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Nagoya Math. J.
Vol. 38 (1970), 41–52

STOCHASTIC INTEGRALS BASED ON MARTINGALES TAKING VALUES IN HILBERT SPACE

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To Professor Katuzi Ono on the occasion of his 60th birthday

Let H be a separable Hilbert space with inner product $(\ ,\ )$ and norm $\|\ \ \|$. We denote by K the set of all linear operators on H. Let $(\Omega, \mathfrak{F}, P)$ be a probability space and suppose we are given a family of σ-fields $\mathfrak{F}_t$, $t \geq 0$ such that $\mathfrak{F}_s \subseteqq \mathfrak{F}_t \subseteqq \mathfrak{F}$ for $0 \leq s \leq t$ and $\bigcap_{\varepsilon>0} \mathfrak{F}_{t+\varepsilon} = \mathfrak{F}_t$. We assume further that each $\mathfrak{F}_t$ is complete relative to the probability measure P. A mapping $X_t(\omega)$; $[0, \infty) \times \Omega \to H$ is called an *H-valued stochastic process* or shortly *H-process* if (f, X_t) is a scalar valued (real or complex) stochastic process for all $f \in H$. In particular, if (f, X_t) is a martingale for every $f \in H$, X_t is called an *H-martingale*.

The purpose of this article is to define two types of stochastic integrals by H-martingale $\int_0^t (\Phi_1(s,\omega), dX_s(\omega))$ and $\int_0^t \Phi_2(s,\omega)\, dX_s(\omega)$ and to establish a formula concerning these stochastic integrals. Here $\Phi_i(s,\omega)$, $i = 1, 2$ is H- or K-process, respectively, with suitable additional conditions. Similar problem concerning Hilbert space valued Brownian motion has been discussed by Daletskii [1].

1. Preliminaries. Let X be an H-random variable. Then $\|X(\omega)\|$ is clearly an $\mathfrak{F}$-measurable real random variable. We suppose $E\|X\| < \infty$. For a given sub σ-field $\mathfrak{G}$ of $\mathfrak{F}$, we define the *conditional expectation* of X relative to $\mathfrak{G}$, denoted by $E(X|\mathfrak{G})$, in the following manner; $E(X|\mathfrak{G})$ is an H-random variable such that $(f, E(X|\mathfrak{G}))$ is $\mathfrak{G}$-measurable and $(f, E(X|\mathfrak{G})) = E((f, X)|\mathfrak{G})$ holds for every $f \in H$. Such $E(X|\mathfrak{G})$ is unique up to measure 0. Then an H-process X_t such that $E\|X_t\| < \infty$, $\forall t \geq 0$, is an H-martingale if and only if $E(X_t|\mathfrak{F}_s) = X_s$ holds for every $t \geq s$.

Received March 31, 1969

PROPOSITION 1. *Let $\mathfrak{G}$ be a sub σ-field of $\mathfrak{F}$. Let X and Y be H-random variables such that $E\|X\|^2<\infty$ and $E\|Y\|^2<\infty$. Then if X is $\mathfrak{G}$-measurable, we have*

$$E((X,Y)|\mathfrak{G})=(X,E(Y|\mathfrak{G})) \quad or \quad E((Y,X)|\mathfrak{G})=(E(Y|\mathfrak{G}),X).$$

Proof. It is enough to prove the proposition in the case where X is a step function, i.e., there exists a $\mathfrak{G}$-measurable partition $\{B_n\}$ of Ω such that $X(\omega)=a_n$ for $\omega\in B_n$, where each a_n is a fixed element of H. Since $(X,Y)=\sum(a_n,Y)I_{B_n}$ (I_B is the indicator function of the set B) and B_n belongs to $\mathfrak{G}$, we have

$$E((X,Y)|\mathfrak{G})=\sum_n E((a_n,Y)|\mathfrak{G})I_{B_2}=\sum_n(a_n,E(Y|\mathfrak{G}))I_{B_n}=(X,E(Y|\mathfrak{G})).$$

The proof of the second equality is quite similar to the above.

The following proposition is easily verified.

PROPOSITION 2. *Let X_t be an H-martingale such that $E\|X_t\|<\infty$ for every $0\leq t<\infty$. Then X_t has a weakly right continuous modification, i.e., there exists an H-martingale X_t^* such that $P(X_t=X_t^*)=1$ for every t and (f,X_t^*) is a right continuous scalar martingale for every $f\in H$.*

From now we shall only consider weakly right continuous H-martingales. We denote by $\mathfrak{M}$ the set of all H-martingales such that $E\|X_t\|^2<\infty$ for $0<t<\infty$ and $X_0=0$ a.e. P. Then for every $X_t\in\mathfrak{M}$, $\|X_t\|^2$ becomes a real submartingale. In fact, using Proposition 1,

$$\begin{aligned}(1)\qquad E(\|X_t-X_s\|^2|\mathfrak{F}_s)&=E(\|X_t\|^2|\mathfrak{F}_s)+\|X_s\|^2-E((X_t,X_s)|\mathfrak{F}_s)-E((X_s,X_t)|\mathfrak{F}_s)\\&=E(\|X_t\|^2|\mathfrak{F}_s)-\|X_s\|^2\geq 0\end{aligned}$$

if $t\geq s$. Let us now introduce the metric ρ to $\mathfrak{M}$ in the following way.

$$\rho(X,Y)=\sum_n\frac{1}{2}\,\frac{E\|X_n-Y_n\|^2}{1+E\|X_n-Y_n\|^2}\,.$$

Then $\mathfrak{M}$ is a complete metric space (c.f. [2]).

PROPOSITION 3. *Let $X\in\mathfrak{M}$. Then, for almost all ω, $X_t(\omega)$ is strongly right continuous and has strong left limits with respect to t. Furthermore, if $X_t(\omega)$ is weakly continuous, it is strongly continuous for almost all ω.*

Proof. Let L be a finite dimensional subspace of H. Then the assertion of the proposition is obvious for L-martingales. Since such L-martingales are dense in $\mathfrak{M}$, it is enough to verify that the limit of a sequence of strongly (right) continuous H-martingales is again strongly (right) continuous. Let $\{X_t^n\}$ be a sequence of $\mathfrak{M}$ converging to X_t. Then, for each $\lambda > 0$ and N,

$$P(\sup_{t\leqq N} \|X_t - X_t^n\| > \lambda) \leqq \frac{1}{\lambda^2} E\|X_N - X_N^n\|^2 \to 0 \quad \text{as} \quad n \to \infty$$

by Doob's inequality. Hence there exists a subsequence $\{X_t^{n_k}\}$ such that

$$P(\sup_{t\leqq N} \|X_t - X_t^{n_k}\| \to 0 \quad \text{as} \quad k \to \infty \text{ for every } N > 0) = 1,$$

by Borel-Cantelli's lemma. It is now obvious that X_t is strongly (right) continuous if so is each $\{X_t^n\}$. The existence of strong left limits is obvious from the above discussion.

2. **Stochastic integral I.** We have shown in the preceding section that $\|X_t\|^2$ is a positive sub-martingale for any $X \in \mathfrak{M}$. Hence, by Meyer's decomposition, there exists a unique natural increasing process $\langle X\rangle_t$ such that $\|X_t\|^2 - \langle X\rangle_t$ is a real martingale. This $\langle X\rangle_t$ plays an important role in the future. For $X, Y \in \mathfrak{M}$, set

$$\langle X, Y\rangle_t = \frac{1}{4}\{\langle X+Y\rangle_t - \langle X-Y\rangle_t\}.$$

in case of real Hilbert space. In case of complex Hilbert space, the definition of $\langle X, Y\rangle_t$ should be modified in an obvious way. Then we have, making use of equality (1),

$$E((X_t - X_s, Y_t - Y_s)|\mathfrak{F}_s) = E(\langle X, Y\rangle_t - \langle X, Y\rangle_s|\mathfrak{F}_s), \qquad t \geqq s. \tag{2}$$

We denote by $L(\langle X\rangle)$ the set of all very well measurable scalar processes ([5]) $\Phi(s,\omega)$ such that $E\left(\int_0^t |\Phi(s,\omega)|^2 d\langle X\rangle_s(\omega)\right) < \infty$ for every $0 \leqq t < \infty$. Then similarly as the one dimensional case, we have

$$\left| E\left(\int_0^t \Phi\Psi d\langle X, Y\rangle\right)\right| \leqq E\left(\int_0^t |\Phi|^2 d\langle X\rangle\right)^{\frac{1}{2}} E\left(\int_0^t |\Psi|^2 d\langle Y\rangle\right)^{\frac{1}{2}}, \tag{3}$$

where $\Phi \in L(\langle X\rangle)$ and $\Psi \in L(\langle Y\rangle)$.

THEOREM 1. *For each* $X \in \mathfrak{M}$ *and* $\Phi \in L(\langle X\rangle)$, *there exists a unique* $Y \in \mathfrak{M}$ *such that*

$$\langle Y, Z\rangle_t(\omega) = \int_0^t \Phi(s,\omega)d\langle X, Z\rangle_s(\omega) \quad \text{for every } Z \in \mathfrak{M}.$$

Further, this Y *satisfies*

$$(f, Y_t) = \int_0^t \bar{\Phi}\, d(f, X_t) \qquad \forall f \in H,$$

where the right hand of the above is the stochastic integral of the scalar martingale (f, X_t), *and* $\bar{\Phi}$ *is the complex conjugate of* Φ.

The proof can be carried out similarly as that of real martingale, making use of inequalities (2) and (3). (See [2]). We shall call the above Y as the stochastic integral of Φ relative to X and denote it as $\int_0^t \Phi dX$.

By virtue of Theorem 1, we can define orthogonality of H-martingales, projection etc. quite similarly as the case of real martingales. Two X and Y are *orthogonal* if $\langle X, Y\rangle \equiv 0$ or equivalently (X_t, Y_t) is a scalar martingale. A subset $\mathfrak{N}$ of $\mathfrak{M}$ is a *stable subspace* if it is a closed subspace of $\mathfrak{M}$ and $\int \Phi dX \in \mathfrak{N}$ whenever $X \in \mathfrak{N}$ and $\Phi \in L(\langle X\rangle)$. Let $\mathfrak{N}$ be a stable subspace. We denote by $\mathfrak{N}^\perp$ the set of all $Y \in \mathfrak{M}$ which is orthogonal to every element of $\mathfrak{N}$. Then each $X \in \mathfrak{M}$ has a unique decomposition $X = X^1 + X^2$, where $X^1 \in \mathfrak{N}$ and $X^2 \in \mathfrak{N}^\perp$. Let $\mathfrak{M}_c$ be the set of all $X_t(\omega) \in \mathfrak{M}$ which is strongly continuous with respect to t for almost all ω. Then $\mathfrak{M}_c$ is a stable subspace of $\mathfrak{M}$. We denote $\mathfrak{M}_c^\perp$ as $\mathfrak{M}_d$.

THEOREM 2. *The set of all* $X \in \mathfrak{M}_d$ *decomposed to the difference of the following two* Y *and* $\tilde{Y}$, *is dense in* $\mathfrak{M}_d$: Y_t *is an* $\mathfrak{F}$*-measurable* H*-process changing the values by jumps only;* $\tilde{Y}_t$ *is a strongly continuous* H*-process such that*

$$\sup \sum_{t_n \leqq t} \|\tilde{Y}_{t_n} - \tilde{Y}_{t_{n-1}}\| < \infty$$

where $0 = t_0 < t_1 < t_2 < \cdots$ *and* sup *is taken for all such* $\{t_n\}$.

Furthermore, we have for every $X \in \mathfrak{M}$,

$$\sum_{t_n \leqq t} \|X_{t_n} - X_{t_{n-1}}\|^2 \to \sum_{\substack{\|\Delta x_s\| > 0 \\ s \leqq t}} \|\Delta X_s\|^2 + \langle X^c\rangle_t \quad \text{in } L^1\text{-sense}$$

as $\limsup_n |t_n - t_{n-1}| = 0$, *where* X^c *is the projection of* X *on* $\mathfrak{M}_c$ and $\Delta X_s = X_s - X_s^-$ $(X_s^- = \lim_{\varepsilon \downarrow 0} X_{s-\varepsilon})$.

Proof. Let L be a finite dimensional subspace of H. We denote by $\mathfrak{M}(L)$ etc. the subset of $\mathfrak{M}$ etc. consisting of L-martingales. Then all the assertions of the theorem is immediate from that of real martingales if $\mathfrak{M}$, $\mathfrak{M}_d$ etc. are replaced by $\mathfrak{M}(L)$, $\mathfrak{M}_d(L)$ etc. (See [2] or [5]). Since $\bigcup_L \mathfrak{M}_d(L)$ is dense in $\mathfrak{M}$, the first assertion is obvious. Now let Y be the projection of $X \in \mathfrak{M}$ to $\mathfrak{M}(L)$. Then making use of orthogonal expansion and Bessel's inequality, it is easily seen that $\|Y_t - Y_s\|^2$ increases to $\|X_t - X_s\|^2$ as L increases to H. Similar fact holds for $\langle Y\rangle_t$ and $\langle X\rangle_t$. On the other hand since $\sum_{t_n^{(k)}\leqq t}\|Y_{t_n^{(k)}} - Y_{t_{n-1}^{(k)}}\|^2$ converges to $\sum_{s\leqq t}\|\Delta Y_s\|^2 + \langle Y^c\rangle_t$ as $\lim_k \sup_n |t_n^{(k)} - t_{n-1}^{(k)}| = 0$, $\lim_k \sum_{t_n^{(k)}} \|X_{t_n^{(k)}} - X_{t_{n-1}^{(k)}}\|^2 \geqq \sum_{s\leqq t}\|\Delta X_s\|^2 + \langle X^c\rangle_t$. To obtain the converse inequality, choose L large enough so that $E\|X_N - Y_N\|^2 < \varepsilon$ for given $\varepsilon > 0$ and $N > 0$. Then for $t \leq N$,

$$
\begin{aligned}
E[\lim \sum_{t_n^{(k)}\leqq t}\|X_{t_n^{(k)}} - X_{t_{n-1}^{(k)}}\|^2] &\leqq E\|Y_t\|^2 + E\|X_t - Y_t\|^2 \\
&\leqq \varepsilon + E(\sum_{s\leqq t}\|\Delta Y_s\|^2 + \langle Y^c\rangle_t) \\
&\leqq \varepsilon + E(\sum_{s\leqq t}\|\Delta X_s\|^2 + \langle X^c\rangle_t).
\end{aligned}
$$

Therefore we have the desired equality.

Remark. Let $X \in \mathfrak{M}$. Then for $\varepsilon > 0$ there sxists an increasing sequence of stopping times $\{T_n^\varepsilon\}$ converging to ∞ a.e. such that $\|X_t - X_s\| < \varepsilon$ holds for all $T_n^\varepsilon \leqq s, t < T_{n+1}^\varepsilon$. Such $\{T_n^\varepsilon\}$ is called an ε-chain for X. The assertion of Theorem 2 holds if we replace $t_n^{(k)}$ by $T_n^{\varepsilon_k}$, where $\{\varepsilon_k\}$ is a sequence converging to 0.

3. Stochastic integral II. An H-process $\Phi(t,\omega)$ is called *very well measurable* if $(f, \Phi(t,\omega))$ is very well measurable for every $f \in H$ ([5]). For a fixed $X \in \mathfrak{M}$, set

$L_H(\langle X\rangle) = \Big\{\Phi \,|\, \Phi$ is very well measurable H-process such that $E\Big(\int_0^t \|\Phi\|^2 d\langle X\rangle\Big) < \infty$ for $0 \leqq t < \infty\Big\}$. Our purpose of this section is to define the stochastic integral $\int_0^t (\Phi, dX)$ for $X \in \mathfrak{M}$ and $\Phi \in L_H(\langle X\rangle)$.

We first consider the case where Φ is a step function, i.e., there exist $0 = t_0 < t_1 < \cdots < t_n < \cdots$ such that $\Phi(t,\omega) = \Phi(t_k,\omega)$ for $t_k \leqq t < t_{k+1}$. Define the scalar process Y_t as

$$Y_t = \sum_{t_{k+1}\leqq t} (\Phi(t_k),\ X_{t_{k+1}} - X_{t_k}) + (\Phi(t_l),\ X_t - X_l),$$

where l is the natural number such that $t_l < t \leqq t_{l+1}$. Applying Proposition 1, it is easily seen that Y_t is a scalar martingale. Moreover,

$$E|Y_t|^2 \leqq \sum_{t_{k+1}\leqq t} E\|\Phi(t_k)\|^2\|X_{t_{k+1}} - X_{t_k}\|^2 + E\|\Phi(t_l)\|^2\|X_t - X_{t_l}\|^2$$
$$\leqq E\left(\int_0^t \|\Phi\|^2 d\langle X\rangle\right).$$

We shall denote by $L_H^0(\langle X\rangle)$ the closure of the step functions in $L_H(\langle X\rangle)$. (The metric of $L_H(\langle X\rangle)$ is defined similarly as that of $\mathfrak{M}$).

Choose a sequence of step functions $\{\Phi^n\}$ of $L_H(\langle X\rangle)$ converging to $\Phi \in L_H^0(\langle X\rangle)$. Set $Y_t^n = \int_0^t (\Phi^n, dX)$. Then

$$E|Y_t^n - Y_t^m|^2 \leqq E\left(\int_0^t \|\Phi^n - \Phi^m\|^2\, d\langle X\rangle\right) \to 0 \quad \text{as} \quad n, m \to \infty.$$

Hence there exists a square integrable martingale Y_t such that $E|Y_t - Y_t^n|^2 \to 0$. It does not depend on the choice of $\{\Phi^n\}$. We shall write this Y as $\int_0^t (\Phi, dX)$.

In order to see that the stochastic integral can be defined for all $\Phi \in L_H(\langle X\rangle)$, it is necessary to verify $L_H^0(\langle X\rangle) = L_H(\langle X\rangle)$. Let $\{f^i\}$ be a complete orthonormal system of H. For an arbitrary Φ of $L_H(\langle X\rangle)$, set $\Phi^i = (\Phi, f^i)$. It is known that for each Φ^i there exists a sequence of step functions $\{\Phi_n^i\}_{n=1,2,\dots}$ such that $E\left(\int_0^t |\Phi_n^i - \Phi^i|^2 d\langle X\rangle\right) \to 0$ as $n \to \infty$ ([5]). Since $\|\Phi\|^2 = \sum_{i=1}^{\infty} |\Phi^i|^2$, Φ can be approximated by step functions of $L_H(\langle X\rangle)$.

Remark. Set $X_t^i = (f^i, X_t)$ and denote the scalar stochastic integral as $\int_0^t \Phi^i dX^i$. Then $\sum_{i=1}^{n} \int_0^t \Phi^i dX^i$ converges to $\int_0^t (\Phi, dX)$ in L^2-norm. In fact, if $\Phi = \sum_{i=1}^{n} (\Phi, f^i)$ (finite sum) is a step function, then

$$\int_0^t (\Phi, dX) = \sum_{t_{k+1}\leqq t} (\Phi(t_k),\ X_{t_{k+1}} - X_{t_k}) + (\Phi(t_l),\ X_t - X_l)$$
$$= \sum_{i=1}^{n} \sum_{k} \Phi^i(t_k)(X_{t_{k+1}}^i - X_{t_k}^i) + \sum_{i=1}^{n} \Phi^i(t_l)(X_t^i - X_l^i)$$
$$= \sum_{i=1}^{n} \int_0^t \Phi^i dX^i.$$

The above holds obviously for arbitrary Φ^i. The convergence of $\sum_{i=1}^{n}\int_0^t \Phi^i dX^i$ to $\int_0^t (\Phi, dX)$ for arbitrary $\Phi \in L_H(\langle X\rangle)$ is now obvious.

4. **Stochastic integral III.** Let $\Phi(t,\omega)$ be a mapping from $[0,\infty)\times\Omega$ to K such that $\Phi(t,\omega)f$ is very well measurable for all $f\in H$. We denote by $\|\Phi\|(t,\omega)$ the norm of the operator $\Phi(t,\omega)$. It is easily seen that $\|\Phi\|(t,\omega)$ is a real very well measurable process. We shall call Φ is a step function if there exists $0=t_0<t_1<\cdots<t_n<$ such that $\Phi(t)=\Phi(t_k)$ for $t_k\leqq t<t_{k+1}$. Set

$$L_K^0(\langle X\rangle)=\left\{\Phi \mid \Phi \text{ is a very well measurable step function such that } E\left(\int_0^t \|\Phi\|^2\, d\langle X\rangle\right)<\infty\right\}$$

and denote by $L_K(\langle X\rangle)$ the closure of $L_K^0(\langle X\rangle)$. We shall define the stochastic integral $Y_t=\int_0^t \Phi dX$, $X\in\mathfrak{M}$, $\Phi\in L_K(\langle X\rangle)$ as an element of $\mathfrak{M}$ in the following way;

$$(f, Y_t)=\int_0^t (\Phi^* f, dX) \quad \text{for} \quad f\in H, \tag{4}$$

where $\Phi^*(t,\omega)$ is the adjoint of $\Phi(t,\omega)$ for each t,ω. The stochastic integral defined in Section 2 is a particular case of this. To verify the existence of such Y, let us first consider the case where $\Phi(t,\omega)$ is a step function. Then

$$Y_t=\sum_{t_{k+1}\leqq t}\Phi(t_k)(X_{t_{k+1}}-X_{t_k})+\Phi(t_l)(X_t-X_{t_l})$$

satisfies (4). Furthermore,

$$E\|Y_t\|^2\leqq E\left(\int_0^t\|\Phi\|^2\, d\langle X\rangle\right).$$

Consequently, the stochastic integral $\int\Phi dX$ can be defined for all $\Phi\in L_K(\langle X\rangle)$ as the limit of $\int\Phi^n dX$, where $\{\Phi^n\}$ is a sequence of step functions such that $E\left(\int_0^t\|\Phi-\Phi^n\|^2\, d\langle X\rangle\right)\to 0$ for $0\leqq t<\infty$.

Remark. The characterization of the space $L_K(\langle X\rangle)$ in an explicit form remains open. We shall give here two sufficient conditions that Φ belongs to $L_K(\langle X\rangle)$: (a) $\Phi(t,\omega)$ is left continuous in t with respect to the operator norm for almost all ω, and $\|\Phi\|(t,\omega)$ is a bounded function; (b) $\Phi(t,\omega)$ is very well measurable and satisfies $E\left(\int_0^t\|\Phi\|_2^2 d\langle X\rangle\right)<\infty$, where $\|\Phi\|_2$ is the Hilbert-Schmidt norm of Φ. The first assertion is obvious. Suppose that Φ satisfies the condition (b). Let $\{f^i\}$ be a complete orthonormal system of H. Set $\Phi^i=\Phi f^i$. Since $\Phi^i\in L_H(\langle X\rangle)$, it can be approximated by step functions of $L_H(\langle X\rangle)$. Define linear operator $\Phi^{(n)}$ by $\Phi^{(n)}f=\sum_{i=1}^n a_i\Phi f^i$, where $a_i=(f,f^i)$. Then $\Phi^{(n)}$ can be approximated by step functions of $L_K(\langle X\rangle)$ from the fact just remarked above. Furthermore, since

$$\|\Phi^{(n)}f-\Phi^{(m)}f\|^2=\sum_{i=m+1}^n a_i^2\|\Phi f^i\|^2\leqq\Big(\sum_{i=m+1}^n\|\Phi f^i\|^2\Big)\|f\|^2,$$

$\|\Phi^{(n)}-\Phi^{(m)}\|\leqq\sum_{i=m+1}^n\|\Phi f^i\|^2$. This inequality shows that $\{\Phi^n\}$ forms a cauchy sequence of $L_K(\langle X\rangle)$. The limit of $\{\Phi^{(n)}\}$ is clearly Φ. Therefore Φ belongs to $L_K(\langle X\rangle)$.

5. Formula on stochastic integral. A mapping F from the Hilbert space to the space of real or complex numbers is called twice differentiable at $x\in H$ if there exists a linear functional $F'(x)$ and linear operator $F''(x)$ of H such that

$$F(x+h)-F(x)=(F'(x),\bar h)+\frac{1}{2}(F''(x)h,\bar h)+o(\|h\|^2),$$

where $o(\|h\|^2)$ means $o(\|h\|^2)/\|h\|^2\to 0$ as $\|h\|\to 0$. Further if $F'(x)$ and $F''(x)$ are continuous in their norms, F is called twice continuously differentiable.

Now let $\varphi_t(\omega)$ be an strongly right continuous H-process such that $\sup_{\{t_k\}}\sum_{t_{k+1}\leqq t}\|\varphi_{t_{k+1}}-\varphi_{t_k}\|<\infty$ for any t, then the Bochner integral $\int_0^t(\Phi,d\varphi)$ is well defined for H-process $\Phi(t,\omega)$, for almost all ω. We shall call such φ is an H-process with finite variation.

THEOREM 3. *Let F be twice continuously differentiable function such that $\|F'(x)\|$ and $\|F''(x)\|$ are bounded. Let X be of $\mathfrak{M}$ and φ be well measurable strongly continuous process with finite variation. Set $A=X+\varphi$. Then we have the following formula*

$$F(A_t)-F(A_0)=\int_0^t(F'(A_s^-),\ d\bar X_s)+\frac{1}{2}\Big\langle\int F''(A^-)dX^c,\ \bar X^c\Big\rangle_t$$

$$+\int_0^t (F'(A_s^-),\ d\bar{\varphi}_s) + \sum_{\substack{\|\Delta x_s\|>0 \\ s\leq t}} [F(A_s) - F(A_s^-) - (F'(A_s^-),\ \bar{X}_s - \bar{X}_s^-)],$$

where $\dot{X}^c$ is the projection of X on $\mathfrak{M}_c$.

Proof. Since the proof is essentially the same as that of one dimensinal case (See [2] or [5]), we shall state here the outline. Let $X = X^c + X^d$ be the orthogonal decomposition such that $X^c \in \mathfrak{M}_c$ and $X^d \in \mathfrak{M}_d$. We shall assume that X^d is written as $Y - \tilde{Y}$ where Y and $\tilde{Y}$ are the processes having the properties of Theorem 2. Let $\{T_n\}$ be an ε-chain of X, Y, $\tilde{Y}$ and φ, i.e., $\{T_n\}$ is an increasing sequence of stopping times converging to ∞ such that for $T_n \leqq t, s < T_{n+1}$, $\|X_t - X_s\|$, $\|Y_t - Y_s\|$, $\|\tilde{Y}_t - \tilde{Y}_s\|$ and $\|\varphi_t - \varphi_s\|$ are all dominated by ε. We shall write $T_n \wedge t$ as T_n for the notational convention. Then

$$F(A_t) - F(A_0) = \sum [F(A_{T_n}^-) - F(A_{T_{n-1}})] + \sum [F(A_{T_n}) - F(A_{T_n}^-)].$$

The first term of the right hand side is written as

$$\sum(F'(A_{T_{n-1}}),\ \bar{A}_{T_n}^- - \bar{A}_{T_{n-1}}) + \frac{1}{2} \sum(F''(A_{T_{n-1}})(A_{T_n}^- - A_{T_{n-1}}),\ \bar{A}_{T_n}^- - \bar{A}_{T_{n-1}})$$
$$+ \sum o(\|A_{T_n} - A_{T_{n-1}}\|^2) = I_1 + I_2 + I_3.$$

Each I_i converges as $\varepsilon \to 0$ in the following way.

$$I_1 = \sum (F'(A_{T_{n-1}}),\ \bar{X}_{T_n} - \bar{X}_{T_{n-1}}) + \sum (F'(A_{T_{n-1}}),\ \bar{\varphi}_{T_n} - \bar{\varphi}_{T_{n-1}}) - \sum (F'(A_{T_{n-1}}), \Delta\bar{X}_{T_n})$$
$$\to \int (F'(A_s^-),\ d\bar{X}_s) + \int_0^t (F'(A_s^-),\ d\bar{\varphi}_s) - \sum_{s\leqq t} (F'(A_s^-),\ \Delta\bar{X}_s).$$

$$I_2 = \frac{1}{2}\sum(F'(A_{T_{n-1}})(X_{T_n}^c - X_{T_{n-1}}^c),\ \bar{X}_{T_n}^c - \bar{X}_{T_{n-1}}^c) + \frac{1}{2}\sum(F''(A_{T_{n-1}})(X_{T_n}^c - X_{T_{n-1}}^c),\ \bar{\psi}_{T_n} - \bar{\psi}_{T_{n-1}})$$
$$+ \frac{1}{2}\sum(F''(A_{T_{n-1}})(\psi_{T_n} - \psi_{T_{n-1}}),\ \bar{X}_{T_n}^c - \bar{X}_{T_{n-1}}^c) + \frac{1}{2}\sum(F''(A_{T_{n-1}})(\psi_{T_n} - \psi_{T_{n-1}}),\ \bar{\psi}_{T_n} - \bar{\psi}_{T_{n-1}}),$$

where $\psi = \varphi - \tilde{Y}$. The first term converges to $\frac{1}{2}\left\langle \int F''(A^-)\, dX^c,\ \bar{X}^c \right\rangle_t$ by virtue of the definition of $\int_0^t F''(A_s^-) dX_s$ and Theorem 2. The other members converge to 0, because of the following estimate; for example,

$$\left| \frac{1}{2} \sum(F''(A_{T_{n-1}})(X_{T_n}^c - X_{T_{n-1}}^c),\ \psi_{T_n} - \psi_{T_{n-1}}) \right| \leqq \frac{1}{2} \varepsilon \sup \|F''(x)\| \sum \|\psi_{T_n} - \psi_{T_{n-1}}\|.$$

It is easily seen that I_3 converges to 0. Summing up all these, we obtain the desired formula.

To prove the general case, choose $\{X^n\}$ converging to X such that for each X^n the above argument is applicable. It will be shown that each mumber of (5) replacing X for X^n converges to the corresponding member of (5).

6. Examples. Additive and linear processes. Let H be a real Hilbert space. An H-process X_t is called additive if $X_{t_4}-X_{t_3}$ and $X_{t_2}-X_{t_1}$ are independent for any $0\leqq t_1<t_2\leqq t_3<t_4$. We assume that $E\|X_t\|^2<\infty$ for all $0\leqq t<\infty$ and X_t is mean continuous i.e., $E\|X_t-X_s\|^2\to 0$ as $t\to s$. The least σ-field in which X_s, $s\leqq t$ are measurable is denoted by $\mathfrak{F}_t$. Then $\mathfrak{F}_t$ is right continuous. Now, since $E(f,X_t)\leqq\|f\|E\|X_t\|$, there exists a unique m_t of H such that $E(f,X_t)=(f,m_t)$ holds for any $f\in H$. Set $Y_t=X_t-m_t$. Then Y is again an additive process such that $E(f,Y_t)=0$ for every $f\in H$. Furthermore, Y_t is an H-martingale because (f,Y_t) is an real martingale. Therefore Y_t has strongly right continuous modification. So we will assume that Y_t is strongly right continuous. Let X_t^c be the projection of Y_t to $\mathfrak{M}_c$ and X_t^d the projection of Y_t to $\mathfrak{M}_d$. Remembering the procedure of defining X_t^c and X_t^d, it is seen that they are additive processes.

Let us now consider the covariance functional of X_t, $E(f,X_t)(g,X_t)$. Then it is a positive definite, symmetric and continuous bilinear form on H. Hence there exists a unique positive definite and symmetric operator S_t such that (S_tf,g) coincides with the above bilinear form. Moreover, S_t has finite trace, because $\sum(S_tf^i,f^i)=\sum E(f^i,X_t)(f^i,X_t)\leqq E\|X_t\|^2$, where $\{f^i\}$ is the complete orthonormal system of H.

Let E be a measurable subset of H such that $0<\rho(0,E)<\infty$, where ρ is the metric induced by the norm of H. Define.

$$P_t(E)=\sum_{\substack{s\leqq t\\ \|\Delta X_s\|>0}} I_E(\Delta X_s)$$

Then we have the following

THEOREM 4. *X_t^c and X_t^d are independent. Furthermore, the characteristic functionals are given by*

$$E(\exp(i(f,X_t^c)))=\exp-\frac{1}{2}(S_tf,f)$$

$$E(\exp(i(f,X_t^c)))=\exp-\left[\int\left\{\exp(i(f,x))-1-i(f,x)\right\}\pi_t(dx)\right]$$

where $\pi_t(dx) = E(P_t(dx))$.

Proof. The function $F(x) = \exp i(f, x)$ has the first derivative $F'(x) = i \cdot \exp(i(f, x)) \cdot f$ and the second derivative $F''(x) = -\exp(i(f, x))f \cdot f$, which are continuous and bounded in their norms. Applying Theorem 3, we obtain

$$\exp(i(f, Y_t)) - 1 = \text{martingale} - \frac{1}{2}\int_0^t \exp(i(f, X_s))d\langle(f, X)\rangle_s$$
$$+ \exp(i(f, X_s^-))[\exp(i(f, \Delta X_s)) - 1 - i(f, \Delta X_s)]$$

Therefore,

$$E(\exp(i(f, X_t))) - 1 = -\frac{1}{2}\int_0^t E(\exp(i(f, X_s)))d(S_t f, f)$$
$$+ \int E(\exp(i(f, X_s)))\int_H (\exp(i(f, X)) - 1 - i(f, x))d_s\pi(s, dx)$$

Consequently,

$$E(\exp(i(f, X_t))) = \exp\left(-\frac{1}{2}(S_t f, f) + \int [\exp(i(f, x)) - 1 - i(f, x)]\pi_t(dx)\right).$$

The independence of X_t^c and X_t^d can be derived similarly as [2], if we replace (f, Y_t) by $(f, X_t^c - X_s^c) + (g, X_t^d - X_s^d)$ in the above discussion.

Now let X_t be a square integrable $(E\|X_t\|^2 < \infty)$ and mean continuous H-process. We denote by M_t (resp. M) the smallest linear manifold containing X_s, $s \leqq t$ (resp. $s < \infty$). Then M_t is a pre-Hilbert space by the inner product $(X, Y) = E(X, Y)$. We shall denote by $\bar{M}_t$ the completion of M_t. The mean continuity of X_t implies that $\bigcap_{\varepsilon>0} \bar{M}_{t+\varepsilon} = \bar{M}_t$ and $\overline{\bigcup_{\varepsilon>0} M_{t-\varepsilon}} = \bar{M}_t$. We shall call the H-process X_t *linear* if for any $X \in M$, the projection $P_t X$ of X to $\bar{M}_t$ is independent of $X - P_t X$. ([3]).

We shall show that the joint distribution of linear process is subject to an infinitely divisible distribution. Let $X \in \bar{M}$. Then $P_t X$ is an additive process. In fact, since the projection of $P_t X$ to $\bar{M}_s$ coincides with $P_s X (t \geqq s)$, $P_t X - P_s X$ has to be independent of $\bar{M}_s$. Now let $X = \sum_{i=1}^n c_i X_{t_i}$. Then $\sum c_i(P_t X_{t_i} - P_t X_{t_i})$ is independent of $\bar{M}_s$. This means that $(P_t X_{t_i} - P_s X_{t_i}, \cdots, P_t X_{t_n} - P_s X_{t_n})$ is independent of $\bar{M}_s$, or equivalently, $(P_t X_{t_i}, i = 1, 2, \cdots, n)$ is an H^n-valued additive process. Thus $(P_t X_{t_i}; i = 1, \cdots, n)$ is subject to infinitely divisible distribution. Taking $t \geqq \max(t_i)$, we see that $(X_{t_i}, \cdots, X_{t_n})$ is also subject to infinite divisible distribution.

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