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ON SELF-INTERSECTION NUMBER OF A SECTION ON A RULED SURFACE

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To Professor K. Ono for his sixtieth birthday

Let E be a non-singular projective curve of genus $g \geq 0$, $\boldsymbol{P}$ the projective line and let F be the surface $E \times \boldsymbol{P}$. Then it is well known that a ruled surface F^* which is birational to F is biregular to a surface which is obtained by successive elementary transformations from F (for the notion of an elementary transformation, see [3]). The main purpose of the present article is to prove the following

THEOREM 1. *For any such F^*, there is a section (i.e., an irreducible curve s on F such that $(s, l) = 1$ for a fibre l of F^*) such that its self-intersection number (s, s) is not greater than g.*

In classifying ruled surface F^*, as was noted by Atiyah [1], it is important to know the minimum value of self-intersection numbers (s, s) of sections of F^*.[1] Our Theorem 1 is important in the respect.

The following is a key to our proof of Theorem 1:

THEOREM 2. *Let d be a non-negative rational integer. If $Q_1, \cdots, Q_{g+2d+1}$ are points[2] of F, then there is a positive divisor D of F such that* (i) *D goes through $Q_1, \cdots, Q_{g+2d+1}$ and* (ii) *D is linearly equivalent to $E \times P + \sum_{i=1}^{g+d} R_i \times \boldsymbol{P}$ with a $P \in \boldsymbol{P}$ and suitable $R_i \in E$.*

In connection with this Theorem 2, we prove the following theorem too:

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1) Atiyah proved that the minimum value is not greater than $2g-1$ if $g>0$. On the other hand, it was remarked by M. Maruyama that there is an F (for every E) which carries only sections s such that $(s, s) \geqq g$ (see [2]).

2) In this theorem, these Q_i need not be ordinary points, namely, some of these Q_i may be infinitely near points of some ordinary points. For the definition of the term "go through" in such a case, see [3].

THEOREM 3. *Let $Q_1^*, \cdots, Q_{t+1}^*$ be independent generic points of F over a field of definition k of F. Let S^* be the set of positive divisors D of F such that* (i) *D goes through $Q_1^*, \cdots, Q_{t+1}^*$ and* (ii) *D is linearly equivalent to $E\times P+\sum_{i=1}^t R_i\times \boldsymbol{P}$ with a $P\in \boldsymbol{P}$ and suitable $R_i\in E$. If $t\leq g$, then S^* is not empty and S^* dose not contain any algebraic family of positive dimension.*

In appendix, we add some remarks on dimensions of algebraic families.

1. Some preliminary results, notation.

Since the case where $g=0$ is obvious, we assume that $g\geq 1$. P (or P') denotes a point of $\boldsymbol{P}$. R (or R_i, R_j', R_t^*, etc.) denotes a point of E. Q (or Q_i, Q_j', etc.) denotes a point of F. k is a field of definition for E and F, and for the sake of simplicity, we assume that k is algebraically closed. $L(R_1, \cdots, R_s)$ is the complete linear system $|E\times P+\sum_{i=1}^s R_i\times \boldsymbol{P}|$. Specializations are understood with reference to k. For fundamentals on specializations of cycles, see [4] and [5].

LEMMA 1. *Let d be the dimension of the complete linear system $|\sum_{i=1}^s R_i|$ on E. Let $\sum_{i=1}^s R_i^*$ be a generic member of the linear system over a field containing k and let C^* be a generic member of $L(R_1, \cdots, R_s)$ over $k(R_1^*, \cdots, R_s^*)$. Then*

(i) $\dim L(R_1, \cdots, R_s)=2d+1$,

(ii) $\text{trans.}\deg_k k(C^*)=d+1+\text{trans.}\deg_k k(R_1^*, \cdots, R_s^*)$,

(iii) *if* $\dim|\sum_{i=1}^s R_i'|=d$ *and if* $(R_1', \cdots, R_s')$ *is a specialization of* $(R_1^*, \cdots, R_s^*)$ *then every member of* $L(R_1', \cdots, R_s')$ *is a specialization of* C^* *over the specialization* $(R_1^*, \cdots, R_s^*)\rightarrow(R_1', \cdots, R_s')$.

Proof. Consider $E'=E\times P$. Then $\dim \mathrm{Tr}_{E'}L(R_1, \cdots, R_s)=d=\dim(L(R_1, \cdots, R_s)-E')$, from which (i) follows readily. Now, consider loci T and U of $(C^*, R_1^*, \cdots, R_s^*)$ and C^* respectively, over k. Then $\dim T=\text{trans.}\deg_k k(R_1^*, \cdots, R_s^*)+\text{trans.}\deg_{k(R_1^*,\ldots,R_s^*)} k(C^*)$, and on the other hand, letting p denote the natural projection from T onto U, we have $\dim p^{-1}(C^*)=\dim|\sum_{i=1}^s R_i|=d$. Therefore $\text{trans.}\deg_k k(C^*)=\dim U=\dim T-d=d+1+\text{trans.}\deg_k k(R_1^*, \cdots, R_s^*)$, which proves (ii). As for (iii), we consider a specialization of $(C^*, R_1^*, \cdots, R_s^*, L(R_1, \cdots, R_s))$ over the specialization $(R_1^*, \cdots, R_s^*)\rightarrow(R_1', \cdots, R_s')$. $E\times P+\sum_i R_i^*\times \boldsymbol{P}$ is specialized to $E\times P'+\sum R_i'\times\boldsymbol{P}$, which must be a member of the specialization L^* of $L(R_1, \cdots, R_s)$. Since $\dim L^*=\dim L(R_1, \cdots, R_s)=d=\dim L(R_1', \cdots, R_s')$ and since all members of L^* are linearly equivalent to each other,[3] we see that $L^* = L(R_1', \cdots, R_s')$. Thus Lemma 1 is proved.

LEMMA 2. *Let V be a surface defined over k. If $M_1, \cdots, M_n$ are points of V and if* trans. $\deg_k k(M_1, \cdots, M_n) \geqq 2n - \alpha$, *then suitable $n - \alpha$ points among $M_1, \cdots, M_n$ are independent generic points of V over k.*

Proof. We use induction argument on n. (1) If M_n is a generic point of V over $k(M_1, \cdots, M_{n-1})$, then trans. $\deg_k k(M_1, \cdots, M_{n-1}) \geqq 2(n-1) - \alpha$. Then, by our induction assumption, there are $n - 1 - \alpha$ independent generic points among $M_1, \cdots, M_{n-1}$ and we see the assertion in this case. (2) Otherwise, we have trans. $\deg_k k(M_1, \cdots, M_{n-1}) \geqq 2(n-1) - (\alpha - 1)$, and we completes the proof by our induction assumption.

2. Proof of Theorem 2.

Let $R_1^*, \cdots, R_{g+d}^*$ be independent generic points of E over k and let C^* be a generic member of $L(R_1^*, \cdots, R_{g+d}^*)$ over $k(R_1^*, \cdots, R_{g+d}^*)$. Let $Q_1^*, \cdots, Q_{2g+2d+1}^*$ be independent generic points of C^* over $k(C^*)$. Then by Lemma 1, trans. $\deg_k k(C^*, Q_1^*, \cdots, Q_{2g+2d+1}^*)$ = trans. $\deg_k k(C^*) + 2g + 2d + 1 = d + 1 + d + g + 2g + 2d + 1 = 3g + 4d + 2 = 2(2g + 2d + 1) - g$. Now we consider locus T of $(C^*, Q_1^*, \cdots, Q_{2g+2d+1}^*)$ and the natural projection pr from T into the $(2g + 2d + 1)$-ple product F'' of F. Since the self-intersection number (C^*, C^*) of C^* is equal to $2g + 2d$, we see that pr is generically a one-one correspondence between T and pr T, which shows that $\dim T = \dim \mathrm{pr}\, T$. Therefore, applying Lemma 2 with $n = 2g + 2d + 1$, we see that there are $g + 2d + 1$ independent generic points of F among $Q_1^*, \cdots, Q_{2g+2d+1}^*$. This proves Theorem 2 in the case where $Q_1, \cdots, Q_{g+2d+1}$ are independent generic points of F. New we complete the proof making use of specializations.

3. Proof of Theorem 1.

As was noted at the beginning, F^* is obtained by successive elementary transformations with centers, say, $P_1, \cdots, P_m$ from F. If $m \leqq g$, then the proper transform of an $E \times P$ has self-intersection number $\leqq g$. Therefore we assume that $m > g$. Then there is d such that $m = g + 2d$ or $m = g + 2d + 1$. By virtue of Theorem 2, there is a positive divisor D of F such that (i)

[3] Note that if D and D' are divisors which are linearly equivalent to each other, and if they are specialized to D_1 and D_1' under the same specialization, then D_1 is linearly equivalent to D_1'.

D goes through $P_1, \cdots, P_m$ and (ii) D is linearly equivalent to $E \times P + \sum_{i=1}^{g+d} R_i \times \boldsymbol{P}$. Then the proper transform D' of D, or more precisely, the divisor of F^* which is the transform of $D - \sum P_i$, has self-intersection number $2g + 2d - m$, which is either g or $g-1$. D' has a section s of F^* as a component, and $(s, s) \leq g$. This completes our proof of Theorem 1.

4. Proof of Theorem 3.

Let P and $R_i (i = 1, \cdots, t)$ be such that $Q_i^* \in R_i \times \boldsymbol{P}$ and $Q_{t+1}^* \in E \times P$. Then $E \times P + \sum_{i=1}^t R_i \times \boldsymbol{P}$ is in S^*, and therefore S^* is not empty. Assume now that there is an irreducible algebraic family S of positive dimension contained in S^*. Let C be a generic member of S over $k(Q_1^*, \cdots, Q_{t+1}^*)$ and let R_i' be such that $C \in L(R_1', \cdots, R_t')$. Let $\sum_{i=1}^t R_i''$ be a generic member of $|\sum_i R_i'|$ over $k(Q_1^*, \cdots, Q_{t+1}^*, R_1', \cdots, R_t')$ and let C'' be a generic member of $L(R_1', \cdots, R_t')$ over $k(Q_1^*, \cdots, Q_{t+1}^*, R_1', \cdots, R_t', R_1'', \cdots, R_t'')$. Let U be the locus of C'' over k and set $d = \dim |\sum_{i=1}^t R_i'|$. Lemma 1 shows that $\dim U = \text{trans.}\deg_k k(C'') = d + 1 + \text{trans.}\deg_k k(R_1'', \cdots, R_t'')$. Set $u = \text{trans.}\deg_k k(R_1'', \cdots, R_t'')$. Then we may assume that $R_1'', \cdots, R_u''$ are independent generic points of E over k. Since $t \leq g$, $\dim |\sum_{i=1}^u R_i''| = 0$, whence $d = \dim |\sum_{i=1}^t R_i''| \leq t - u$. Thus we have that $\dim U \leq t - u + 1 + u = t + 1$. Since U is defined over k and since $Q_1^*, \cdots, Q_{t+1}^*$ are independent generic points, $\dim S \leq t + 1 - (t + 1) = 0$. This completes our proof of Theorem 3.

Appendix

Our proof of Theorem 2 above really gives a proof of the following fact:

THEOREM A1. *Let $\mathfrak{F}$ be an algebraic family of positive divisors on a projective variety V. If $\dim \mathfrak{F} \geq d$ and if $P_1, \cdots, P_d$ are points of V, then there is a member D of $\mathfrak{F}$ such that $P_i \in D$ for all i.*

If $\mathfrak{F}$ is a linear system, then, for a point P of V, $\{D \in \mathfrak{F} | P \in D\}$ forms a hyperplane of $\mathfrak{F}$ if $\mathfrak{F}$ is viewed as a projective space of dimension d. Therefore if $\mathfrak{F}$ is a linear system, then Theorem A1 is obvious and is well known. But, in the general case, the same reasoning cannot be given. Furthermore, if $\mathfrak{F}$ is an algebraic family of r-cycles ($\neq$ divisors), then the dimension defect by the condition to go through one point is not uniform. For instance, let V be the projective space of dimension n and let $\mathfrak{F}$ be the family of m points which are colinear $(m \geq 3)$, then $\dim \mathfrak{F} = 2(n-1) + m$.

For $\mathfrak{F}' = \{D \in \mathfrak{F} | P \in D\}$ (where P is a point of V), $\dim \mathfrak{F}' = \dim \mathfrak{F} - n$. For $\mathfrak{F}'' = \{D \in \mathfrak{F}' | P' \in D\}$ (where P' is a point of V which is different from P), $\dim \mathfrak{F}'' = \dim \mathfrak{F}' - n$. But then, if P'' is a point of V which is different from P, P', (i) if P'' is in outside of the line going through P, P', then $\mathfrak{F}^* = \{D \in \mathfrak{F}'' | \mathfrak{P}'' \in D\}$ is empty, (ii) otherwise, $\dim \mathfrak{F}^* = \dim \mathfrak{F}'' - 1$.

Here we shall discuss such dimension defect in the general case. Our result will give another proof of Theorem A1 above.

From now on, let V be a projective variety of dimension n and let $\mathfrak{F}$ be an (irreducible) algebraic family of positive r-cycles on V. We fix an algebraically closed, common field of definition k for V and $\mathfrak{F}$. Let C^* be a generic member of $\mathfrak{F}$ over k and let P be a point of V. Denote by $\mathfrak{F} - P$ the set $\{C \in \mathfrak{F} | P \in C\}$.

Assume that there is a member C of $\mathfrak{F} - P$. Then there is a point P^* of C^* such that (C^*, P^*) is specialized to (C, P). Let U be the locus of P^* over k. Then

THEOREM A2. *There is an algebraic family $\mathfrak{F}'$ such that* (1) $C \in \mathfrak{F}' \subseteq \mathfrak{F} - P$ *and* (2) $\dim \mathfrak{F}' = \dim \mathfrak{F} + \dim (U \cap C^*) - \dim U$.

Proof. To begin with, we may assume that P^* is a generic point of an arbitrarily fixed component of $C^* \cap U$ over $k(C^*)$, whence we may assume that $\dim (U \cap C^*) = \text{trans.}\deg_{k(C^*)} k(C^*, P^*)$. Let W and T be the locus of C^* over $k(P^*)$ and the locus of (C^*, P^*) over k respectively. Then $\dim U + \dim W = \text{trans.}\deg k(P^*) + \text{trans.}\deg_{k(P^*)} k(P^*, C^*) = \dim T = \dim F + \dim (U \cap C^*)$. Thus $\dim W = \dim F + \dim (U \cap C^*) - \dim U$. Consider a specialization $W \to W'$ over $(C^*, P^*) \to (C, P)$. Then, since $C^* \in W$, we have $C \in W'$. Thus it is enough to set $\mathfrak{F}' = W'$.

From our Theorem A2, we get the following result immediately:

Let C_i^* $(i = 1, \cdots, t)$ be all of the irreducible components of C^* and let P_i^* be a generic point of C_i^* over $k(C_1^*, \cdots, C_t^*)$. Let V_i be the locus of P_i^* over k for each i. Then

THEOREM A3. *For $P \in V$, we have*

(1) *if P is not in any of V_i, then $\mathfrak{F} - P$ is empty,*

(2) *otherwise, let p be the maximum of* $\dim U_i$ *where U_i ranges over all V_i which goes through P, then the dimension of every component of $F - P$ is not less than* $\dim \mathfrak{F} + r - p$.

Now, our Theorem A1 is a corollary to this.

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