Yoshiaki Ikeda
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THE CAUCHY PROBLEM OF LINEAR PARABOLIC EQUATIONS WITH DISCONTINUOUS AND UNBOUNDED COEFFICIENTS

YOSHIAKI IKEDA

§1. Introduction.

In this article we shall prove the uniqueness and existense of a weak solution for the Cauchy problem of linear parabolic equations with discontinuous and unbounded coefficients

$$
(1.1) \qquad Lu = u_t - \{\sum_{i,j=1}^{n}(a_{ij}(x,t)u_{x_i})_{x_j} + \sum_{j=1}^{n} b_j(x,t)u_{x_j} + c(x,t)u\}
$$
$$
= \sum_{j=1}^{n}(f_j(x,t))_{x_j} + g(x,t).
$$

In the case where the coefficients are bounded for large $|x|$, Aronson [1] proved the uniqueness and existence using the weighted energy type estimates for weak solutions. Bodanko [2] also discussed the questions of a regular solution for the Cauchy problem of linear parabolic equations

$$
(1.2) \qquad Lu = \sum_{i,j=1}^{n} a_{i,j}(x,t)u_{x_i x_j} + \sum_{j=1}^{n} b_j(x,t)u_{x_j} + c(x,t)u - u_t
$$
$$
= f(x,t)
$$

with unbounded coefficients under some assumption.

In §2 we shall state some notations and definitions. §3 is devoted to derive the energy estimates for weak solutions of the equation (1.1) and we prove main theorems in §4.

§2. Some notations and definitions.

Let $x = (x_1, \cdots, x_n)$ be a point in the n-dimensional Euclidean space E^n and t a point on the real line. Let T be a fixed positive number and set $Q = E^n \times (0, T]$. For some fixed positive number R_0 we put $Q_0 = \sum_{R_0} \times (0, T]$,

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where $\sum_{R_0} = \{x \in E^n \mid |x| < R_0\}$. Now let ω be a domain in E^n. The space $L^q[0,T; L^p(\omega)]$ is the set of real functions $w(x,t)$ with the following properties;

(i) w is defined and measurable in $\tilde{\omega} = \omega \times (0,T]$,

(ii) $w(x,t) \in L^p(\omega)$ for almost all $t \in (0,T]$,

(iii) $\|w\|_{L^p(\omega)}(t) \in L^q((0,T))$.

The space $L^q[0,T; L^p(\omega)]$ is denoted by $L^{p,q}(\tilde{\omega})$ for simplicity. For $w \in L^{p,q}(\tilde{\omega})$ with $1 \leqq p, q < \infty$ we define the norm by

$$\|w\|_{p,q,\tilde{\omega}} = \left\{ \int_0^T \left(\int_\omega |w|^p \, dx \right)^{q/p} dt \right\}^{1/q}.$$

In the case either p or q is infinte, $\|w\|_{p,q,\omega}$ is defined in a similar fashion using L^∞-norms rather than integrals.

We shall consider the following Cauchy problem:

$$(2.1) \qquad \begin{cases} Lu = \sum_{j=1}^n (f_j)_{x_j} + g & \text{for} \quad (x,t) \in Q, \\ u(x,0) = u_0(x) & \text{for} \quad x \in E^n, \end{cases}$$

where the coefficients a_{ij}, b_j and c are measurable, real valued functions in Q, and f_j and g are given functions in Q. We assume the following conditions:

Condition A.

(A.1) For all $\xi \in E^n$ and for almost all (x,t) there exist positive constants k and K_1 such that

$$k(1+|x|^2)^{1-\lambda}|\xi|^2 \leqq \sum a_{ij}(x,t)\xi_i\xi_j \leqq K_1(1+|x|^2)^{1-\lambda}|\xi|^2$$

where λ is any fixed number with $0 \leqq \lambda \leqq 1$.

(A.2) The restriction of every coefficient b_j to Q_0 belongs to some space $L^{p_j,q_j}(Q_0)$, where p_j and q_j satisfy

$$(*) \qquad 2 < q_j, p_j \leqq \infty \quad \text{and} \quad \frac{n}{2p_j} + \frac{1}{q_j} < \frac{1}{2},$$

and there exists a non-negative constant K_2 such that

$$|b_j(x,t)| \leqq K_2(1+|x|^2)^{\frac{1}{2}} \quad \text{for} \quad (x,t) \in Q - Q_0.$$

(A.3) The restriction of c to Q_0 belongs to $L^{p,q}(Q_0)$, where p and q

satisfy

$$(**)\qquad 1 < p, q \leqq \infty \quad \text{and} \quad \frac{n}{2p} + \frac{1}{q} < 1,$$

and $c(x, t) \leq K_3(1 + |x|^2)^\lambda$ in $(x, t) \in Q - Q_0$ for a non-negative constant K_3.

A function $u(x)$ defined and measurable in ω is said to belong to $H^{1,p}(\omega)$ if u possesses a distribution derivative $(u_{x_1}, \cdots, u_{x_n})$ and $\|u\|_{L^p(\omega)} + \|u_x\|_{L^p(\omega)} < \infty$, where $\|u_x\|_{L^p}^p = \sum_{i=1}^n \|u_{x_i}\|_{L^p}^p$.

The space $H_0^{1,p}(\omega)$ is the completion of the $C_0^\infty(\omega)$ functions in this norm. The space $H^{1,p}(E^n)$ is the completion of the $C_0^\infty(E^n)$ functions in the norm

$$\|\varphi\|_{L^p(E^n)} + \|\varphi_x\|_{L^p(E^n)}.$$

DEFINITION 2.1. A function $u = u(x, t)$ is said to be a weak solution of the problem (2.1) in Q for the initial data $u_0 \in L^2_{loc}(E^n)$ if u belongs to $L^\infty[0, T; L^2(\omega)] \cap L^2[0, T; H^{1,2}(\omega)]$ for any ω compact in E^n, that is, if

$$u \in L^\infty[0, T; L^2_{loc}(E^n)] \cap L^2[0, T; H^{1,2}_{loc}(E^n)]$$

and if u satisfies

$$(2.2)\qquad \iint_Q \{-u\varphi_t + \sum a_{ij} u_{x_i} \varphi_{x_j} - \sum b_j u_{x_j} \varphi - cu\varphi + \sum f_j \varphi_{x_j} - g\varphi\}\, dxdt = 0$$

for any $\varphi \in C_0^1(Q)$ and further if

$$(2.3)\qquad \lim_{t \to 0} \int_{E^n} u(x, t)\psi(x)dx = \int_{E^n} u_0(x)\psi(x)dx \quad \text{for all } \psi \in C_0^1(E^n).$$

At the end of this section we state a lemma which is often used in the following sections.

LEMMA 2.1. (*Aronson* [1]). *If* $w \in L^\infty[0, T; L^2(E^n)] \cap L^2[0, T; H_0^{1,2}(E^n)]$, *then* $w \in L^{2p', 2q'}(Q)$ *for all values of* p' *and* q' *whose Hölder conjugates* p *and* q *satisfy*

$$\frac{n}{2p} + \frac{1}{q} \leqq 1,$$

where if $n = 2$, *then the strict inequality holds. Moreover*

$$\|w\|^2_{2p', 2q', Q} \leqq KT^\theta \{\|w\|^2_{2,\infty,Q} + \|w_x\|^2_{2,2,Q}\},$$

where $|w_x|^2 = \sum_{j=1}^{n} w_{x_j}^2$, $\theta = 1 - \frac{1}{q} - \frac{n}{2p}$, *and K is a positive constant which depends only on n for* $n \neq 2$ *and only on p for* $n = 2$.

§ 3. Energy Estimates.

Let Ω be any bounded domain in E^n such that $\Omega \subset \sum_{R_0}$, and set $Q_1 = \Omega \times (0, T]$. In Q_1 we consider the equation

$$Lu = \sum (f_j)_{x_j} + g, \tag{3.1}$$

where $f_j \in L^{2,2}(Q_1)$ and $g \in L^{p,q}(Q_1)$ for any p and q satisfying (**).

LEMMA 3.1. *Let* $u \in L^\infty[0, T; L^2(\Omega)] \cap L^2[0, T; H^{1,2}(\Omega)]$ *be a weak solution of* (3.1) *with the initial data* u_0, *that is, let u satisfy*

$$\iint_{Q_1} \{-u\varphi_t + \sum a_{ij} u_{x_i} \varphi_{x_j} - \sum b_j u_{x_j} \varphi - cu\varphi + \sum f_j \varphi_{x_j} - g\varphi\}\, dxdt = 0 \tag{3.2}$$

for any $\varphi \in C_0^1(Q_1)$ *and* (2.3), *and let* $\zeta = \zeta(x)$ *be a non-negative smooth function such that* $\zeta u \in L^2[0, T; \dot{H}_0^{1,2}(\Omega)]$. *Then for any positive number* μ_0, *there exist positive constants* $\mathscr{C}$ *and* μ *such that*

$$\begin{aligned} &\|\zeta e^{-\mu(1+|x|^2)^\lambda} u\|_{2,\infty,Q_1'}^2 + \|\zeta e^{-\mu(1+|x|^2)^\lambda} u_x\|_{2,2,Q_1'}^2 \\ &\leqq \mathscr{C}\Big(\int_\Omega \zeta^2 e^{-2\mu_0(1+|x|^2)^\lambda} u_0^2 dx + \|\zeta_x e^{-\mu_0(1+|x|^2)^\lambda} \\ &\times (1+|x|^2)^{\frac{1-\lambda}{2}} u\|_{2,2,Q_1'}^2 + \|\zeta e^{-\mu_0(1+|x|^2)^\lambda} f_j\|_{2,2,Q'}^2 \\ &+ \zeta e^{-\mu_0(1+|x|^2)^\lambda} g\|_{\frac{2p}{p+1}, \frac{2q}{q+1}, Q_1'}\Big) \end{aligned} \tag{3.3}$$

where $Q_1' = \Omega \times (0, T')$ *with* $T' < \frac{\pi}{2}$. *The constants* T', $\mathscr{C}$ *depend only on* k, K_1, K_2, K_3, λ, μ_0, $\|b_j\|_{p_j, q_j, Q_0}$ *and* $\|c\|_{p,q,Q_0}$; *and* μ *depends on* μ_0 *and* T'.

Proof. Let $h(x, t) = -\alpha(t)(1+|x|^2)^\lambda$, where $\alpha(t) \in C^1(\tau', \tau' + \sigma)$ for any τ' and a sufficiently small number σ with $0 \leqq \tau' < \tau' + \sigma \leqq T$. Then we obtain for any $\tau \in [\tau', \tau' + \sigma]$

$$\begin{aligned} &\frac{1}{2}\int_\Omega \zeta^2 e^{2h} u^2 dx\,|_{t=\tau} + \int_{\tau'}^{\tau}\int_\Omega \zeta^2 e^{2h}\{\sum a_{ij} u_{x_i} u_{x_j} - \sum b_j u u_{x_j} \\ &+ (-h_t - c)u^2 + \sum f_j u_{x_j} - gu\}\, dxdt + 2\int_{\tau'}^{\tau}\int_\Omega \zeta e^{2h} u(\sum a_{ij} u_{x_i} \zeta_{x_j} \end{aligned} \tag{3.4}$$

$$+ \sum a_{ij} u_{x_i} \zeta h_{x_j}) + \zeta e^{2h} u \sum f_j(\zeta_{x_j} + \zeta h_{x_j})\} \, dxdt$$
$$= \frac{1}{2} \int_\Omega \zeta^2 e^{2h} u^2 dx \,|_{t=\tau'}$$

(See § 2 in [1]).

Now we shall estimate each term of (3.4) by using Hölder's inequality and Young's inequality together with Lemma 2.1.

First we see from Condition (A.1) that

$$(3.5) \qquad \iint \zeta^2 e^{2h} \sum a_{ij} u_{x_i} u_{x_j} dxdt \geqq k \iint \zeta^2 e^{2h} (1 + |x|^2)^{1-\lambda} |u_x|^2 dxdt,$$

$$(3.6) \qquad 2\iint \sum a_{ij} u_{x_i} u \zeta \zeta_{x_j} e^{2h} \, dxdt \leqq \frac{k}{10} \iint \zeta^2 e^{2h} (1 + |x|^2)^{1-\lambda} |u_x|^2 dxdt$$
$$+ \frac{10 n^2 K_1^2}{k} \iint |\zeta_x|^2 e^{2h} (1 + |x|^2)^{1-\lambda} u^2 dxdt,$$

and

$$(3.7) \qquad 2\iint \sum a_{ij} \zeta^2 e^{2h} u u_{x_i} h_{x_j} dxdt \leqq \frac{k}{10} \iint \zeta^2 e^{2h} (1 + |x|^2)^{1-\lambda} |u_x|^2 dxdt$$
$$+ \frac{10 n^2 K_1^2}{k} \iint \zeta^2 e^{2h} (1 + |x|^2)^{1-\lambda} u^2 |h_x|^2 dxdt.$$

Next we have

$$\int_{\tau'}^{\tau} \int_\Omega \sum b_i \zeta^2 e^{2h} u_{x_j} u dxdt \leqq \int_{\tau'}^{\tau} \int_{\Sigma_{R_0}} \sum |b_i| \zeta^2 e^{2h} |u_{x_j}| \; |u| \, dxdt$$
$$+ K_2 \int_{\tau'}^{\tau} \int_\Omega \zeta^2 e^{2h} (1 + |x|^2)^{\frac{1}{2}} |u| \sum |u_{x_j}| \, dxdt$$
$$\equiv B_1 + B_2, \quad \text{say.}$$

By the inequalities of Hölder and Young it is clear that

$$B_1 \leqq \frac{k}{10} \|\zeta e^h u_x\|_{2,2}^2 + \frac{5}{k} \sum \|b_j^2\|_{p,q} \cdot \|\zeta e^h u\|_{2p',2q'}^2,$$

where p and q satisfy (**). Thus by Lemma 2.1, we see

$$B_1 \leqq \frac{k}{10} \|\zeta e^h u_x\|_{2,2}^2 + \frac{5}{k} MK\sigma^\theta \{\|\zeta e^h u\|_{2,\infty}^2 + \|(\zeta e^h u)_x\|_{2,2}^2\},$$

where we assumed $\sum \|b_j^2\|_{p,q} \leqq M$.

Using Young's inequality again we see

$$B_2 \leqq \frac{k}{10} \iint \zeta^2 e^{2h}(1+|x|^2)^{1-\lambda}|u_x|^2\,dxdt + \frac{5K_2^2 n}{k} \iint \zeta^2 e^{2h}(1+|x|^2)u^2 dxdt.$$

Thus we obtain

$$(3.8) \qquad \sum\iint b_j \zeta^2 e^{2h} u_{x_j} u dxdt \leqq \frac{k}{5} \iint \zeta^2 e^{2h}(1+|x|^2)^{1-\lambda}|u_x|^2 dxdt$$

$$+ \frac{5}{k} MK\sigma^\theta \{\|\zeta e^h u\|_{2,\infty}^2 + \|(\zeta e^h u)_x\|_{2,2}^2\}$$

$$+ \frac{5K_2^2 n}{k} \iint \zeta^2 e^{2h}(1+|x|^2)^\lambda u^2 dxdt.$$

Similarly we have

$$(3.9) \qquad \iint c\zeta^2 e^{2h} u^2 dxdt \leqq MK\sigma^\theta\ \{\|\zeta e^h u\|_{2,\infty}^2 + \|(\zeta e^h u)_x\|_{2,2}^2\}$$

$$+ K_3 \iint \zeta^2 e^{2h}(1+|x|^2)^\lambda u^2 dxdt,$$

where we assumed $\|c\|_{p,q} \leqq M$.

It is easily seen that

$$(3.10) \qquad 2\iint \sum f_i \zeta e^{2h}(\zeta_{x_j} + \zeta h_{x_j}) u dxdt \leqq$$

$$2\sum \|\zeta e^h f_j\|_{2,2}^2 + \|\zeta_x e^h u\|_{2,2}^2 + \|\zeta e^h u h_x\|_{2,2}^2,$$

$$(3.11) \qquad \iint \sum f_j \zeta^2 e^{2h} u_{x_j} dxdt \leqq \frac{k}{10} \|\zeta e^h u_x\|_{2,2}^2 + \frac{10}{k} \sum \|\zeta e^h f_j\|_{2,2}^2,$$

and

$$(3.12) \qquad \iint \zeta^2 e^{2h} g u dxdt \leqq \|\zeta e^h g\|_{\frac{2p}{p+1},\ \frac{2q}{q+1}} \|\zeta e^h u\|_{2p',\,2q'}$$

$$\leqq \frac{1}{2M} \|\zeta e^h g\|^2_{\frac{2p}{p+1},\ \frac{2q}{q+1}} + \frac{M}{2} K\sigma^\theta \{\|\zeta e^h u\|_{2,\infty}^2 + \|(\zeta e^h u)_x\|_{2,2}^2\}.$$

Finally we note that

$$(3.13) \qquad \|(\zeta e^h u_x)\|_{2,2}^2 \leqq 2\|\zeta e^h u_x\|_{2,2}^2 + 4\|u e^h \zeta_x\|_{2,2}^2 + 4\|\zeta e^h u h_x\|_{2,2}^2.$$

Combining these estimates from (3.5) to (3.13), we have

$$(3.14) \qquad \frac{1}{2}\int_\Omega \zeta^2 e^{2h} u^2 dx|_{t=\tau} + \left\{\frac{k}{2} - \left(\frac{10}{k} + 3\right) MK\sigma\right\} \int_{\tau'}^{\tau} \int_\Omega \zeta^2 e^{2h}$$

$$(1+|x|^2)^{1-\lambda}|u_x|^2 dxdt + \int_{\tau'}^{\tau} \int_\Omega \zeta^2 e^{2h} u^2 \left\{ -h_t - \left[\frac{10n^2 K_1^2}{k} (1+|x|^2)^{1-\lambda}|h_x|^2\right.\right.$$

$$+\Big(\frac{20}{k}n^2MK\sigma^\theta+6MK\sigma^\theta+1\Big)|h_x|^2+\Big(\frac{5K_2^2n}{k}+K_3\Big)(1+|x|^2)^\lambda\Big]\Big\}dxdt$$

$$\leqq\frac{1}{2}\Big(\frac{10}{k}+3\Big)MK\sigma^\theta\|\zeta e^h u\|_{2,\infty}^2+\frac{1}{2}\int_\Omega\zeta^2e^{2h}u^2dx|_{t=\tau'}$$

$$+C_1\Big(\int_{\tau'}^{\tau}\int_\Omega|\zeta_x|^2e^{2h}(1+|x|^2)^{1-\lambda}u^2dxdt+\sum\|\zeta e^h f_j\|_{2,2}^2+\|\zeta e^h g\|^2_{\frac{2p}{p+1},\frac{2q}{q+1}}\Big),$$

where we used the fact that $1\leqq(1+|x|^2)^{1-\lambda}$.

Since $h(x,t)=-\alpha(t)(1+|x|^2)^\lambda$, we see

$$-h_t(x,t)=\alpha(t)(1+|x|^2)^\lambda \text{ and } |h_x|^2\leqq\alpha^2(t)4\lambda^2(1+|x|^2)^{2\lambda-1}.$$

Noting that $2\lambda-1\leqq\lambda$, that is, $(1+|x|^2)^{2\lambda-1}\leqq(1+|x|^2)^\lambda$, we obtain

$$-h_t-\Big[\frac{10n^2K_1^2}{k}(1+|x|^2)^{1-\lambda}|h_x|^2+\Big(\frac{20}{k}n^2MK\sigma^\theta+6MK\sigma^\theta+1\Big)|h_x|^2$$

$$+\Big(\frac{5K_2^2n}{k}+K_3\Big)(1+|x|^2)^\lambda\Big]\geqq(1+|x|^2)^\lambda\{\alpha'(t)-A\alpha^2(t)-B\},$$

where

$$A=4\Big(\frac{10n^2K_1^2+20n^2MK\sigma^\theta}{k}+6MK\sigma+1\Big)\lambda^2,\text{ and } B=\frac{5K_2^2n}{k}+K_3.$$

If $\lambda\neq0$ we put $\alpha(t)=\sqrt{\frac{B}{A}}\tan\{\sqrt{AB}\,t+m\}$ for $0\leqq t\leqq\tau'$, where $m(>0)$ is a constant with $\sqrt{\frac{B}{A}}\tan m=\mu_0\Big(0<m<\frac{\pi}{2}\Big)$. Then

$$\alpha'(t)-A\alpha^2(t)-B=0.$$

Now we put σ so small that $\frac{k}{2}-\Big(\frac{10}{k}+3\Big)MK\sigma>0$, $\frac{1}{2}-\frac{1}{2}\Big(\frac{10}{k}+3\Big)$. $MK\sigma>0$ and $m+\sqrt{AB}\,\sigma<\min\Big(\frac{\pi}{2},\ T\Big)$. Then from (3.14) we have

$$\max_{\tau'\leq t\leq\tau'+\sigma}\int_\Omega\zeta^2e^{2h}u^2dx+\int_\tau^{\tau'+\sigma}\int_\Omega\zeta^2e^{2h}|u_x|^2dxdt \tag{3.15}$$

$$\leq\mathscr{C}_2\Big(\int_\Omega\zeta^2e^{2h}u^2dx|_{t=\tau'}+\|\zeta_xe^h(1+|x|^2)^{\frac{1-\lambda}{2}}u\|_{2,2}^2$$

$$+\sum\|\zeta e^hf_j\|_{2,2}^2+\|\zeta e^hg\|^2_{\frac{2p}{p+1},\frac{2q}{q+1}}\Big),$$

where $h=h(x,t)=-\sqrt{\frac{B}{A}}\tan(\sqrt{AB}\,t+m)(1+|x|^2)$.

Let $X(t)=\int\zeta^2(x)e^{2h(x,t)}u^2(x,t)dx$ and $J=\|\zeta_x e^h(1+|x|^2)^{\frac{1-\lambda}{2}}u\|_{2,2}^2$
$+\sum\|\zeta e^h f_j\|_{2,2}^2+\|\zeta e^h g\|^2_{\frac{2p}{p+1},\frac{2q}{q+1}}$.

Then from (3.14) we see

$$X(t)\leqq \mathscr{C}_2\{X(\tau')+J\} \text{ for } \tau'<t\leqq\tau'+\sigma.$$

If $(j-1)\sigma\leqq t<j\sigma$, it follows by iteration that

$$X(t)\leqq\mathscr{C}_2^jX(0)+\frac{\mathscr{C}_2(\mathscr{C}_2^j-1)}{\mathscr{C}_2-1}J \text{ for } (j-1)\sigma<t\leqq j\sigma. \tag{3.16}$$

On the other hand,

$$\int_{(j-1)\sigma}^{j\sigma}\int_\Omega\zeta^2e^{2h}|u_x|^2dxdt\leqq\mathscr{C}_2\{X((j-1)\sigma)+J\} \tag{3.17}$$
$$\leqq\mathscr{C}_2^jX(0)+\frac{\mathscr{C}_2^j-1}{\mathscr{C}_2-1}J.$$

Suppose that $(l-1)\sigma<T'<l\sigma$ for some integer l, where $\sqrt{AB}\,T'+m<\frac{\pi}{2}$. Then summing (3.16) on j from 1 to l, we have

$$\|\zeta e^hu\|_{2,\infty,Q_1'}^2\leqq\mathscr{C}_3\{X(0)+J\}.$$

Similarly from (3.16),

$$\|\zeta e^hu_x\|_{2,2,Q_1'}^2\leqq\mathscr{C}_4\{X(0)+J\}.$$

Putting $\sqrt{\frac{B}{A}}\tan(\sqrt{AB}\,l+m)=\mu$, and combining these two inequalities we obtain (3.3).

If $\lambda=0$, then we put $\alpha(t)=Bt+\mu_0$ and we have (3.3) in the same manner as above.

§ 4. The Cauchy problem.

In this section we consider the Cauchy problem (2.1). A measurable function $u(x,t)$ on Q is said to be in the class $\mathscr{E}_\mu^\lambda(Q)$ if there exist numbers $\lambda\geqq0$ and $\mu>0$ such that

$$\iint_\Omega e^{-2\mu|x|^{2\lambda}}u^2dxdt<\infty.$$

THEOREM 1. *If there are solutions u of the Cauchy problem* (2.1) *in the class $\mathscr{E}_{\mu_1}^\lambda(Q)$ for some positive constant $\mu_1<\mu_0$, then u is uniquely determined in Q.*

Proof. Let us assume that there are two solutions u_1, u_2 of the Problem (2.1) *in* *the class* $\mathscr{E}^{\lambda}_{\mu_1}(Q)$. *Put* $u = u_1 - u_2$. *Then we see that* u *is in* $\mathscr{E}^{\lambda}_{\mu_1}(Q)$ *and* u *is a weaksolution of the problem*

$$Lu = 0 \quad \textit{for} \quad (x, t) \in Q,$$
$$u(x, 0) = 0, \quad \textit{for} \quad x \in E^n.$$

For each $R \geqq R_0$, we define a function $\zeta_R(x)$ in such a way that

(i) $\quad |\zeta_R(x)| \leqq 1 \quad \text{in} \quad (-\infty, \infty),$

(ii) $\quad \zeta_R(x) = \begin{cases} 1 & |x| \leqq R, \\ 0 & |x| \geqq R+1, \end{cases}$

(iii) $\quad |\zeta_{R_x}(x)| \leqq C$ in $(-\infty, \infty)$, where C is independent of R.

By Lemma 3.1,

$$(4.1) \qquad \|\zeta_R e^{-\mu(1+|x|^2)^\lambda} u\|^2_{2,\infty}$$
$$\leq \|\zeta_{R_x} e^{-\mu_0(1+|x|^2)^\lambda} (1+|x|^2)^{\frac{1-\lambda}{2}} u\|^2_{2,2,Q'}$$
$$\leqq \mathscr{C}_1 \int_\tau^{\tau'} \int_{|x| \geqq R} e^{-2\mu_0(1+|x|^2)^\lambda} (1+|x|^2)^{1-\lambda} u^2 dx dt,$$

where $\mathscr{C}_1$ is independent of R, and $\mu_1 < \mu_0 < \mu$.
Since

$$e^{-\mu_1(1+|x|^2)^\lambda} u \in L^2[0, T'; L^2(E^n)] \quad \text{for} \quad \mu_0 > \mu_1,$$

we see

$$e^{-\mu_0(1+|x|^2)^\lambda} (1+|x|^2)^{1-\lambda} u \in L^2[0, T'; L^2(E^n)].$$

Therefore the integral on the right in (4.1) tends to zero as $R \to \infty$. Hence

$$\max_{0 \leq t \leq \tau'} \int_{|x| \leq \rho} e^{-2\mu(1+|x|^2)^\lambda} u^2 dx = 0$$

for an arbitrary $\rho > 0$. This means that $u \equiv 0$ in $E^n \times (0, T']$ that is, $u_1 \equiv u_2$ in $E^n \times (0, T']$.

Repeating the same argument on $E^n \times (NT', (N+1)T']$ inductively, we conclude that $u_1 \equiv u_2$ in $E^n \times (0, T]$.

THEOREM 2. *Suppose that* $e^{-\mu_0 |x|^{2\lambda}} f_j \in L^{2,2}(Q)$, $e^{-\mu_0 |x|^{2\lambda}} g \in L^{p,q}(Q)$ *with* p *and* q *satisfying* (**) *and* $e^{-\mu_0 |x|^{2\lambda}} u_0 \in L^2(E^n)$. *Then there exists a weak solution* u *of the Cauchy problem* (2.1) *in* $Q' = E^n \times (0, T']$, *where* T' *depends on the constant in*

Condition (A) *and* μ_0. *Moreover there exists a constant* μ *depending on* T' *and* μ_0 *such that*

$$\begin{aligned}&\|e^{-\mu(1+|x|^2)^\lambda}u\|^2_{2,\infty,Q'}+\|e^{-\mu(1+|x|^2)^\lambda}u_x\|^2_{2,2,Q'}\\&\leqq \mathscr{C}\{\|e^{-\mu_0(1+|x|^2)^\lambda}u_0\|^2_{L^2(E^n)}+\sum\|e^{-\mu_0(1+|x|^2)^\lambda}f_j\|^2_{2,2,Q'}\\&\quad+\|e^{-\mu_0(1+|x|^2)^\lambda}g\|_{p,q,Q'}\}.\end{aligned}$$

Since the proof of Theorem 2 is almost parallel to that of [1], we omit it here.

Remark. In Lemma 3.1, if $\lambda=0$, then we put $\alpha(t)=Bt+\mu_0$. Thus Lemma 3 is valid for $Q=E^n\times(0,T]$ if $\lambda=0$. Therefore if $\lambda=0$, a weak solution of the problem (2.1) exists in $Q=E^n\times(0,T]$.

References

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Nagoya University