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ON REDUCIBILITY OF PROVABILITY IN THE PRIMITIVE LOGIC [LO]

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Dedicated to Professor Katuzi Ono on his 60th birthday

Introduction.

In the present paper, we would like to show a theorem concerning with *reducibility of provability* in the primitive logic. This theorem seems to suggest a procedure to find the proof-note of a given proposition which is provable in the primitive logic.

The formulations of the primitive logic ***LO*** and [***LO***] have been introduced in Ono [1], [2], [3], and [5]. The primitive logic is the logic having only two logical constants IMPLICATION $\rightarrow$ and UNIVERSAL QUANTIFICATION ( ), and has an interesting property that any logic belonging to intuitionistic series or to classical series can be faithfully interpreted in it (Ono [3], [4]).

In the proof-note of the logic [***LO***], there are some propositions enclosed in pairs of brackets. Any proposition in proof-notes of [***LO***] is said to be CLAD or BARE according as it is enclosed or is not enclosed in a pair of brackets. The followings are the inference rules of the logic [***LO***]:

[***F***]: *The step* $\mathfrak{A}$ *can be deduced from the step* [$\mathfrak{A}$].

[***I***]: *The step* [$\mathfrak{B}$] *can be deduced from the steps* $\mathfrak{A}$ *and* [$\mathfrak{A}\rightarrow\mathfrak{B}$].

[***I****]: *The step* $\mathfrak{A}\rightarrow\mathfrak{B}$ *can be deduced from the fact that* $\mathfrak{B}$ *is deducible from* [$\mathfrak{A}$].

[***U***]: *The step* [$\mathfrak{A}(t)$] *can be deduced from the step* [$(x)\mathfrak{A}(x)$] *as far as* $\mathfrak{A}(u)$ *contains no free variable* x *at all.*

[***U****]: *The step* $(x)\mathfrak{A}(x)$ *can be deduced from the fact that* $\mathfrak{A}(t)$ *is deducible for any variable* t *whatever, i.e. from the fact that the step* $\mathfrak{A}(t)$ *is deducible from the step* $\forall t$:.

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Any proof-note is understood as a practical discription of steps of reasonings. Every step is introduced and denoted by its introductory index-word. Any index-word is a sequence of letters, such as $\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, \cdots$ assuming the usual alphabetical order between them, including the null sequence. We denote a sequence of letters by an underlined single letter such as $\underline{\boldsymbol{p}}$ or $\underline{\boldsymbol{t}}$, and especially null sequence by $\emptyset$.

Here, we must further refer to [5] on some technical terms such as ORDER (Natural, Fundamental and Basis) of steps, REFERENCE STEP and ASSUMPTION STEP. In [5], K. Ono showed a characteristic feature of the logic [***LO***] by proving the theorem:

Any proposition occurring in a wasteless proof-note of a proposition $\mathfrak{P}$ *is a subformula of the proposition* $\mathfrak{P}$.

In this paper, we shall show that this theorem may be described more precisely from a view-point on reducibility of provability.

In section 1, we shall introduce some expressions on ***LO-formulas*** such as REGULAR PARSING-FORM, τ-KERNEL and σ-CONSTRUCTION with their PARSING-FACTORS and KERNEL.

In section 2, we shall prove the main theorem on reducibility of provability such as the theorem:

The provability of a given proposition $\mathfrak{A}$ *is reducible to the fact that there are finite sequences of factors* $\{\boldsymbol{\mu}\}$ *and* $\{\boldsymbol{\nu}\}$ *which satisfy the following conditions;* 1) *the* τ*-kernel of* $\mathfrak{A}$ *by factors* $\{\boldsymbol{\mu}\}$ *is deducible from the assumptions* $\{\boldsymbol{\mu}\}$, 2) $\{\boldsymbol{\mu}\}$ *contains a* σ*-construction by factors* $\{\boldsymbol{\nu}\}$ *with that* τ*-kernel of* $\mathfrak{A}$ *by factors* $\{\boldsymbol{\mu}\}$ *as its kernel, and* 3) *any formula of* $\{\boldsymbol{\nu}\}$ *is deducible from the assumptions* $\{\boldsymbol{\mu}\}$.

1. Parsing expression.

Any formula which has no logical constants other than *implication* and *universal quantification* is called an *LO-formula*.

Firstly, we would like to introduce the parsing expressions.

(**1. 1**) *Regular parsing-form*

We introduce a PARSING-FORM with its FACTOR and KERNEL as follows;

An expression $\langle\lambda\rangle\mathfrak{A}$ which is called a *parsing-form* by a π*-factor* λ with *kernel* $\mathfrak{A}$, is defined by

1) if λ is a formula $\mathfrak{C}$, $\langle\mathfrak{C}\rangle\mathfrak{A} \equiv$[1] $\mathfrak{C}\rightarrow\mathfrak{A}$:

1) $P\equiv Q$ means "P is defined by Q".

2) if λ is a variable x, $\langle x\rangle\mathfrak{A}\equiv(x)\mathfrak{A}(x)$:

3) otherwise, $\langle\lambda\rangle\mathfrak{A}$ is undefined.

Especially, if λ is empty or denoted by λ_0, $\langle\ \rangle\mathfrak{A}$ or $\langle\lambda_0\rangle\mathfrak{A}\equiv\mathfrak{A}$ is called a parsing-form by the *empty-factor* or λ_0 with kernel $\mathfrak{A}$.

For any $i(i\geqq 1)$, a parsing-form with kernel $\mathfrak{A}$ by π-factors $\{\lambda_0,\cdots,\lambda_i\}$, $\langle\lambda_0,\lambda_1,\cdots\cdot,\lambda_i\rangle\mathfrak{A}\equiv\langle\lambda_i\rangle(\langle\lambda_0,\cdots\cdot,\lambda_{i-1}\rangle\mathfrak{A})$ recursively, where λ_0 is empty and each of $\lambda_i(i\geqq 1)$ is a formula or a variable, respectively. In the case that λ_i is a formula each of λ_i is called a *π-formula,* otherwise a *π-variable.*

From the definition of parsing-forms we have the following lemmas.

LEMMA 1. *Any parsing-form which has an LO-formula as its kernel is also an LO-formula.*

LEMMA 2. *Any LO-formula is expressible in a parsing-form by π-factors.*

Accordingly, corresponding to any *LO*-formula $\mathfrak{P}$, there is one and only one parsing-form which has an *elementary* formula as its kernel. This expression is called a *regular parsing-form* of $\mathfrak{P}$.

(**1. 2**) *τ-Kernel.*

Let $\mathfrak{A}$ be a formula whose regular parsing-form is $\langle\lambda_0,\cdots,\lambda_n\rangle E$. The expression $\tau\langle\mu\rangle\mathfrak{A}$ which is called a τ-kernel of $\mathfrak{A}$ by a τ-factor μ, is defined as follows:

1) If μ is a formula $\mathfrak{C}$, and λ_n is $\mathfrak{C}$, $\tau\langle\mathfrak{C}\rangle(\langle\mathfrak{C}\rangle\mathfrak{B})\equiv\mathfrak{B}$,

2) If μ is a free variable t and λ_n is a variable x, $\tau\langle t\rangle(\langle x\rangle\mathfrak{B}(x))\equiv\mathfrak{B}(t)$,

3) otherwise, $\tau\langle\mu\rangle\mathfrak{A}$ is undefined,

where $\mathfrak{B}$ or $\mathfrak{B}(x)$ denotes $\langle\lambda_0,\cdots,\lambda_{n-1}\rangle E$. The τ-factor μ is called a *τ-formula* or a *τ-variable* according as μ is a formula or a variable, respectively.

Especially, if μ is empty or denoted by μ_0, $\tau\langle\ \rangle\mathfrak{A}$ or $\tau\langle\mu_0\rangle\mathfrak{A}\equiv\mathfrak{A}$, is called a τ-kernel of $\mathfrak{A}$ by the empty-factor or μ_0.

For any $i(i\geqq 1)$, a τ-kernel of $\mathfrak{A}$ by τ-factors $\{\mu_0,\cdots,\mu_i\}$

$$\tau\langle\mu_0,\mu_1,\cdots,\mu_i\rangle\mathfrak{A}\equiv\tau\langle\mu_i\rangle(\tau\langle\mu_0,\cdots,\mu_{i-1}\rangle\mathfrak{A})$$

recursively, where μ_0 is empty and each of $\mu_j(i\geqq j\geqq 1)$ is a *τ-formula* or a *τ-variable* according as λ_{n-j-1} is a formula or a variable, respectively.

(1. 3) *σ-construction.*

An expression $\sigma\langle\nu\rangle\mathfrak{A}$ which is called a *σ-construction* with *kernel* $\mathfrak{A}$ by a *σ-factor* ν, is defined as follows;

1) If ν is a formula $\mathfrak{C}$, $\sigma\langle\mathfrak{C}\rangle\mathfrak{A}\equiv\langle\mathfrak{C}\rangle\mathfrak{A}$.

2) If ν is a variable t, $\sigma\langle t\rangle\mathfrak{A}(t)\equiv\langle x\rangle\mathfrak{A}(x)$

where $\mathfrak{A}(x)$ denotes the derived formula from $\mathfrak{A}(t)$ replacing t in its position by x which has no occurrence in $\mathfrak{A}(t)$. The σ-factor ν is called *σ-formula* or *σ-variable* according as ν is a formula or a variable, respectively.

Especially, if ν is empty or denoted by ν_0, $\sigma\langle\ \rangle\mathfrak{A}$ or $\sigma\langle\nu_0\rangle\mathfrak{A}\equiv\mathfrak{A}$ is called a σ-construction with kernel $\mathfrak{A}$ by the empty-factor or ν_0. For any i ($i\geqq 1$), a σ-construction with kernel $\mathfrak{A}$ by σ-factors $\{\nu_0,\cdots,\nu_i\}$,

$$\sigma\langle\nu_0,\nu_1,\cdots,\nu_i\rangle\mathfrak{A}\equiv\sigma\langle\nu_i\rangle(\sigma\langle\nu_0,\cdots,\nu_{i-1}\rangle\mathfrak{A})$$

recursively, where ν_0 is empty and each of $\nu_j (j\geqq 1)$ is a *σ-formula* or a *σ-variable*.

(1. 4) *Modulation of inference rules.*

Using our expressions the inference rules of the logic [***LO***] are modulated as follows;

[***F***]: *The step* $\mathfrak{A}$ *can be deduced from the step* $[\mathfrak{A}]$.

[***I***]: *The step* $[\mathfrak{B}]$ *can be deduced from the steps* $\mathfrak{A}$ *and* $[\sigma\langle\mathfrak{A}\rangle\mathfrak{B}]$.

[***I****]: *The step* $\langle\mathfrak{A}\rangle\mathfrak{B}$ *can be deduced from the fact that* $\tau\langle\mathfrak{A}\rangle(\langle\mathfrak{A}\rangle\mathfrak{B})$ *is deducible from* $[\mathfrak{A}]$.

[***U***]: *The step* $[\mathfrak{A}(t)]$ *can be deduced from the step* $[\sigma\langle t\rangle\mathfrak{A}(t)]$.

[***U****]: *The step* $\langle x\rangle\mathfrak{A}(x)$ *can be deduced from the fact that* $\tau\langle t\rangle(\langle x\rangle\mathfrak{A}(x))$ *is deducible for any variable t whatever, i.e., from the fact that the step* $\tau\langle t\rangle(\langle x\rangle\mathfrak{A}(x))$ *is deducible from the step* $[t]$.[2)]

Let Γ be an ordered set of clad formulas or variables which belong to assumption steps of a step $\underline{s}$, arranged its index-word in the fundamental order. Then, the step $\underline{s}$: $\underline{s}$) $\mathfrak{A}$ or $[\mathfrak{A}]$ is said to be *deducible* from the *assumption* Γ. We would like to denote this by $\Gamma\vdash\mathfrak{A}$ or $\Gamma\vdash[\mathfrak{A}]$, respectively. Especially, if $\underline{s}$ is $\emptyset$, the formula of $\underline{s}$ is bare and Γ is empty. We denote this by $\vdash\mathfrak{A}$.

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2) We will use clad variable $[t]$ instead of $\forall t$:, because in a proof-note, denominating quantifier $\forall t$: as well as clad formula $[\mathfrak{A}]$ has an assumptional character.

Any proposition $\mathfrak{P}$ is said to be *provable* in [***LO***] if and only if the step $\mathfrak{P}$ is deducible in [***LO***].

2. Reducibility of provability.

(2. 1) *Lemmas.*

Now, we are going to prove the following lemmas preparatory to the main theorem.

LEMMA 3. $\vdash \langle\lambda\rangle \mathfrak{A}$ *if and only if* $[\mu] \vdash \tau\langle\mu\rangle(\langle\lambda\rangle\mathfrak{A})$, *where* μ *is a* τ*-variable which has no occurrence in* $\langle\lambda\rangle\mathfrak{A}$ *or a* τ*-formula according as* λ *is a variable or a formula, respectively.*

Proof. Firstly we assume $\vdash \langle\lambda\rangle \mathfrak{A}$. In this proof-note, the assumption step of the step $\emptyset$ is empty, therefore the inference rule for the step $\emptyset$ is [U^*] or [I^*]. Accordingly, there are steps $A)[\mu]$ and $\epsilon)\tau\langle\mu\rangle(\langle\lambda\rangle\mathfrak{A})$ in this proof-note, where μ is a τ-variable which has no occurrence in $\langle\lambda\rangle\mathfrak{A}$ or a τ-formula according as λ is a variable or a formula. Thus we have $[\mu] \vdash \tau\langle\mu\rangle(\langle\lambda\rangle\mathfrak{A})$. Conversely, we assume $[\mu] \vdash \tau\langle\mu\rangle(\langle\lambda\rangle\mathfrak{A})$ with the μ-condition. Adding to this proof-note the step $\emptyset)\langle\lambda\rangle\mathfrak{A}$, we have $\vdash \langle\lambda\rangle\mathfrak{A}$ by [U^*] or [I^*] according as λ is a variable or a formula, respectively.

In the following lemmas, let $\langle\lambda_0, \cdots, \lambda_n\rangle E$ be the regular parsing-form of $\mathfrak{A}$.

LEMMA 4. *For any* k $(n \geqq k \geqq 1)$, *if* $[\mu_1], \cdots, [\mu_k] \vdash \tau\langle\mu_0, \cdots, \mu_k\rangle\mathfrak{A}$, *then* $\vdash \mathfrak{A}$, *where the* μ*-condition*: μ_i $(k \geqq i \geqq 1)$ *is a* τ*-variable which has no occurrence in any* $\tau\langle\mu_0, \cdots, \mu_{j-1}\rangle\mathfrak{A}$ $(i \geqq j \geqq 1)$ *or a* τ*-formula according as* λ_{n-i+1} *is a variable or a formula, respectively, holds.*

Proof. For $k=1$, the case is the last part of lemma 3.

Now, we assume this lemma for any number less than k, and prove it for k by induction. In this proof-note, there are the step $\underline{s})$ $\tau\langle\mu_0, \cdots, \mu_k\rangle\mathfrak{A}$ and its assumption step $\underline{uA})$ $[\mu_k]$ with the μ-condition. Therefore, adding the step $\underline{u})$ $\tau\langle\mu_0, \cdots, \mu_{k-1}\rangle\mathfrak{A}$ to this proof-note, we have $[\mu_1], \cdots, [\mu_{k-1}] \vdash \tau\langle\mu_0, \cdots, \mu_{k-1}\rangle\mathfrak{A}$ which leads up to $\vdash \mathfrak{A}$ by assumption of induction.

LEMMA 5. $\vdash \mathfrak{A}$ *if and only if there is a sequence of factors* $\{\mu_0, \cdots, \mu_k\}$ $(n \geqq k \geqq 1)$ *such as*

$$[\tau\langle\mu_0, \cdots, \mu_k\rangle\mathfrak{A}] \in \{[\mu_1], \cdots, [\mu_k]\}$$

$$\textit{or}\quad [\mu_1], \cdots, [\mu_k] \vdash [\tau\langle\mu_0, \cdots, \mu_k\rangle\mathfrak{A}]$$

where the μ-condition holds.

Proof. Firstly by assuming $\vdash \mathfrak{A}$, we have $[\mu_1] \vdash \tau\langle\mu_0, \mu_1\rangle\mathfrak{A}$ by lemma 3. If the inference for the step $\underline{s}$) $\tau\langle\mu_0, \cdots, \mu_{m-1}\rangle\mathfrak{A}$ is not [*F*], it must be [I^*] or [U^*] and then there are steps $\underline{s}A$) $[\mu_m]$ and $\underline{s}_t$)$\tau\langle\mu_0, \cdots, \mu_m\rangle\mathfrak{A}$ in this proof-note, where the μ-condition holds. However, m can not exceed n so that there is k $(k \leqq n)$ such as the inference for the step $\underline{p}$) $\tau\langle\mu_0, \cdots, \mu_k\rangle\mathfrak{A}$ is [*F*]. And therefore, $[\tau\langle\mu_0, \cdots, \mu_k\rangle\mathfrak{A}]$ belongs to the set of assumption steps of the step $\underline{p}$ or is deducible from the assumption steps of the step $\underline{p}$.

Conversely, by assuming the condition, from the step s) $[\tau\langle\mu_0, \cdots, \mu_k\rangle\mathfrak{A}]$ we have the step $\underline{p}$) $\tau\langle\mu_0, \cdots, \mu_k\rangle\mathfrak{A}$ by [*F*], and the assumption steps of the step $\underline{s}$ is also those of the step $\underline{p}$ so that we have $\vdash \mathfrak{A}$ by lemma 4.

LEMMA 6. *$\Gamma \vdash [\mathfrak{B}]$ if and only if there is a sequence of factor $\{\nu_0, \cdots, \nu_m\}$ such as*

1) $[\sigma\langle\nu_0, \cdots, \nu_m\rangle\mathfrak{B}] \in \Gamma$

and 2) *for any σ-formula ν_j $(1 \leqq j \leqq m)$ $\Gamma \vdash \nu_j$.*

Proof. Firstly we assume $\Gamma \vdash [\mathfrak{B}]$, and let the final step of this proof-note which is arranged in the fundamental order of its index-words be $\underline{s}$ (i.e., $\underline{s}$)$[\mathfrak{B}]$). We may prove the conclusion by induction referring to the step $\underline{s}$.

If the step $\underline{s}$ is the first step, the step $\underline{s}$ is the assumption step of itself and the σ-construction with kernel $\mathfrak{B}$ by the empty factor i.e., $\sigma\langle\ \rangle\mathfrak{B}$. In this case, $m = 0$ and the condition 2) is omitted.

Otherwise, we assume the assertion for any step $\underline{r}$ which takes precedence of $\underline{s}$ in the fundamental order.

The formula of the step $\underline{s}$ is clad so that the step $\underline{s}$ is either an assumption step of itself (case 1), or a step deduced from a step $\underline{u}$ by [*U*] (case 2), or from steps $\underline{u}$ and $\underline{v}$ by [*I*] (case 3).

In the case 1, the assertion holds evidently.

In the case 2, the step $\underline{u}$ is a clad formula and is a σ-construction with kernel $\mathfrak{B}$ by a σ-variable i.e., $\underline{u}$) $[\sigma\langle\nu\rangle\mathfrak{B}]$. And the step $\underline{u}$ takes precedence of $\underline{s}$ in the fundamental order so that there is an assumption step $\underline{p}$ of $\underline{u}$ which is a σ-construction with kernel $\sigma\langle\nu\rangle\mathfrak{B}$, by our assumption of induction. The step $\underline{p}$ is an assumption step of $\underline{s}$, and a σ-construction with kernel $\sigma\langle\nu\rangle\mathfrak{B}$ is also a σ-construction with kernel $\mathfrak{B}$. Thus the assertion holds in the step $\underline{s}$.

In the case 3, one of the steps $\underline{u}$ and $\underline{v}$ is a clad formula and the other is a bare formula by $[\boldsymbol{I}]$, and both of them take precedences of $\underline{s}$ in the fundamental order.

We would like to suppose the step $\underline{u}$ is clad, then there is an assumption step of $\underline{s}$ whose formula is σ-construction with kernel $\mathfrak{B}$ as we had it in the case 1, where ν is a σ-formula in this case. And then, the step $\underline{v}$ is the same bare formula as ν and is deduced from its assumption steps. The assumption steps of $\underline{v}$ are the assumption steps of $\underline{s}$ too. Thus the assertion also holds in this case.

Conversely, we have $\Gamma\vdash[\mathfrak{B}]$, by assuming the fact that there is a sequence of factor $\{\nu_0, \cdots, \nu_m\}$ such as the conditions 1) and 2) hold for any formula $\mathfrak{B}$, by induction as follows;

If $m=0$, the assertion is trivial.

We assume the assertion holds for any i $(i<m)$.

If ν_m is a free variable, we can deduce the step $\underline{s})[\sigma\langle\nu_0, \cdots, \nu_{m-1}\rangle\mathfrak{B}]$ from the step $\underline{u})$ $[\sigma\langle\nu_0, \cdots, \nu_{m-1}, \nu_m\rangle\mathfrak{B}]$ of Γ by $[\boldsymbol{U}]$.

If ν_m is a formula, according to the condition 2), we have a step $\underline{v})$ ν_m which is deduced from its assumption steps Γ, and the step $\underline{u})$ $[\sigma\langle\nu_0, \cdots, \nu_{m-1}, \nu_m\rangle\mathfrak{B}]$ of Γ by the condition 1). Then we can deduce the step $\underline{s})$ $[\sigma\langle\nu_0, \cdots, \nu_{m-1}\rangle\mathfrak{B}]$ from the steps $\underline{u}$ and $\underline{v}$ by $[\boldsymbol{I}]$. In the both cases of ν_m, the steps of Γ are also the assumption steps of $\underline{s}$.

Thus we have the conclusion.

(2. 2) *Theorem.*

From the preceding Lemmas 5 and 6 we have the main theorem and its corollaries.

THEOREM. *Let $\mathfrak{A}$ be any LO-formula which has its regular parsing form by π-factors $\{\lambda_0, \cdots, \lambda_n\}$. In the primitive logic $[\boldsymbol{LO}]$, the provability of a given proposition $\mathfrak{A}$ is reducible to the fact that there are τ-factors $\{\mu_0, \cdots, \mu_k\}$ $(k\leqq n)$ and σ-factors $\{\nu_0, \cdots, \nu_m\}$ which satisfy the following conditions;*

1) *The τ-kernel of $\mathfrak{A}$ by τ-factors $\{\mu_0, \cdots, \mu_k\}$ which satisfies the $\boldsymbol{\mu}$-condition of lemma 4, is deducible from the assumptions $\{[\mu_1], \cdots, [\mu_k]\}$,*

2) *$\{[\mu_1], \cdots, [\mu_k]\}$ contains a σ-construction by σ-factors $\{\nu_0, \cdots, \nu_m\}$ with that τ-kernel of $\mathfrak{A}$ by τ-factors $\{\mu_0, \cdots, \mu_k\}$ as its kernel,*

3) *Any σ-formula ν_j such as $1 \leqq j \leqq m$, is deducible from the assumptions* $\{[\mu_1], \cdots, [\mu_k]\}$.

COROLLARY 1. *If there is no σ-construction of τ-kernel of $\mathfrak{A}$ in the set of τ-factors of $\mathfrak{A}$, $\mathfrak{A}$ is unprovable.*

COROLLARY 2. *Without assumption, any elementary proposition is unprovable.*

This theorem seems to suggest a procedure to find a proof-note of a given proposition which is provable in the primitive logic [***LO***]. But we would like to mention about it in our later paper ([1]-part 3).

Finally, I would like to thank Professor Katuzi Ono for his guidance and his helpful suggestions.

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