∞ *is that there is a non-negative continuous function* $\beta(u)$ $(u \geqq u_0)$ *such that*

(i) $\left\{w : u = \Re w \geqq u_0;\ |\mathscr{I} w| < \frac{\pi}{2} - \beta(u)\right\} \subset S,$

(ii) $\int_{u_0}^{\infty} \beta(u)du < +\infty,$

(iii) $\int_{u-\beta(u)}^{u+\beta(u)} \beta(\tau)d\tau \geqq c\beta^2(u)$ *for some fixed* $c>0$, *and all large* u.

Theorem 1 indicates that Warschawski's condition is necessary when $S\in\mathscr{D}$ since (iii) of Theorem 1 implies (iii) of Theorem B. The condition is not necessary however if $S\in\mathscr{D}'$. Consider the domain R which consists of a union of rectangles

$$R_n = \left\{w = u + iv : \hat{u}_n < u < \hat{u}_{n+1};\ -\frac{\pi}{2} + h_n < v < \frac{\pi}{2} + h_n\right\}$$
$$\left(n = 1, 2, \cdots;\ 0 < |h_n| < \frac{\pi}{2}\right)$$

together with segments of $\Re w = \hat{u}_n (n = 1, 2, \cdots)$, where $\{\hat{u}_n\}_1^\infty$ is an unbounded increasing sequence. Then $R\in\mathscr{D}'$ but $R\notin\mathscr{D}$. If $\sum_{n=1}^\infty \nu_n^{3/2} < +\infty$, where $\nu_n = |h_{n+1} - h_n|$, then R has a finite angular derivative at w_∞.[3] By taking e.g. $\hat{u}_{n+1} - \hat{u}_n = 1$, $\sum_{n=1}^\infty \nu_n = +\infty$, we see that R omits an infinite amount of area in $\{|\mathscr{I}w|\{<\pi/2\}$ and so Theorem B (ii) can never be satisfied for R.

Since $\mathscr{D}\subset\mathscr{D}'$, Theorem 1 (sufficiency) follows from Theorem B.

For the necessity (§ 4), we first establish (Theorem 2, § 2) another necessary condition. Theorem 2 shows, in particular, that for domains consisting of the strip $|\mathscr{I}w| < \frac{\pi}{2}$ slit along the segments $\left\{\Re w = u_n;\ |\mathscr{I}w| \geqq \geqq \frac{\pi}{2} - \lambda_n\right\}$, $u_n \uparrow \infty\ (n\to\infty)$, and $u_{n+1} - u_n > c\lambda_n^\alpha$ (all n, $c>0$, $\alpha\geqq 0$), a necessary condition for a finite angular derivative at w_∞ is the convergence of $\sum_{n=1}^\infty \lambda_n^\gamma$ where

$$\gamma = \max(2, 1+\alpha).^{4)}$$

Ahlfors ([1] p. 40) notes that $\sum \lambda_n^2 < +\infty$ is necessary if $\alpha = 0$, and Wolff [6] proves, independently of the spacing restriction on the slits, that this condition is also sufficient.

2. The condition C and Theorem 2.

We assume $S\in\mathscr{D}$ and has a finite angular derivative at w_∞. Then given $\Psi\left(0<\Psi<\frac{\pi}{2}\right)$ we can find

3) This follows for instance from [4], p. 194, (4). It is now known that the convergence of $\sum_1^\infty \nu_n^2 \log \nu_n^{-1}$ is necessary and sufficient for R to have an angular derivative at W_∞. (Comment. Math. Helv. to appear)

4) For $0\leqq\alpha\leqq 1$, this is an unpublished observation of Warschawski.

$u(\Psi)$ such that $\{w : \Re w \geqq u_0;\ |\mathscr{I}w| < \Psi\} \subset S$. Let Γ_1, Γ_2 denote the part of ∂S in $\left\{w : \Re w \geqq u\left(\frac{\pi}{4}\right); \mathscr{I}w > 0\right\}$, $\left\{w : \Re w \leqq u\left(\frac{\pi}{4}\right); \mathscr{I}w < 0\right\}$ respectively. Γ_1, Γ_2 are not necessarily connected.

Let $\{w_n = u_n + iv_n\}_1^\infty$ be any sequence of points on Γ_1 for which $u_n \uparrow \infty$ $\left(n \to \infty;\ u_1 \geqq u\left(\frac{\pi}{4}\right)\right)$ and which satisfies the following conditions to be denoted by C:

C (i) $\quad v_n = \frac{\pi}{2} - \lambda_n < \frac{\pi}{2}$, all n,

C (ii) $\quad u_{n+1} - u_n \geqq c\lambda_n^{\alpha_n}$, $\quad (\alpha_n \geqq 1$ all n; some fixed $c > 0)$,

C (iii) $\quad \min\limits_{\substack{u+iv \in \Gamma_1 \\ u \in I_n}} v = \frac{\pi}{2} - \lambda_n$, where I_n is a closed interval of length $c\lambda_n^{\alpha_n}$,

containing u_n (possibly as an endpoint) and the intervals $\{I_n\}_1^\infty$ have disjoint interiors.

Such sequences $\{w_n\}_1^\infty$, $\{I_n\}_1^\infty$ can always be found except when all points of Γ_1 with sufficiently large real part lie in $v \geqq \frac{\pi}{2}$. As Theorem 2 (below) does not concern such S we suppose this not to be the case. To produce examples of $\{w_n\}_1^\infty$, $\{I_n\}_1^\infty$ we may take u_n to be the largest value of u for which $u + i\left(\frac{\pi}{2} - \lambda_n\right) \in \Gamma_1$ and $I_n = [u_n, u_n + c\lambda_n^{\alpha_n}]$, λ_n being given small enough. The largest value of u exists since S has a finite angular derivative at w_∞. The $\{\alpha_n\}_1^\infty$ are introduced in C (ii) to allow us to take the w_n close together and we note that 1 is the smallest value of α_n which it is necessary to permit.

THEOREM 2. *Suppose that $S \in \mathscr{D}$ has a finite angular derivative at w_∞ and $\{w_n\}_1^\infty$ is a sequence of points on ∂S satisfying condition C, then* $\sum\limits_{n=1}^{\infty} \lambda_n^{1+\alpha_n} < +\infty$.

3. Proof of Theorem 2. If condition C is satisfied for some $c > 0$ it is satisfied for any smaller c, and we assume that $0 < c < \frac{2}{3\pi}$. We work with the crosscuts θ_u defined as follows. If $u \notin \bigcup\limits_{n=1}^{\infty} I_n$, we take $\theta_u \equiv \Theta_u$.

If $u \in I_n$, θ_u consists of a straight line segment from $u + iv_n$ to $u - it(u)$ where $t(u)$ is the smallest positive number such that $u - it(u) \in \partial S$, together with the arc of a circle centred on $u_n + iv_n$, of radius $|u - u_n|$, which begins at $u_n + iv_n$, lies initially in $\mathscr{I} w \geqq v_n$ and terminates at the first point of intersection with ∂S.

Then θ_{u_1}, θ_{u_2} are disjoint in S if $u_1 \neq u_2$ (the simple proof being analogous to [3], § 2).

Suppose $x_1(u)$, $x_2(u)$ are respectively the infimum, supremum of $\mathfrak{R} z$ for $z \in z\ \{\theta_u\}$. By Ahlfors' well known application of the length-area principle ([1], pp. 8-10), we obtain, for $u\left(\frac{\pi}{4}\right) < u_1 < u_2$,

$$x_2(u_2) - x_1(u_1) \geqq \pi \int_{u_1}^{u_2} \frac{du}{\theta(u)},$$

$$x_1(u_2) - u_2 \geqq x_1(u_1) - (x_2(u_2) - x_1(u_2)) + \int_{u_1}^{u_2} \frac{\pi - \theta(u)}{\theta(u)}\, du - u_1.$$

Since S has a finite angular derivative at w_∞, it follows, in particular, that:

$$x(u_2) - u_2 \text{ tends to a finite limit as } u_2 \to +\infty;$$

S is semi-conformal at w_∞ and therefore $x_2(u_2) - x_1(u_2) \to 0$ as $u_2 \to \infty$, (for a proof, see e.g. [3] § 5 or [5], p. 92).

Then we have

$$\overline{\lim_{u_2 \to +\infty}} \int_{u_1}^{u_2} \frac{\pi - \theta(u)}{\theta(u)}\, du < +\infty.^{5)} \tag{2}$$

Let

$$E_-(u_1, u_2) = [u_1, u_2] \setminus \left(\bigcup_{n=1}^{\infty} I_n \cap [u_1, u_2] \right),$$

so that

$$\int_{E_-(u_1,u_2)} \frac{\pi - \theta(u) du}{\theta(u)} > \frac{-2}{\pi} \int_{E_-(u_1,u_2)} (\theta(u) - \pi) du \geqq -\frac{2}{\pi} \int_{E_-(u_1,u_2)} \max(\theta(u) - \pi, 0) du,$$

and this remains bounded below as $u_2 \to +\infty$. Thus (2) implies

$$\overline{\lim_{N \to \infty}} \sum_{n=1}^{N} \int_{I_n} (\pi - \theta(u))\, du < +\infty.$$

Next, $\sum_{n=1}^{\infty} \int_{I_n} \max\left(t(u) - \frac{\pi}{2},\ 0\right) du$ is finite if $S \in \mathscr{D}$ and, using the estimate,

5) Using the ideas of [2], we may replace $\overline{\lim}$ by lim, but we do not need this fact here.

$$\pi - \theta(u) \geqq \lambda_n - \frac{3\pi}{2}|u-u_n| + \left(\frac{\pi}{2} - t(u)\right), \quad u \in I_n,$$

we find

$$\sum_{n=1}^{\infty}\int_{I_n}\left(\lambda_n - \frac{3\pi}{2}|u-u_n|\right)du < +\infty$$

whence Theorem 2 since

$$\int_{I_n}\left(\lambda_n - \frac{3\pi}{2}|u-u_n|\right)du \geqq \lambda_n|I_n| - \frac{3\pi}{4}|I_n|^2 \geqq$$

$$\geqq \frac{1}{4}(4-3\pi c)c\lambda_n^{1+\alpha_n} > 0.$$

Remark. Taking $\alpha_n = \max(1, \alpha)$, $w_n = u_n + i\left(\frac{\pi}{2} - \lambda_n\right)$ for the domain $|v| < \frac{\pi}{2}$ slit along $\left\{w: \Re w = u_n;\ |\mathscr{I} w| \geqq \frac{\pi}{2} - \lambda_n;\ n = 1, 2, \cdots\right\}$, we find that Theorem 2 gives the observation at the end of §1.

4. Proof of Theorem 1 (necessity). The idea of the construction of $\beta(u)$ is to apply Theorem 2 ($\alpha_n = 1$, all n) to a sequence of boundary points satisfying condition C. Each point of ∂S in $\left\{w: \Re w > u\left(\frac{\pi}{4}\right); 0 < \mathscr{I} w < \frac{\pi}{2}\right\}$ will be "close to" a boundary point which belongs to the sequence. Theorem 2 will show that the subdomain of S, lying in $\left\{w: \Re w > u\left(\frac{\pi}{4}\right);\ 0 < \mathscr{I} w < \frac{\pi}{2}\right\}$, whose boundary has sides parallel to the coordinate axes and which is naturally associated with condition C, omits only a finite amount of area in $\left\{w: \Re w > u\left(\frac{\pi}{4}\right); 0 < \mathscr{I} w < \frac{\pi}{2}\right\}$. After applying similar considerations to produce a subdomain of S in $0 > \mathscr{I} w > -\frac{\pi}{2}$ we obtain a boundary of the required smoothness by omitting a further finite amount of area.

All points $w \in \partial S$ with $\Re w \geqq u_0' \geqq u\left(\frac{\pi}{4}\right)$ have $|\mathscr{I} w| \geqq \frac{\pi}{2} - 1$. We consider first those points of ∂S in $\left\{w: \Re w \geqq u_0';\ \mathscr{I} w \geqq \frac{\pi}{2} - 1\right\}$. Let $E_1 = \left\{u:\right.$ there is a point $w \in \partial S$ with $\Re w = u \geqq u_0'$ and $\left.2^{-1} < \frac{\pi}{2} - \mathscr{I} w \leqq 2^0\right\}$, and set, if $E_1 \neq \phi$,

$$u_{11} = \inf_{u \in E_1} u,$$

$$i_{11} = [u_{11},\ u_{11} + 1],$$

$$\lambda_{11} = \sup_{\substack{w=u+iv\in\partial S \\ u\in i_{11},\, v>0}} \left(\frac{\pi}{2} - v\right).$$

Then $2^{-1} < \lambda_{11} \leqq 1$. Since the distance from $w = \hat{u}$ to the nearest point $\hat{u} + iv \in \Gamma_1$ is a lower semi-continuous function of $\hat{u}$, there is a smallest number $\mathring{u}_{11}$, say, in the closed interval i_{11} such that $\mathring{u}_{11} + i\left(\frac{\pi}{2} - \lambda_{11}\right) \in \Gamma_1$. Now define

$$u_{12} = \inf u \ \text{ for } \ u \in E_1 \cap [u_{11} + 2, \infty),$$

$$i_{12} = [u_{12},\ u_{12} + 1],$$

$$\lambda_{12} = \sup_{\substack{w=u+iv\in\partial S \\ u\in i_{12},\, v>0}} \left(\frac{\pi}{2} - v\right),$$

λ_{12} being attained at $u = \mathring{u}_{12} \in i_{12}$, $\mathring{u}_{12}$ minimal. Proceeding in this way, we construct a finite number (zero, if E_1 is empty) of intervals i_{1j} $(1 \leqq j \leqq n_1)$ such that

(i) $E_1 \cap [u_{n_1} + 2, \infty) = \phi$,

(ii) the intervals $i^*_{1j} \equiv [u_{1j},\ u_{1j} + 2]\,(1 \leqq j \leqq n_1)$ have disjoint interiors and cover E_1,

(iii) $\mathring{u}_{1j} + i\left(\frac{\pi}{2} - \lambda_{1j}\right) \in \partial S$ $(1 \leqq j \leqq n_1)$,

(iv) we can find a closed subinterval I_{1j} of i_{1j} of length λ_{1j} such that $u = \mathring{u}_{1j} \in I_{1j}$ $(1 \leqq j \leqq n_1)$. Then $\{I_{1j}\}_{j=1}^{n_1}$ satisfy C (iii) with $c = 1$, $\alpha_j = 1$ $(1 \leqq j \leqq n_1)$,

(v) $\mathring{u}_{1,j+1} - \mathring{u}_{1,j} \geqq 1 \geqq \lambda_{1j}$ $(1 \leqq j \leqq n_1 - 1)$.

Next we introduce

$$E_2 = \Big\{u\colon \text{there is a } w \in \partial S \text{ with } \Re w = u \geqq u'_0 \text{ and } 2^{-2} < \frac{\pi}{2} - \mathscr{I} w \leqq 2^{-1};$$

$$|u - \mu| \geqq 2^0 \ \text{ if } \ \mu \in \bigcup_{j=1}^{n_1} i^*_{1j} \Big\}.$$

As above, we find intervals i_{2j} $(1 \leqq j \leqq n_2 < +\infty)$ of length 2^{-1}; points $\mathring{u}_{2j} \in i_{2j}$ for which $\mathring{u}_{2j} + i\left(\frac{\pi}{2} - \lambda_{2j}\right) \in \partial S$, and such that $u \in i_{2j}, u + iv \in \partial S$ imply $v \geqq \frac{\pi}{2} - \lambda_{2j}$. The subinterval I_{2j} of i_{2j} of length λ_{2j} is determined as in (iv) above. The closed intervals i^*_{2j} $(1 \leqq j \leqq n_2)$ formed by extending

i_{2j} to the right a distance 2^{-1} do not necessarily cover the set of u outside $\bigcup_{j=1}^{n_1} i_{1j}^*$ for which a v can be found with $u+iv\in\partial S$ and $2^{-2}<\frac{\pi}{2}-v\leqq 2^{-1}$. The intervals i_{1j}^* $(1\leqq j\leqq n_1)$ are now extended to both right and left by the largest amount possible not in excess of 2^0 so that the new closed intervals J_{1j} $(1\leqq j\leqq n_1)$ have disjoint interiors, and $2\leqq |J_{1j}|\leqq 4$ $(1\leqq j\leqq n_1)$. Then, for $u\geqq u_0'$ and outside the set $\bigcup_{j=1}^{n_1} J_{1j}\cup\bigcup_{j=1}^{n_2} i_{2j}^*$, any point $u+iv\in\partial S$ $(v>0)$ has $v\geqq\frac{\pi}{2}-2^{-2}$.

Taking

$$E_3=\Big\{u\colon \text{there is a } w\in\partial S \text{ with } \Re w=u\geqq u_0' \text{ and } 2^{-3}<\frac{\pi}{2}-\mathscr{I}w\leqq 2^{-2};$$
$$|u-\mu|\geqq 2^{-1} \text{ if } \mu\in\bigcup_{j=1}^{n_1} J_{1j}\cup\bigcup_{j=1}^{n_2} i_{2j}^*\Big\},$$

we follow the process outlined above and define intervals I_{mj}, J_{mj} $(1\leqq j\leqq n_m<+\infty;\ m=1,2,\cdots)$ inductively so that, for each j $(1\leqq j\leqq n_m)$ we have

(a) $2\cdot 2^{1-m}\leqq |J_{mj}|\leqq 4\cdot 2^{1-m}$, $|I_{mj}|=\lambda_{mj}$,

(b) $\mathring{u}_{mj}\in I_{mj}\subseteqq i_{mj}\subset i_{mj}^*\subseteqq J_{mj}$ and $\mathring{u}_{mj}+i\left(\frac{\pi}{2}-\lambda_{mj}\right)\in\partial S$,

(c) if $u\in I_{mj}$, $u+iv\in\partial S$, then $v\geqq\frac{\pi}{2}-\lambda_{mj}$,

(d) $2^{-m}<\lambda_{mj}\leqq 2^{1-m}$ so that $2\lambda_{mj}\leqq |J_{mj}|<8\lambda_{mj}$,

(e) $\bigcup_{m=1}^{M}\bigcup_{j=1}^{n_m} J_{mj}\cup\bigcup_{j=1}^{n_{M+1}} i_{M+1,j}^*$ covers the set of $u(\geqq u_0')$ for which a $v(>0)$ can be found so that $u+iv\in\partial S$ and $v<\frac{\pi}{2}-2^{-M-1}$.

Then each value $u(\geqq u_0')$ for which a $v\left(0<v<\frac{\pi}{2}\right)$ can be found such that $u+iv\in\partial S$ lies in some J_{mj}. Suppose $J_{mj}=[u_{mj}', u_{mj}'']$ and denote by A the set of accumulation points of $\{u_{mj}'\}$ $(1\leqq j\leqq n_m;\ m=1,2,\cdots)$. Define inductively

$$\sigma_1=\inf_{u\in A} u,\qquad \sigma_2=\inf_{u\in A\cap[\sigma_1+1,\infty)} u,$$
$$\sigma_3=\inf_{u\in A\cap[\sigma_2+2^{-1},\infty)} u,\cdots,\qquad \sigma_{n+1}=\inf_{u\in A\cap[\sigma_n+n^{-1},\infty)} u,\cdots.$$

If $A\cap[\sigma_{n_0}+n_0^{-1},\infty)=\phi$ for some n_0, then there will be a finite number of values σ_n. Otherwise $\{\sigma_n\}_1^\infty$ is a monotonically increasing sequence with

$\sigma_n \to +\infty$ as $n \to +\infty$. We set

$$K_1^* = [\sigma_1 - 1,\ \sigma_1 + 1] \cap [u_0', \infty),$$
$$K_2^* = [\sigma_2 - 2^{-1},\ \sigma_2 + 2^{-1}] \cap [\sigma_1 + 1, \infty), \cdots$$
$$K_n^* = [\sigma_n - n^{-1},\ \sigma_n + n^{-1}] \cap [\sigma_{n-1} + (n-1)^{-1}, \infty), \cdots\cdots;$$

a finite or countable number of intervals having disjoint interiors, and ordered so that $\mu_1 \in K_m^*$ separates $\mu_2 \in K_n^*$ from $+\infty$ in $[u_0', \infty)$ if $m > n$ and K_m^*, K_n^* are not empty. If $u \in K_n^*$ and $u + iv \in \partial S$, $v > 0$, it follows from (c) and (d) that $v \geqq \frac{\pi}{2} - \frac{1}{2n}$. Thus the area of

$$\bigcup_n \left\{ w\colon \mathfrak{R} w \in K_n^*;\ \frac{\pi}{2} - \frac{1}{2n} \leqq \mathscr{I} w \leqq \frac{\pi}{2} \right\}$$

is finite, and we also have

$$\bigcup_n \left\{ w\colon \mathfrak{R} w \in K_n^*;\ 0 \leqq \mathscr{I} w < \frac{\pi}{2} - \frac{1}{2n} \right\} \subset S.$$

There are no members of A in $[u_0', \infty) \setminus \cup K_n^*$ and so we can define a reordering

$$K_n = [\tau_n, \tau_n'] \quad (\tau_n' \leqq \tau_{n+1},\ n = 1, 2, \cdots;\ \tau_n \to \infty \text{ as } n \to \infty)$$

of those intervals J_{mj} which are outside, or have a subinterval outside, $\bigcup_n K_n^*$. The subinterval of K_n arising from the I_{mj} is denoted by I_n, and we also set

$$\lambda_{mj} = \lambda_n,\ \mathring{u}_{mj} = u_n \in I_n \quad \text{when} \quad J_{mj} = K_n.$$

By construction, condition C (with $c = 1$, $\alpha_n = 1$ all n) is satisfied by the sequence of boundary points $w_n = u_n + i\left(\frac{\pi}{2} - \lambda_n\right)$ and the intervals I_n. Theorem 2 indicates that

$$\sum_{n=1}^{\infty} \lambda_n^2 < +\infty.$$

Put

$$\min_{\substack{u + iv \in \partial S,\, v > 0 \\ u \in K_n}} v = \nu_n,$$

so that

$$\lambda_n \leqq \frac{\pi}{2} - \nu_n \leqq 2\lambda_n.$$

We define a subdomain S_1 of $S \cap \{\mathscr{I}w > 0\} \cap \{\Re w > u_0'\}$. For $u \in K_n$ $(n = 1, 2, \cdots)$, the points $u + iv \in S_1$ if $0 < v < \nu_n$; if $u \notin \bigcup_{n=1}^{\infty} K_n$, but $u \in K_m^*$ for some m, then $u + iv \in S_1$ if $0 < v < \frac{\pi}{2} - \frac{1}{2m}$; for other values of $u (\geqq u_0')$, $u + iv \in S_1$ if $0 < v < \frac{\pi}{2}$. Then ∂S_1 consists of $[u_0', \infty)$ together with straight line segments parallel to the coordinate axes. Further the area of $\left\{w\colon \Re w \geqq u_0';\ 0 < \mathscr{I}w < \frac{\pi}{2}\right\} \setminus S_1$ is finite.

Given $\varepsilon > 0$, we draw straight line segments in S_1, making angles ε or $\pi - \varepsilon$ with the real axis, from the vertices of the polygonal line ∂S_1 with positive imaginary part. This removes from S_1 a finite area of magnitude $O(\varepsilon^{-1} \sum \lambda_n^2)$, and the boundary of the new subdomain, S_2, consists of $\{w\colon \Re w > u_0';\ \mathscr{I}w = 0\}$, a segment of $\Re w = u_0'$, and a polygonal line none of whose sides makes an angle greater than ε with both directions of the real axis.

Using a sequence of boundary points on Γ_2 and the method described above we construct $S_2' \subset S \cap \left\{w\colon \Re w > u_0';\ -\frac{\pi}{2} < \mathscr{I}w < 0\right\}$ such that the area of $\left\{w\colon \Re w > u_0';\ -\frac{\pi}{2} < \mathscr{I}w < 0\right\} \setminus S_2'$ is finite.

The boundary of the largest subdomain of $\{w\colon \Re w > u_0';\ \mathscr{I}w = 0\} \cup S_2 \cup S_2'$ which is symmetric about $\mathscr{I}w = 0$ will be described by a function $v = \beta(u)$ having the desired properties. This completes the proof of Theorem 1 (necessity).

References

[1] L.V. Ahlfors, Untersuchungen zur Theorie der konformen Abbildung und der ganzen Funktionen, Acta Soc. Sci. Fenn. N.S. (A), No. **9**, (1930), 40 pp.

[2] B.G. Eke, On the angular derivative of regular functions, Math. Scand. **21**, (1967), 122–127.

[3] B.G. Eke and S.E. Warschawski, On the distortion of conformal maps at the boundary, J. London Math. Soc. **44**, (1969), 625–630.

[4] J. Lelong-Ferrand, Représentation conforme et transformations à intégrale de Dirichlet bornée, Gauthier-Villars, Paris, 1955.

[5] S.E. Warschawski, On the boundary behaviour of conformal maps, Nagoya Math. J. **30** (1967), 83–101.

[6] J. Wolff, Sur la représentation d'un demi-plan sur un demi-plan a une infinité d'incisions circulaires, C.R. Acad. Sci. **200**, (1935), 630–632.

Queen's University, Belfast,
Northern Ireland.