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REMARKS ON THE ANGULAR DERIVATIVE*)

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Introduction. Suppose that Ω is a simply connected domain in the w-plane, $w = u + iv$, and that w_∞ is an accessible boundary point of Ω located at $w = \infty$. Suppose $w = W(z) = U(z) + iV(z)$ maps the strip $\sum = \{z = x + iy\colon\ -\infty < x < \infty,\ 0 < y < \pi\}$ conformally onto Ω such that $\lim_{x\to+\infty} W\left(x + i\frac{\pi}{2}\right) = w_\infty$. If in any sub-strip $\{z = x + iy\colon -\infty < x < \infty,\ \delta \leqq y \leqq \pi - \delta\}$, $0 < \delta < \frac{\pi}{2}$,

$$\lim_{x\to+\infty} [W(z) - z] = \kappa \quad \text{exists and is finite,} \tag{1}$$

then $W(z)$ is said to have an angular derivative at $z = +\infty$.[1] The problem of finding geometrical conditions on Ω which ensure the existence of the angular derivative has received considerable attention ever since Carathéodory introduced this notion in the study of the boundary behavior of conformal maps in 1929 (cf. [5], Chapter III, [4], Chapter VI, in particular pp. 204–217, and [6], Theorem 6). In this note we present another such criterion, which for a wide class of domains yields a sharper sufficient condition than the earlier results. The basis for this criterion is the following more special result.

Suppose $\{u_n\}$, $\{v_n\}$, $\{v'_n\}$ are sequences of real numbers such that

$$u_{n+1} - u_n \geqq d > 0,\ \lim_{n\to\infty} v_n = 0,\ \lim_{n\to\infty} v'_n = \pi \tag{2}$$

and let S denote the interior of the union of the rectangles

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1) If Ω is mapped conformally onto a domain D such that w_∞ corresponds to a finite boundary point of D and $\sum$ onto the unit disk $\{|\zeta| < 1\}$ such that $z = +\infty$ corresponds to $\zeta = 1$, then the conformal mapping of the disk onto D has a non-vanishing finite derivative at $\zeta = 1$ for approach in a Stolz angle.

$$S_n = \{w = u + iv\colon\ u_n \leqq u \leqq u_{n+1},\ v_n \leqq v \leqq v_n'\},\ n = 1, 2, \cdots$$

and the half-strip

$$S_0 = \{w = u + iv\colon\ -\infty < u \leqq u_1,\ v_1 \leqq v \leqq v_1'\},$$

i.e.

$$S = \operatorname{Int} \bigcup_{n=0}^{\infty} S_n. \tag{3}$$

Suppose $w = W(z) = (Uz) + iV(z)$ maps $\sum$ conformally onto S such that $\lim_{x\to+\infty} U\left(x + i\frac{\pi}{2}\right) = +\infty$ and $\lim_{x\to-\infty} U\left(x + i\frac{\pi}{2}\right) = -\infty$. Then we prove first the following theorem:

THEOREM 1. *Let* $\theta_n = v_n' - v_n$ *and* $\lambda_n = Max[|v_{n+1} - v_n|,\ |v_{n+1}' - v_n'|]$. *If*

(a) $$\sum_{n=1}^{\infty} |\pi - \theta_n|(u_{n+1} - u_n) < \infty$$

and

(b) $$\sum_{n=1}^{\infty} \lambda_n^2 \log \frac{1}{\lambda_n} < \infty,$$

then, for unrestricted approach for $z \in \sum$,

$$\lim_{x\to+\infty} [W(z) - z] = \kappa \quad \textit{exists and} \quad -\infty < \kappa < +\infty. \tag{1'}$$

The essential step in the proof of this theorem is an estimate of the oscillation $\omega(x)$ of $U(x + iy)$ on a vertical segment $\Re z = x$ of $\sum$ (Lemma 2).

The above mentioned criterion for more general domains is then obtained from Theorem 1 by using S as an "interior comparison domain" (Theorem 2, section 4). To indicate the scope of Theorem 1 we mention an example considered by J. Ferrand in [2] and jointly with J. Dufresnoy in [3], viz. the special case of the domain S where $v_n' = v_n + \pi$, so that $\theta_n \equiv \pi$ and $|v_{n+1} - v_n| = |v_{n+1}' - v_n'| = \lambda_n$. In [3] they proved that $\sum_{\nu=1}^{\infty} \lambda_\nu^2 < \infty$ is necessary for the existence of (1′) and that a *sufficient* condition is $\sum_{\nu=1}^{\infty} \lambda_\nu^{3/2} < \infty$. All present criteria known to the author do not appear to yield a sharper sufficient condition. *Theorem* 1 *shows that* $\sum_{\nu=1}^{\infty} \lambda_\nu^2 \log \frac{1}{\lambda_\nu} < \infty$ *is sufficient for the existence of* (1′).

1. Semiconformality. Since the boundary curves of S have the lines $v=0$ and $v=\pi$ as asymptotes as $u\to+\infty$ it follows that $\lim\limits_{x\to\infty}(V(z)-y)=0$ exists, uniformly for $0\leqq y\leqq\pi$; in particular, the map $z\to W(z)$ is semiconformal at $z=+\infty$. This has a number of useful consequences. Let $z=Z(w)=X(w)+iY(w)$ be the inverse function of $W(z)$. *Then for any* $w', w''\in\bar{S}$, $\Re(w')=u'$, $\Re(w'')=u''$, $u'<u''$,

$$X(w'')-X(w')=(1+o(u',u''))(u''-u')+o(u',u'') \tag{1.1}$$

where $o(u',u'')\to 0$ *as* $u'\to+\infty$, *uniformly in* $\bar{S}$. This follows e.g. from Corollary 1 of Theorem 1a and Theorem 2 of [6].

Let w_n and w_n^* denote the vertices u_n+iv_{n-1} and u_n+iv_n on the lower boundary of S and w_n', $w_n'^*$ those on the upper, u_n+iv_{n-1}' and u_n+iv_n', respectively $(n\geqq 2)$. Under the map $w\to Z(w)$, w_n, w_n^* correspond to points x_n, x_n^*, $x_n<x_n^*$, and w_n', $w_n'^*$ to points $x_n'+i\pi$, $x_n'^*+i\pi$ with $x_n'<x_n'^*$. Since $u_{n+1}-u_n\geqq d>0$ we have from (1.1) for all sufficiently large n

$$x_{n+1}-x_n\geqq\frac{3d}{4} \quad\text{and}\quad x_{n+1}'-x_n'\geqq\frac{3d}{4},$$

and therefore there exists a constant $k>0$ such that for *all* $n=1,2,\cdots$

$$x_{n+1}-x_n\geqq k \quad\text{and}\quad x_{n+1}'-x_n'\geqq k. \tag{1.2}$$

Furthermore, (1.1) shows that the octagon

$$\left\{w=u+iv : w\in\bar{S},\ |u-u_n|\leqq\frac{d}{2}\right\}$$

is mapped onto a curvilinear rectangle contained in the rectangles

$$\left\{z=x+iy : |x-x_n|\leqq\frac{5}{8}d,\ 0\leqq y\leqq\pi\right\}$$
and
$$\left\{z=x+iy : |x-x_n'|\leqq\frac{5}{8}d,\ 0\leqq y\leqq\pi\right\} \tag{1.3}$$

provided n is sufficiently large, say $n>N_0$.

We also assume N_0 so large that $\lambda_n<\frac{\pi}{16}$ and $|\theta_n-\pi|<\frac{\pi}{8}$ for $n>N_0$.

Finally, it follows from Theorem 5 of [6], under the hypothesis (a) of Theorem 1 (which ensures condition (5.1) of [6]), since $\lim\limits_{n\to\infty}\theta_n=\pi$ that for $w\in S$

$$\lim_{u\to+\infty}[Z(w)-w]=\Lambda \tag{1.4}$$

exists and that $-\infty < \Lambda \leqq +\infty$. It remains thus for us to show that $\Lambda < +\infty$ if $\sum_{\nu=1}^{\infty} \lambda_\nu^2 \log \frac{1}{\lambda_\nu}$ converges.

For the proof we shall need two lemmas.

LEMMA 1. *Let* $a_n = \frac{1}{2}(w_n + w_n^*)$, $n > N_0$. *If*

$$w', w'' \in \{|w - a_n| \leqq r\} \cap \bar{S}, \quad \textit{where } 2r \leqq r_0 = \mathit{Min}\left(\frac{d}{4}, \frac{\pi}{2}\right), \tag{1.5}$$

then

$$|Z(w') - Z(w'')| \leqq cr \quad \textit{where} \quad c = \frac{2\pi}{r_0}\sqrt{\frac{2d}{\log 2}} \geqq 1.$$

An analogous statement holds for $a_n' = \frac{1}{2}(w_n' + w_n'^*)$ *in place of* a_n. *In particular, Lemma* 1 *implies that*

$$x_n^* - x_n \leqq c\lambda_n, \quad x_n'^* - x_n' \leqq c\lambda_n \textit{ for } n > N_0. \tag{1.6}$$

Proof. Let $\gamma_\rho = \{|w - a_n| = \rho\} \cap \bar{S}$ for $\rho \leqq r_0$; because of the symmetrical location of a_n, γ_ρ is a semicircle. If l_ρ denotes the length of $\Gamma_\rho = Z(\gamma_\rho)$ we have

$$l_\rho^2 = \left(\int_{\gamma_\rho} |Z'(a_n + \rho e^{i\theta})|\,\rho d\theta\right)^2 \leqq \int_{\gamma_\rho} |Z'(a_n + \rho e^{i\theta})|^2 \rho d\theta \cdot \pi\rho$$

and therefore $(r \leqq r_0)$

$$\int_0^r \frac{l_\rho^2}{\rho}\, d\rho \leqq \pi \int_0^r \int_{\gamma_\rho} |Z'(a_n + \rho e^{i\theta})|\,\rho d\theta d\rho = \pi A(r) \tag{1.7}$$

where $A(r)$ is the area of the domain Δ_r bounded by Γ_r and a segment of the real axis which contains $Z(a_n)$. We reflect Γ_r with respect to the real axis obtaining an arc $\bar{\Gamma}_r$ and consider the interior of the closed Jordan curve bounded by $\Gamma_r \cup \bar{\Gamma}_r$. By the isoperimetric inequality:

$$2A(r) \leqq \frac{(2l_r)^2}{4\pi}$$

and thus

$$A(r) \leqq \frac{l_r^2}{2\pi} \leqq \frac{rA'(r)}{2},$$

$$\frac{2}{r} \leqq \frac{A'(r)}{A(r)} \quad \text{or} \quad \frac{A(r_1)}{r_1^2} \leqq \frac{A(r_2)}{r_2^2} \qquad (r_1 < r_2 \leqq r_0).$$

Since for $r < r_0$, Δ_r is surely contained in the rectangles (1.3), $A(r_0) \leqq 2\pi d$ and therefore, for any $r \leqq r_0$

$$A(r) \leqq \frac{2\pi d}{r_0^2} r^2.$$

Thus, from (1.7), for $2r < r_0$

$$\int_r^{2r} \frac{l_\rho^2}{\rho} d\rho < \frac{2\pi^2 d}{r_0^2} (2r)^2.$$

Hence there exists a ρ_1, $r \leqq \rho_1 \leqq 2r$, such that

$$l_{\rho_1}^2 \leqq \frac{8\pi^2 d}{r_0^2 \log 2} r^2.$$

Now, if w' and w'' satisfy (1.5), $Z(w')$, $Z(w'') \in \Delta_{\rho_1}$ whose diameter is $\leqq l_{\rho_1} \leqq cr$, $c = \frac{2\pi}{r_0}\sqrt{\frac{2d}{\log 2}} > 1$. This proves the conclusion.

2. Estimate of the oscillation $\omega(x)$. We return to the function $w = W(z) = U(z) + iV(z)$ and define for $-\infty < x < \infty$

$$\omega(x) = \operatorname*{Max}_{0 \leqq y, y' \leqq \pi} |U(x+iy) - U(x+iy')|.$$

Clearly

$$\omega(x) \leqq \int_0^\pi \left| \frac{\partial U(x+iy)}{\partial y} \right| dy = \int_0^\pi \left| \frac{\partial V(x+iy)}{\partial x} \right| dy \tag{2.1}$$

by the Cauchy-Riemann differential equation. We obtain an estimate for $\omega(x)$ by estimating the latter integral.

LEMMA 2. *Suppose, for some n, x is a point in the interval*

$$x_{n-1} + \frac{k}{2} < x < x_{n+1} - \frac{k}{2} \tag{2.2}$$

which has at least the distance δ, $0 < \delta < \frac{k}{4}$ from the intervals $I_n = [x_n, x_n^*]$ and $I_n' = [x_n', x_n'^*]$. Here k is the constant defined in (1.2). Then there exists an N such that for $n > N$

$$\omega(x) \leqq \frac{2}{\pi} \left\{ \lambda_n \log \frac{2e^k}{\delta} + \left[\sinh \frac{k}{8} \right]^{-1} \sigma_n \right\} \equiv \mu_n$$

where $\sigma_n > 0$, $\lim_{n\to\infty} \sigma_n = 0$ and $\sum_{n=1}^\infty \sigma_n \lambda_n$ converges if $\sum \lambda_n^2$ converges. In fact, if $\sum \lambda_n^2 = A$, then

$$\sum_{n=1}^{\infty} \sigma_n \lambda_n \leqq AR \frac{R+1}{R-1}, \quad R = e^k. \tag{2.3}$$

Proof. Since $V(z)$ is harmonic and bounded in $\sum$ and has continuous boundary values, except when $x \to \pm\infty$, we have by the Poisson integral

$$V(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} V(\xi) d \arctan\left(\frac{e^\xi - e^x \cos y}{e^x \sin y}\right) + \frac{1}{\pi} \int_{-\infty}^{\infty} V(\xi + i\pi) d \arctan\left(\frac{e^\xi + e^x \cos y}{e^x \sin y}\right).$$

Since $\lim_{\xi \to +\infty} V(\xi) = 0$, $\lim_{\xi \to -\infty} V(\xi) = v_1$, $\lim_{\xi \to +\infty} V(\xi + i\pi) = \pi$, $\lim_{\xi \to -\infty} V(\xi + i\pi) = v_1'$ we obtain by integration by parts, with $2c = \pi - (v_1' - v_1)$, $\pi c_1 = v_1' - v_1$,

$$V(z) = c + c_1 y - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dV(\xi)}{d\xi} \arctan \frac{e^\xi - e^x \cos y}{e^x \sin y}\, d\xi - \frac{1}{\pi} \int_{-\infty}^{\infty} \times$$
$$\frac{dV(\xi + i\pi)}{d\xi} \arctan \frac{e^\xi + e^x \cos y}{e^x \sin y}\, d\xi.$$

Hence

$$\frac{\partial V(z)}{\partial x} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dV(\xi)}{d\xi} \frac{e^{\xi + x} \sin y\, d\xi}{e^{2\xi} + e^{2x} - 2e^{(\xi + x)} \cos y} - \frac{1}{\pi} \int_{-\infty}^{\infty} \times$$
$$\frac{dV(\xi + i\pi)}{d\xi} \frac{e^{\xi + x}(-\sin y)}{e^{2x} + e^{2\xi} + 2e^{(\xi + x)} \cos y}\, d\xi.$$

Using the equation $\dfrac{\partial V}{\partial x} = -\dfrac{\partial U}{\partial y}$ we obtain

$$\int_0^\pi \left|\frac{\partial U}{\partial y}\right| dy \leqq \frac{1}{2\pi} \int_{-\infty}^{\infty} \left|\frac{dV(\xi)}{d\xi}\right| \log\left[\frac{e^x + e^\xi}{e^x - e^\xi}\right]^2 d\xi + \frac{1}{2\pi} \int_{-\infty}^{\infty} \left|\frac{dV(\xi + i\pi)}{d\xi}\right| \log\left[\frac{e^x + e^\xi}{e^x - e^\xi}\right]^2 d\xi. \tag{2.4}$$

We note now that $\dfrac{dV(\xi)}{d\xi} = 0$ outside of I_ν and $\dfrac{dV(\xi + i\pi)}{d\xi} = 0$ outside of I_ν' $(\nu = 1, 2, \cdots)$; in I_ν and I_ν' V is a monotone function and

$$\int_{I_\nu} \left|\frac{dV(\xi)}{d\xi}\right| d\xi \leqq \lambda_\nu, \quad \int_{I_\nu'} \left|\frac{dV(\xi + i\pi)}{d\xi}\right| d\xi \leqq \lambda_\nu \quad (\nu = 1, 2, \cdots). \tag{2.5}$$

We estimate therefore $\log\left|\dfrac{e^x + e^\xi}{e^x - e^\xi}\right|$ for $\xi \in I_\nu$ and $\xi \in I_\nu'$. We choose $N > N_0$ and so large that for $n > N$: $|x_{n-1}' - x_{n-1}| < \dfrac{k}{8}$ and $8c\lambda_{n-1} < k$. We consider the cases $\nu = n$, $\nu > n$, and $\nu < n$ separately.

(a) Let $\nu = n$. Assume first $\xi \in I_n$ and

$$x_n - \frac{k}{2} \leqq x \leqq x_n + \frac{k}{2} \tag{2.6}$$

which is a subinterval of $\left[x_{n-1}+\frac{k}{2},\ x_{n+1}-\frac{k}{2}\right]$. Then either $x_n^*+\delta \leqq x \leqq x_n+\frac{k}{2}$ or $x_n-\frac{k}{2} \leqq x \leqq x_n-\delta$. In the first instance

$$0<\frac{e^x+e^\xi}{e^x-e^\xi} \leqq \frac{e^{x_n+\frac{k}{2}}+e^{x_n^*}}{e^{x_n^*+\delta_n}-e^{x_n^*}} \leqq \frac{e^{\frac{k}{2}}+1}{e^\delta-1} \leqq \frac{e^{\frac{k}{2}}+1}{\delta}$$

and in the second, using (1.6),

$$0<\frac{e^\xi+e^x}{e^\xi-e^x} \leqq \frac{e^{x_n+c\lambda_n}+e^{x_n-\delta}}{e^{x_n}-e^{x_n-\delta}} = \frac{e^{\delta+c\lambda_n}+1}{e^\delta-1} \leqq \frac{e^{\frac{k}{2}}+1}{\delta}.$$

If x is outside the interval (2.6) then for $x \geqq x_n+\frac{k}{2}$

$$e^{\xi-x} \leqq e^{x_n+c\lambda_n-\left(x_n+\frac{k}{2}\right)} \leqq e^{-\frac{k}{4}},$$

and for $x \leqq x_n-\frac{k}{2}$

$$e^{x-\xi} \leqq e^{x_n-\frac{k}{2}-x_n} = e^{-\frac{k}{2}},$$

so that

$$\left|\frac{e^x+e^\xi}{e^x-e^\xi}\right| \leqq \frac{1+e^{-\frac{k}{4}}}{1-e^{-\frac{k}{4}}} \leqq \frac{e^{\frac{k}{4}}+1}{\delta} \leqq \frac{e^{\frac{k}{2}}+1}{\delta} \leqq \frac{2e^k}{\delta}. \tag{2.7}$$

Thus (2.7) holds for $\nu=n$, $\xi \in I_\nu$ and x in (2.1).

When $\xi \in I_\nu'$ we note that our assumption $|x_{n-1}'-x_{n-1}|<\frac{k}{8}$ for $n>N$ implies that for any x in (2.2) we also have $x \in \left[x_{n-1}'+\frac{k}{4},\ x_{n+1}'-\frac{k}{4}\right]$. Hence the same argument shows that (2.7) is satisfied for $\xi \in I_\nu'$.

(b) When $\nu>n$ (and also when $\nu<n$) we use the inequality

$$\log\frac{1+u}{1-u} \leqq 2u\,\frac{1}{1-a^2} \quad \text{for} \quad 0<u \leqq a<1. \tag{2.8}$$

For x in (2.2) and $\xi \in I_\nu$, $\nu>n$, we have

$$e^{x-\xi} \leqq e^{x_{n+1}-\frac{k}{2}-x_\nu} \leqq \frac{e^{-\frac{k}{2}}}{e^{k(\nu-n-1)}} = \frac{e^{-\frac{k}{2}}}{R^{\nu-(n+1)}}, \quad R=e^k.$$

Applying (2.8) with $a = e^{-\frac{k}{2}}$ we have

$$\log\left|\frac{e^x + e^\xi}{e^x - e^\xi}\right| \leqq \frac{2e^{-\frac{k}{2}}}{R^{\nu-(n+1)}} \frac{1}{1-e^{-k}} = \frac{1}{R^{\nu-(n+1)}} \frac{1}{\sinh\left(\frac{k}{2}\right)}.$$

When $\xi \in I'_\nu$ we observe again that for x in (2.2), we have $x \in \left[x'_{n-1} + \frac{k}{4}, x'_{n+1} - \frac{k}{4}\right]$, and therefore

$$\log\left|\frac{e^x+e^\xi}{e^x-e^\xi}\right| = \log\frac{1+e^{x-\xi}}{1-e^{x-\xi}} \leqq \frac{2e^{-\frac{k}{4}}}{R^{\nu-n-1}} \frac{1}{1-e^{-\frac{k}{2}}} < \frac{1}{R^{\nu-n-1}} \frac{1}{\sinh\left(\frac{k}{4}\right)}. \tag{2.9}$$

Thus (2.9) holds for $\nu > n$ for $\xi \in I_\nu \cup I'_\nu$.

(c) When $\nu < n$ and $\xi \in I_\nu$ we have for $\nu \leqq n-2$

$$e^{\xi-k} \leqq e^{x_\nu^* - x_{n-1} - \frac{k}{2}} \leqq e^{x_{\nu+1} - x_{n-1} - \frac{k}{2}} \leqq e^{-\frac{k}{2} - k(n-\nu-2)} = \frac{e^{-\frac{k}{2}}}{R^{n-\nu-2}} \leqq e^{-\frac{k}{2}}$$

and for $\nu = n-1$, using (1.6),

$$e^{\xi-x} \leqq e^{x_{n-1}^* - x_{n-1} - \frac{k}{2}} \leqq e^{c\lambda_{n-1} - \frac{k}{2}} \leqq e^{-\frac{k}{4}}.$$

Thus, by (2.8)

$$\log\left|\frac{e^x + e^\xi}{e^x - e^\xi}\right| \leqq \begin{cases} \dfrac{1}{R^{n-\nu-2}} \dfrac{1}{\sinh\frac{k}{2}}, & \text{when } \nu \leqq n-2 \\[2ex] \dfrac{1}{\sinh\frac{k}{4}}, & \text{when } \nu = n-1 \end{cases}. \tag{2.10}$$

When $\nu - n$ and $\xi \in I'_\nu$ we have for $\nu \leqq n-2$

$$e^{\xi-x} \leqq e^{x_\nu'^* - x'_{n-1} - \frac{k}{4}} \leqq e^{x'_{\nu+1} - x'_{n-1} - \frac{k}{4}} \leqq \frac{e^{-\frac{k}{4}}}{R^{n-\nu-2}}$$

and for $\nu = n-1$, again using (1.6),

$$e^{\xi-x} \leqq e^{x_{n-1}'^* - x'_{n-1} - \frac{k}{4}} \leqq e^{x'_{n-1} + c\lambda_{n-1} - x'_{n-1} - \frac{k}{4}} \leqq e^{-\frac{k}{8}}.$$

Thus, again by (2.8)

$$\log\left|\frac{e^x+e^\xi}{e^x-e^\xi}\right| \leqq \begin{cases} \dfrac{1}{R^{n-\nu-2}}\dfrac{1}{\sinh\frac{k}{4}}, & \text{when } \nu \leqq n-2 \\ \dfrac{1}{\sinh\frac{k}{8}}, & \text{when } \nu = n-1. \end{cases} \tag{2.11}$$

We obtain then from (2.1), (2.4), (2.7), (2.9), (2.10) and (2.11)

$$\omega(x) \leqq \int_0^\pi \left|\frac{\partial U(x+iy)}{\partial y}\right| dy \leqq \frac{2}{\pi}\left[\lambda_n \log\frac{2e^k}{\delta} + \left(\sinh\frac{k}{8}\right)^{-1}\sigma_n\right] \equiv \mu_n$$

where

$$\sigma_n = R^{n+1}\sum_{\nu=n+1}^{\infty}\frac{\lambda_\nu}{R^\nu} + \frac{1}{R^{n-2}}\sum_{\nu=1}^{n-1}\lambda_\nu R^\nu = s_n' + s_n''.$$

It is easily seen that $\lim_{n\to\infty}\lambda_n = 0$ implies $\lim_{n\to\infty}\sigma_n = 0$. To prove [2.3) we write

$$\begin{aligned}\sum_{n=1}^{\infty}\lambda_n s_n' = R^2\lambda_1&\left[\frac{\lambda_2}{R^2}+\frac{\lambda_3}{R^3}+\frac{\lambda_4}{R^4}+\cdots\right]\\ +R^3\lambda_2&\left[\qquad\frac{\lambda_3}{R^3}+\frac{\lambda_4}{R^4}+\frac{\lambda_5}{R^5}+\cdots\right]\\ +R^4\lambda_3&\left[\qquad\qquad\frac{\lambda_4}{R^4}+\frac{\lambda_5}{R^5}+\frac{\lambda_6}{R^6}+\cdots\right]\\ +&\cdots\cdots\cdots\cdots\cdots\cdots\end{aligned}$$

and taking the sum on the right "by diagonals" we find

$$\sum_{n=1}^{\infty}\lambda_n s_n' = \sum_{\nu=1}^{\infty}\lambda_\nu\lambda_{\nu+1} + \frac{1}{R}\sum_{\nu=1}^{\infty}\lambda_\nu\lambda_{\nu+2} + \frac{1}{R^2}\sum_{\nu=1}^{\infty}\lambda_\nu\lambda_{\nu+3} + \cdots .$$

Now, if $\sum_{\nu=1}^{\infty}\lambda_\nu^2 = A$ then, by the Cauchy-Schwarz inequality,

$$\sum_{\nu=1}^{\infty}(\lambda_\nu\lambda_{\nu+k}) \leqq \left(\sum_{\nu=1}^{\infty}\lambda_\nu^2\sum_{\nu=1}^{\infty}\lambda_{\nu+k}^2\right)^{\frac{1}{2}} \leqq A$$

and therefore

$$\sum_{n=1}^{\infty}\lambda_n s_n' \leqq A\frac{1}{1-\frac{1}{R}} = \frac{AR}{R-1}. \tag{2.12}$$

Similarly

$$\sum_{n=2}^{\infty} \lambda_n s_n'' = \lambda_2[\lambda_1 R]$$

$$+\frac{\lambda_3}{R}[\lambda_1 R + \lambda_2 R^2]$$

$$+\frac{\lambda_4}{R^2}[\lambda_1 R + \lambda_2 R^2 + \lambda_3 R^3]$$

$$+\frac{\lambda_5}{R^3}[\lambda_1 R + \lambda_2 R^2 + \lambda_3 R^3 + \lambda_4 R^4]$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots .$$

Taking again the sum "by diagonals" we find

$$\begin{aligned}\sum_{n=2}^{\infty} \lambda_n s_n'' &= R\sum_{\nu=1}^{\infty}(\lambda_\nu \lambda_{\nu+1}) + \sum_{\nu=1}^{\infty}(\lambda_\nu\lambda_{\nu+2}) + \frac{1}{R}\sum_{\nu=1}^{\infty}(\lambda_\nu \lambda_{\lambda+3}) + \cdots \\ &\leqq A\left(R+1+\frac{1}{R}+\frac{1}{R^2}+\cdots\right) = \frac{AR^2}{R-1}.\end{aligned} \tag{2.13}$$

The estimate for $\sum(\sigma_n\lambda_n)$ follows now from (2.12) and (2.13).

3. Proof of Theorem 1. We choose $N \geqq N_0$ such that Lemmas 1 and 2 apply. Suppose for an $n > N$, $I_n \cap I_n' = \phi$, so that either $x_n^* < x_n'$ or $x_n'^* < x_n$. Assume the former; *then we assert: if* $x_n' - x_n^* > 2\lambda_n$ *then* $x_n' - x_n^* \leqq 4c\mu_n$ *where* $\mu_n = \frac{2}{\pi}\left\{\lambda_n \log\frac{2e^k}{\lambda_n} + \left(\sinh\frac{k}{8}\right)^{-1}\sigma_n\right\}$.

If $x_n' - x_n^* > 4c\mu_n$ we choose an $x \in [x_n^*, x_n']$ at the distance $\geqq 2c\mu_n$ from both endpoints. If follows from Lemma 1 that the point $W(x) \in \partial S$ has a distance $> \mu_n$ from w_n^*. For if this distance were $\leqq \mu_n$ then $W(x)$ would lie within a circle of radius $\left(\mu_n + \frac{1}{2}\lambda_n\right)$ about a_n and, therefore, by Lemma 1, we would have $|x - x_n^*| < c(\mu_n + \lambda_n) < 2c\mu_n$. (Note that for $n > N_0$, $\lambda_n < \frac{\pi}{16}$ and therefore $\frac{2e^k}{\lambda_n} > e^2$ so that $\mu_n > \lambda_n$.) Now, by Lemma 2, $\omega(x) \leqq \mu_n$, and therefore the image l_x of the segment $\{z \mid \Re z = x,\ 0 \leqq \Im m\ z \leqq \pi\}$ under the mapping $z \to W(z)$ must lie in the half-plane $\Re w > u_n$, since $\Re W(x) > u_n + \mu_n$. This contradicts the fact that $x_n^* < x < x_n'$ implies that l_x must cross the line $\Re w = u_n$ in S. Thus we must have $x_n' - x_n^* \leqq 4c\mu_n$. (An analogous result holds if $x_n'^* < x_n$.)

For each $n > N$ let J_n denote the smallest interval containing $I_n \cup I_n'$ (e.g. if $x_n^* < x_n'$, $J_n = [x_n, x_n'^*]$). We can choose N so large that the length of J_n is $\leqq \frac{k}{2}$ for all $n > N$. If x is a point exterior to all J_ν and between J_n and J_{n+1}, then l_x connects a point on $\{u_n \leqq u \leqq u_{n+1},\ v = v_n\}$ to a point on $\{u_n \leqq u \leqq u_{n+1},\ v = v_n'\}$ and therefore its length $l(x) \geqq \theta_n$. For $x \in J_n$, $l(x) \geqq \theta_n - 2\lambda_n$.

Let $x_N < x' < x''$ and x', x'' exterior to any J_ν. Let $\{J_m\}_{m=p}^{q}$ be all of these intervals contained in (x', x''). The set $[x', x''] \setminus \bigcup_{n=p}^{q} J_n$ consists of $q-p+1$ intervals J_n', $n = p-1, p, \cdots, q$, where J_{p-1}' precedes J_p, and J_n' follows J_n, $n = p, p+1, \cdots, q$.

By the arc length-area inequality (see [1], p. 13)

$$\int_{x'}^{x''} l^2(x)dx \leqq \pi \int_{x'}^{x''}\int_0^{\pi} |W'(x+iy)|^2 dy\ dx \leqq \pi^2(\underline{u}'' - \bar{u}') + \pi^2(\omega(x') + \omega(x'')) \tag{3.1}$$

where

$$\bar{u}' = \operatorname*{Max}_{0 \leqq y \leqq \pi} U(x'+iy), \quad \underline{u}'' = \operatorname*{Min}_{0 \leqq y \leqq \pi} U(x''+iy).$$

We write

$$\int_{x'}^{x''} l^2(x)dx = \sum_{n=p-1}^{q} \int_{J_n'} l^2(x)dx + \sum_{n=p}^{q} \int_{J_n} l^2(x)dx\,. \tag{3.2}$$

Now

$$\int_{J_n'} l^2(x)\,dx \geqq \int_{J_n'} \theta_n^2\,dx \geqq \int_{J_n'} (\pi + \theta_n - \pi)^2 dx \geqq [\pi^2 + 2\pi(\theta_n - \pi)] \int_{J_n'} dx \tag{3.3}$$

and

$$\left.\begin{aligned} \int_{J_n} l^2(x)\,dx &\geqq \int_{J_n} (\theta_n - 2\lambda_n)^2 dx \geqq (\theta_n^2 - 4\lambda_n\theta_n) \int_{J_n} dx \\ &\geqq \pi^2 \int_{J_n} dx + [2\pi(\theta_n - \pi) - 4\lambda_n\theta_n] \int_{J_n} dx \end{aligned}\right\} . \tag{3.4}$$

Thus from (3.2), (3.3), and (3.4), if $m(J_n)$ and $m(J_n')$ denote the lengths of J_n and J_n',

$$\left.\begin{aligned} \int_{x'}^{x''} l^2(x)\,dx \geqq \pi^2(x'' - x') &- 2\pi \sum_{\substack{n=p-1 \\ \theta_n \leqq \pi}}^{q} (\pi - \theta_n) m(J_n') \\ - 2\pi \sum_{\substack{n=p \\ \theta_n \leqq \pi}}^{q} (\pi - \theta_n) m(J_n) &- 4 \sum_{n=p}^{q} \theta_n \lambda_n m(J_n) \end{aligned}\right\} . \tag{3.5}$$

Since $J'_n \subset [x_n^*, x_{n+1}]$ we have $m(J'_n) \leqq x_{n+1} - x_n^*$ and since, for $n > N$, $m(J_n) < \frac{k}{2}$ and $x_{n+1} - x_n^* \geqq \frac{k}{2}$ we have also $m(J_n) \leqq x_{n+1} - x_n^*$. Hence the absolute value of the sum of the second and third terms on the right hand side of (3.5) doses not exceed

$$4\pi \sum_{\substack{n=p-1 \\ \theta_n \leqq \pi}}^{q} (\pi - \theta_n)(x_{n+1} - x_n^*).$$

By (1.1), since $u_{n+1} - u_n \geqq d > 0$, we have

$$x_{n+1} - x_n^* = (u_{n+1} - u_n)(1 + \varepsilon_n), \quad \lim_{n \to \infty} \varepsilon_n = 0$$

and therefore

$$\int_{x'}^{x''} l^2(x)\, dx \geqq \pi^2(x'' - x') - 4\pi \sum_{\substack{n=p-1 \\ \theta_n \leqq \pi}}^{\infty} (\pi - \theta_n)(u_{n+1} - u_n)(1 + \varepsilon_n) - 24c \sum_{n=p}^{\infty} \lambda_n \mu_n \theta_n$$

(recalling that $m(J_n) \leqq 6c\mu_n$). The last two series converge by hypotheses (a) and (b). Thus by (3.1) for $z' = x' + iy'$, $z'' = x'' + iy'' \in \Sigma$

$$U(z'') - x'' \geqq U(z') - x' - [\omega(x') + \omega(x'')] + \delta(x')$$

where $\delta(x') \to 0$ as $x' \to \infty$. Since we already know from (1.4) that $\lim_{x \to \infty} (W(z) - z) = \kappa < \infty$ exists and that $\lim_{x'' \to \infty} \omega(x'') = 0$, this shows that $\kappa > -\infty$ and Theorem 1 is proved.

4. **Criterion for angular derivative.** We come now to the application of Theorem 1. Suppose Ω is a simply connected domain which has a boundary point w_∞ at $w = \infty$, accessible along a ray L parallel to the real axis, say $L = \left\{u \geqq u_0,\ v = \frac{\pi}{2}\right\} \subset \Omega$. For $u \geqq u_0$ let θ_u denote the largest open segment on the line $\Re w = u$ which intersects L and is contained in Ω and $\theta(u)$ $(\leqq \infty)$ its length. We denote the endpoints of θ_u by $v(u)$ and $v'(u)$, $v(u) < v'(u)$. Let $\{u_n\}$ be a sequence with $u_{n+1} - u_n \geqq d > 0$, $u_1 > u_0$ and let

$$v_n = \operatorname*{Sup}_{u_n \leqq u \leqq u_{n+1}} v(u), \quad v'_n = \operatorname*{Inf}_{u_n \leqq u \leqq u_{n+1}} v'(u), \quad \theta_n = v'_n - v_n,$$

$$\lambda_n = \operatorname{Max}[|v_{n+1} - v_n|,\ |v'_{n+1} - v'_n|].$$

THEOREM 2. *Suppose there exists a sequence* $\{u_n\}$ *such that*

(a) $\lim_{n \to \infty} v_n = 0, \quad \lim_{n \to \infty} v'_n = \pi,$

(b) $$\sum_{\theta_n \leqq \pi} (\pi - \theta_n)(u_{n+1} - u_n) < \infty,$$

(c) $$\sum_{n=1}^{\infty} \lambda_n^2 \log \frac{1}{\lambda_n} < \infty,$$

and suppose that for all $u > u_0$

(d) $$\int_{u_0}^{u} (\theta(t) - \pi)\, dt \leqq M.$$

If $W(z) = U(z) + iV(z)$ *maps the strip* $\sum$ *conformally onto* Ω *such that* $\lim_{x \to +\infty} W\left(x + i\frac{\pi}{2}\right) = w_\infty$ *then* $W(z)$ *has an angular derivative at* $z = +\infty$.

Remark. If S is the domain (3) constructed with the data $\{u_n\}$, v_n, v_n' of Theorem 2, then the part of S in $u \geqq u_1$ is contained in Ω. For our purposes it is no restriction of generality to assume that the *whole* domain $S \subset \Omega$. We note that S is not required to be contained in a parallel strip of width π (a restriction frequently imposed on an "interior comparison domain"). Under that restriction hypotheses (b) and (d) alone form a sufficient condition for (1), and this is essentially the criterion given by Ahlfors [1], p. 36, the first important criterion in the literature. In this case (b) implies that $\sum_{n=1}^{\infty} \lambda_n < \infty$. In our theorem that restriction has been replaced by the considerably weaker assumptions (a) and (c). It is difficult to compare directly our theorem with some other criteria which use a different geometrical characterization of $\partial\Omega$ (e.g. théorème VI, 16a in [4] p. 208 and those derived from it pp. 209–211), but such comparisons may be made in special cases to which both apply, such as the example described in our introduction.

Proof of Theorem 2. Condition (a) implies that for every η, $0 < \eta < \frac{\pi}{2}$, there exists an $R_\eta \geqq u_0$ such that the half-strip $S_\eta = \{w = u + iv : u \geqq R, \eta \leqq v \leqq \pi - \eta\} \subset \Omega$. Let $E_+ = \{u_0 \leqq u < \infty : \theta(u) - \pi > 0\}$ and $E_- = \{u_0 \leqq u < \infty : \theta(u) - \pi \leqq 0\}$. Then it follows from (b) that

$$\int_{E_-} (\theta(u) - \pi) du \quad \text{converges,}$$

and therefore from (d) that

$$\int_{E_+} (\theta(u) - \pi) \leqq M - \int_{E_-} (\theta(u) - \pi)\, du < \infty.$$

Thus Ω satisfies the hypotheses of Theorem 5 in [6], and if $Z(w) = X(w) + iY(w)$ is the inverse function of $W(z)$, we have for $w \in S_\eta$ for any η, $0 < \eta < \frac{\pi}{2}$,

$$\lim_{u \to \infty} [Z(w) - w] = \Lambda \quad \text{exists and} \quad -\infty < \Lambda \leqq +\infty.$$

As indicated above we may assume that $S \subset \Omega$, where S is the domain (3) constructed with the data of the theorem. If $Z_1(w)$ maps S conformally onto $\sum$ such that $\lim_{u \to +\infty} \Re Z_1\left(u + i\,\frac{\pi}{2}\right) = +\infty$, we know by Theorem 1, that for $w \in S$,

$$\lim_{u \to +\infty} [Z_1(w) - w] = \Lambda_1 \quad \text{exists and is finite.}$$

If $Z_1(w)$ is so normalized that, for some $w_0 \in S$, $Z_1(w_0) = Z(w_0)$, then $S \subset \Omega$ implies $\Lambda \leqq \Lambda_1 < +\infty$. This completes the proof.

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