T. Asai
Nagoya Math. J.
Vol. 37 (1970), 67–80

THE RECIPROCITY OF DEDEKIND SUMS AND THE FACTOR SET FOR THE UNIVERSAL COVERING GROUP OF $SL(2, \boldsymbol{R})$

TETSUYA ASAI

Dedicated to Professor Katuzi Ono on his 60th birthday

By explicit studying on theta-multipliers (i.e. the multipliers which appear in theta-transformation formulas under general modular substitutions), we can naturally get the reciprocity law of Gauss sums or quadratic residue symbols. This remarkable fact, by Cauchy, Kronecker, Hecke and others, is very classical, but its theoretical meaning has not been sufficiently clear yet.

Recently, Kubota made some investigation on this subject and he suggested a certain new point of view that these relations between reciprocity laws and automorphic forms are based on the "automorphic property" of the residue symbol itself (not necessarily quadratic) which leads to somewhat generalized reciprocity law.

Now, a similar discussion is possible even in the case of Dedekind sums. In fact, we shall show in the present paper, briefly to say, the following: *the generalized reciprocity law of Dedekind sums is equivalent to the proposition that $SL(2, \boldsymbol{Z})$ is a splitting*[*)] *subgroup for the factor set of the universal covering group of $SL(2, \boldsymbol{R})$.*

Since Dedekind sums, unlike Gauss sums, were primarily introduced for the purpose of the explicit and finite description of $\log\eta$-multipliers, the above assertion may be rather classical results. By our method, however, the mechanism of the subject about Dedekind sum and its reciprocity becomes very clear and simple. Furthermore, as a by-product, we shall get in the sequel, the explicit form of $\log\eta$-multipliers by purely arithmetical steps without the theory of functions.

Received December 5, 1968

[*)] The term "splitting" is used in a slightly wider sense than usual.

In the case of quadratic residue symbols there is an adelerized expression for the reciprocity, namely, the product formula of Hilbert symbols. But in our case it is an open question if such interpretation is possible or not.

The author wishes to express his gratitude to Professor Kubota who gave him many valuable advices very kindly.

1. The universal covering group of $SL(2, \boldsymbol{R})$ and its factor set.

In this section, we shall construct a universal covering group of $SL(2, \boldsymbol{R})$; the 2×2 special linear group over the real number field $\boldsymbol{R}$, with a convenient parametrization for our purpose, and we shall give the explicit value of the factor set for that extension over $SL(2, \boldsymbol{R})$, which has a very simple expression.

In the notation used here, we especially notice about the choice of argument of a complex number, i.e. for an arbitrary complex number $z(\neq 0)$, $arg(z)$ is always chosen in the fixed interval $[-\pi, \pi)$. Moreover, for any real number θ, a symbol θ' means a unique number in $[-\pi, \pi)$ such that $\theta' \equiv \theta \pmod{2\pi}$. For instance, $arg(e^{i\theta})$ is equal to θ'. For the abbreviation, we denote the group $SL(2, \boldsymbol{R})$ and the above interval $[-\pi, \pi)$ by G and $\boldsymbol{T}$, respectively.

1-1. First, let us recall a non-matric representation of G based on Iwasawa decomposition. Any element σ of G is uniquely decomposed as follows:

$$G \ni \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & \\ & \frac{1}{\sqrt{y}} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \frac{1}{\sqrt{y}} \begin{pmatrix} x\sin\theta + y\cos\theta & x\cos\theta - y\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

where, $z = x + iy = \dfrac{ai+b}{ci+d} = \sigma\langle i\rangle$ is a point of the upper-half-plane $\mathscr{H}$ (hereafter, the operation of $\sigma \in G$ on $z \in \mathscr{H}$ is expressed by $\sigma\langle z\rangle$, i.e. $\sigma\langle z\rangle = \dfrac{az+b}{cz+d}$ for $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$), and $\theta = arg(ci+d)$ is a point of $\boldsymbol{T}$ with torus topology. In this way, the group G can be identified with the $\mathscr{H} \times \boldsymbol{T}$ as a topological group, while an element σ is also parametrized as (z, θ).

By a short calculation, we can see the group multiplication on $G = \mathscr{H} \times \boldsymbol{T}$ under the (z, θ)-parametrization to be as follows:

For $\sigma_\nu = (z_\nu, \theta_\nu)$, $\nu = 1, 2, 3$, *and* $\sigma_1\sigma_2 = \sigma_3$,

$$\left\{ \begin{aligned} z_3 &= \frac{z_2(x_1\sin\theta_1 + y_1\cos\theta_1) + (x_1\cos\theta_1 - y_1\sin\theta_1)}{z_2\sin\theta_1 + \cos\theta_1} \\ \theta_3 &= (\theta_2 + arg(z_2\sin\theta_1 + \cos\theta_1))'. \end{aligned} \right. \tag{1}$$

Because of the notation θ', the difference between θ_3 and $\theta_2 + arg(z_2 \sin\theta_1 + \cos\theta_1)$ must be an integral multiple of 2π. We represent this integer by $w(\sigma_1, \sigma_2)$, i.e.

$$\theta_3 = \theta_2 + arg(z_2 \sin\theta_1 + \cos\theta_1) - 2\pi w(\sigma_1, \sigma_2).$$

Or, by using $\theta' = arg(e^{i\theta})$, $w(\sigma_1, \sigma_2)$ can be represented such as

$$w(\sigma_1, \sigma_2) = \frac{1}{2\pi}\{arg(c_2 i + d_2) + arg(c_1(\sigma_2\langle i\rangle) + d_1) - arg(c_3 i + d_3)\}, \tag{2}$$

where $\sigma_\nu = \begin{pmatrix} a_\nu & b_\nu \\ c_\nu & d_\nu \end{pmatrix}$.

It should be remarked that $w(\sigma_1, \sigma_2)$ is a rational integer, and more precisely, its value is 1, 0 or -1, determined only by σ_1 and σ_2.

More generally than (2), the following lemma is valid:

LEMMA 1. *For any choice of z in $\mathscr{H}$,*

$$w(\sigma_1, \sigma_2) = \frac{1}{2\pi}\{arg(c_2 z + d_2) + arg(c_1(\sigma_2\langle z\rangle) + d_1) - arg(c_3 z + d_3)\}. \tag{3}$$

Proof. Set $w_z(\sigma_1, \sigma_2)$ for the right-hand-side of (3), and let $f(z)$ be the function $exp(2\pi i r w_z(\sigma_1, \sigma_2))$ for any fixed $r \in \boldsymbol{R}$. Then, $f(z)$ is analytic on $\mathscr{H}$, and its absolute value is 1. So, $f(z)$ must be a constant function, and especially $f(z) = f(i)$, i.e. for any $r \in \boldsymbol{R}$, $r w_z(\sigma_1, \sigma_2) \equiv r w(\sigma_1, \sigma_2)$ $(mod.\ 2)$. Therefore, $w_z(\sigma_1, \sigma_2)$ must be equal to $w(\sigma_1, \sigma_2)$.

LEMMA 2. *$w(\sigma_1, \sigma_2)$ satisfies factor set relation, i.e.*

$$w(\sigma_1, \sigma_2) + w(\sigma_1\sigma_2, \sigma_3) = w(\sigma_1, \sigma_2\sigma_3) + w(\sigma_2, \sigma_3) \tag{4}$$

for any $\sigma_\nu \in G$, $\nu = 1, 2, 3$.

Proof. Because of lemma 1, it can be easily seen by direct calculation.

Remark 1. The factor set $w(\sigma_1, \sigma_2)$, that will play an important role in our discussion, was firstly investigated by Hans Petersson in a different way ([5]).

Remark 2. Lemma 2 results also from the associativity of the group multiplication (1), under lemma 1. On the other hand, lemma 1 follows directly from lemma 2, too.

1-2. Now, we define a new group $\tilde{G}$ whose multiplication is defined similarly to (1), and that becomes a universal covering group of G, besides its factor set as the group extension $\tilde{G}$ over G will coincide exactly with $w(\sigma_1, \sigma_2)$ introduced above.

Let $\tilde{G}$ be the product set $\mathscr{H} \times \boldsymbol{R} = \{(z, \theta);\ z \in \mathscr{H},\ \theta \in \boldsymbol{R}\}$. On the topological space $\tilde{G}$, we define a multiplication as follows:

For $\tilde{\sigma}_\nu = (z_\nu, \theta_\nu) \in \tilde{G}$, $\nu = 1, 2, 3$, *then* $\tilde{\sigma}_1\tilde{\sigma}_2 = \tilde{\sigma}_3$ *means that*

$$(5)\qquad \begin{cases} z_3 = \dfrac{z_2(x_1 \sin\theta_1 + y_1 \cos\theta_1) + (x_1 \cos\theta_1 - y_1 \sin\theta_1)}{z_2 \sin\theta_1 + \cos\theta_1} \\ \theta_3 = \theta_2 + arg(z_2 \sin\theta_1 + \cos\theta_1) + \theta_1 - \theta_1{}', \end{cases}$$

where $z_\nu = x_\nu + iy_\nu$.

As compared with (1), it can be easily seen that $\tilde{G}$ with the above multiplication becomes a group, for instance, the associativity of θ-part follows from lemma 1. Moreover, because of the multiplication (5) which is some different from (1) in θ-part, $\tilde{G}$ is surely a topological group. In this way, we have obtained a simply connected group $\tilde{G}$, which is homeomorphic to $\boldsymbol{R}^3$. On the other hand, it is obvious that the mapping:

$$\tilde{G} \ni \tilde{\sigma} = (z, \theta) \longrightarrow \sigma = (z, \theta') \in G$$

is a locally-isomorphic homomorphism from $\tilde{G}$ onto G. And its kernel is $\{(i, 2k\pi);\ k \in \boldsymbol{Z}\}$, which is contained in the center of $\tilde{G}$ and isomorphic to the additive group of rational integers $\boldsymbol{Z}$.

Thus $\tilde{G}$ is a universal covering group of G, while $\tilde{G}$ is a central group extension of G with the kernel $\boldsymbol{Z}$.

For the purpose of inquiring into the factor set of that extension, we denote an element $(z, \theta) = (z, \theta' + 2k\pi)$ of $\tilde{G}$ by (σ, k) afresh, where $\sigma = (z, \theta')$ is an element of G, i.e. we identify $\tilde{G}$ with $G \times \boldsymbol{Z}$. By calculating the multiplication under this parametrization as following: let $\tilde{\sigma}_\nu = (z_\nu, \theta_\nu) = ((z_\nu, \theta'_\nu),\ k_\nu) = (\sigma_\nu, k_\nu) \in \tilde{G}$, $\nu = 1, 2, 3$, then, by using (5) for $\tilde{\sigma}_1\tilde{\sigma}_2 = \tilde{\sigma}_3$,

$$\begin{aligned} \theta_3 &= \theta_2 + arg(z_2 \sin\theta_1 + \cos\theta_1) + \theta_1 - \theta'_1 \\ &= \{\theta'_2 + arg(z_2 \sin\theta_1 + \cos\theta_1) - 2\pi w(\sigma_1, \sigma_2)\} + 2\pi\{k_1 + k_2 + w(\sigma_1, \sigma_2)\} \\ &= \theta'_3 + 2\pi\{k_1 + k_2 + w(\sigma_1, \sigma_2)\}, \end{aligned}$$

we get

$$k_3 = k_1 + k_2 + w(\sigma_1, \sigma_2).$$

Namely, it was proved that the factor set of the group extension $\tilde{G}$ over G is nothing but $w(\sigma_1, \sigma_2)$ of (3) itself.

1-3. We show here that the factor set $w(\sigma_1, \sigma_2)$ does not split, and more strongly, never split even modulo any natural number $n \geq 2$.

THEOREM 1. *There exists no mapping v's from G into $\boldsymbol{R}$, such that: for any $\sigma_\nu \in G$, $\nu = 1, 2$,*

$$w(\sigma_1, \sigma_2) \equiv v(\sigma_1\sigma_2) - v(\sigma_1) - v(\sigma_2)\ (mod.\ n). \tag{6}$$

We prove the theorem after the next lemma, which can be verified by the direct calculation using (3).

LEMMA 3. *For any $\sigma_\nu \in G$, $\nu = 1, 2$, and any $\tau \in H$, where H is a subgroup of all elements of the type $\begin{pmatrix} a & b \\ & d \end{pmatrix}$; $a > 0$,*

$$w(\sigma_1, \sigma_2) = w(\tau\sigma_1, \sigma_2) = w(\sigma_1, \sigma_2\tau),$$
$$w(\sigma_1\tau, \sigma_2) = w(\sigma_1, \tau\sigma_2),$$
$$w(\sigma_1, \tau) = w(\tau, \sigma_2) = 0.$$

Proof of theorem 1. Let v be a mapping satisfying the relation (6). Because of lemma 3, for any $\tau_\nu \in H$, $\nu = 1, 2$,

$$v(\tau_1\tau_2) \equiv v(\tau_1) + (\tau_2)\ (mod.\ n).$$

In particular, from the following matrices' equality

$$\begin{pmatrix} 2 & \\ & 1/2 \end{pmatrix}\begin{pmatrix} 1 & 1/3 \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4/3 \\ & 1 \end{pmatrix}\begin{pmatrix} 2 & \\ & 1/2 \end{pmatrix},$$

we get

$$v\left(\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}\right) \equiv v\left(\begin{pmatrix} 1 & 4/3 \\ & 1 \end{pmatrix}\right) - v\left(\begin{pmatrix} 1 & 1/3 \\ & 1 \end{pmatrix}\right) \equiv 0\ (mod.\ n).$$

On the other hand, we can prove by another method which we shall mention about in lemma 4, $12v\left(\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}\right)$ is congruent to 1 $(mod.\ n)$. This is a contradiction, for $n \geq 2$. This completes the proof.

Remark. Theorem 1 gives a way to construct an n-fold covering group of G. Namely, we can define the group $\tilde{G}_n$ as the image of the homomorphism:

$$\tilde{G} \ni \tilde{\sigma} = (\sigma, k) \longrightarrow (\sigma, k \ (mod.\ n)),$$

then, $\tilde{G}_n$ is surely an n-fold covering group of G. Since its factor set $w(\sigma_1, \sigma_2)$ $(mod.\ n)$ does not split, $\tilde{G}_n$ is a non-trivial covering of G. The universal covering group $\tilde{G}$ can be considered as the projective limit of these $\tilde{G}_n$.

1-4. Finally in this section, let us determine the explicit value of our factor set of (3). It was firstly given by H. Petersson in [5], but his results are not handy but rather complicated. These are caused by the choice of $arg(cz+d)$, and Petersson calculated it by choosing $arg(cz+d)$ in $(-\pi, \pi]$. On the contrary, we choose $arg(cz+d)$ in $[-\pi, \pi)$ as stated at the first of this section, then the results become in a wonderfully simple form. Before our statement, it is convenient to introduce the following notation: for any two numbers c, d, we define a symbol $c(d)$ to be equal to c whenever $c \neq 0$, and to d only when $c = 0$.

Now, our results are as follows:

THEOREM 2. *The explicit value of* $w(\sigma_1, \sigma_2)$ *is given by the following table, where* $\sigma_\nu = \begin{pmatrix} a_\nu & b_\nu \\ c_\nu & d_\nu \end{pmatrix}$, $\nu = 1, 2, 3$, *and* $\sigma_1\sigma_2 = \sigma_3$:

$c_1(d_1)$	$c_2(d_2)$	$c_3(d_3)$	$w(\sigma_1, \sigma_2)$
$+$	$+$	$-$	$+1$
$-$	$-$	$+$	-1
	otherwise		0

Proof. For any $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, the following decomposition is easily seen:

$$\sigma = \begin{pmatrix} c^{-1} & a \\ & c \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \begin{pmatrix} 1 & dc^{-1} \\ & 1 \end{pmatrix}, \quad \text{if } c > 0,$$

$$\sigma = \begin{pmatrix} -c^{-1} & -a \\ & -c \end{pmatrix} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} 1 & dc^{-1} \\ & 1 \end{pmatrix}, \quad \text{if } c < 0,$$

and if $c = 0$, σ or $-\sigma$ is an element of the subgroup H defined in lemma 3. Combining these decompositions with lemma 3, the calculation of an arbitrary $w(\sigma_1, \sigma_2)$ is reduced to the cases of $\sigma_2 = I$, $-I$, S and $-S$; where I is the unit matrix and S is the matrix $\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$. And by direct calculations of these cases with the expression (2), all results are summarized as above. Of course, the method by Petersson in [5] can be applicable, too.

COROLLARY. *In particular, for* $S=\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$, $T=\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$, $I=\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ *and for any* $\sigma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in G$:

$$w(\sigma, I) = w(I, \sigma) = 0,$$
$$w(\sigma, -I) = w(-I, \sigma) = -1 \textit{ if } c(d) < 0, \textit{ and } 0 \textit{ if } c(d) > 0,$$
$$w(T^n, \sigma) = w(\sigma, T^n) = 0,$$
$$w(\sigma, S) = 1 \textit{ if } c(d) > 0, \textit{ and } d(-c) < 0, \textit{ and } 0 \textit{ otherwise},$$
$$w(\sigma, -\sigma) = 0,$$
$$w(\sigma_1, \sigma_2) = w(\sigma_2, \sigma_1) \textit{ whenever } \sigma_1\sigma_2 = \sigma_2\sigma_1.$$

This corollary will be used in the next section.

Remark 1. By theorem 2, it can be easily verified that the factor set for the double covering group $w(\sigma_1, \sigma_2)\,(mod.\ 2)$, or equivalently $(-1)^{w(\sigma_1, \sigma_2)}$ in multiplicative expression, does exactly coincide with Kutoba's factor set $a(\sigma_1, \sigma_2)$ in the real number field case ([4]).

Remark 2. Furthermore, if we define a new symbol $\langle a, b\rangle$ like Hilbert symbol, to be equal to 1 only when $a<0$ and $b<0$, and to 0 otherwise, then by theorem 2,

$$w(\sigma_1, \sigma_2) = -\langle c_1(d_1),\ c_2(d_2)\rangle + \langle -c_1(d_1)c_2(d_2),\ c_3(d_3)\rangle.$$

This expression is quite similar to the construction of $a(\sigma_1, \sigma_2)$ by Kubota ([4]).

2. The splitting formula of $w(\sigma_1, \sigma_2)$ on $SL(2, \boldsymbol{Z})$ and Dedekind sums.

As we have observed in theorem 1, the factor set $w(\sigma_1, \sigma_2)$ does not split on the whole group G. It, however, may split on a certain subgroup of G, and especially, we are interested in the case of an arithmetic subgroup. In particular, we shall deal with the very case of $SL(2, \boldsymbol{Z})$ (hereafter, we denote this group by Γ) in this section. We should notice here that the term "splitting" is used in a wider sense than usual, i.e. in the sense of theorem 3. In fact, there does not exist $\boldsymbol{Z}$-valued "v" satisfying the relation (9), but only rational valued "v". And such "v" is very interesting in our discussion, for it is naturally related to the theory of $log\eta$-multipliers or Dedekind sums.

2-1. We here fix the branch of $log(z)$ for an arbitrary complex number $z(\neq 0)$ as follows: $log(z) = log|z| + i\,arg(z)$, where $log|z|$ is in the real branch and $-\pi \leq arg(z) < \pi$ as before. Then, $log(cz+d)$ is a one-valued analytic function of z on the upper-half-plane $\mathscr{H}$, for any real pair $(c, d) \neq (0, 0)$.

In the above notation, (3) can be stated in another way, i.e.

$$w(\sigma_1, \sigma_2) = \frac{1}{2\pi i}\{log(c_2 z + d_2) + log(c_1(\sigma_2\langle z\rangle) + d_1) - log(c_3 z + d_3)\}. \tag{7}$$

Now, let us quote a little from the theory of modular forms. It is well-known that there is a unique (up to constant multiple) cusp form $\Delta(z)$ of dimension -12 for Γ and that is expressed as 24-th power of Dedekind η-function:

$$\eta(z) = e^{\pi iz/12}\prod_{n=1}^{\infty}(1 - e^{2\pi inz}), \quad z \in \mathscr{H}.$$

We need only the following transformation-equality:

$$\Delta(\sigma\langle z\rangle) = (cz+d)^{12}\Delta(z), \quad \text{for any } \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

Next, we choose the branch of $log\,\Delta(z)$ independently as follows:

$$log\,\Delta(z) = 2\pi iz - 24\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} e^{2\pi imnz}/m,$$

then $log\,\Delta(z)$ is also a one-valued analytic function. Therefore

$$N(\sigma) = \frac{1}{2\pi i}\{log\,\Delta(\sigma\langle z\rangle) - log\,\Delta(z) - 12\,log(cz+d)\} \tag{8}$$

is a rational integer determined only by σ, and independent of $z \in \mathscr{H}$.

Now, let $\sigma_\nu = \begin{pmatrix} a_\nu & b_\nu \\ c_\nu & d_\nu \end{pmatrix}$, $\nu = 1, 2, 3$, and $\sigma_1\sigma_2 = \sigma_3$, then,

$$log\,\Delta(\sigma_3\langle z\rangle) = log\,\Delta(z) + 12\,log(c_3 z + d_3) + 2\pi iN(\sigma_3).$$

On the other hand, the right-hand-side is also equal to

$$\begin{aligned} log\,\Delta(\sigma_1\langle\sigma_2\langle z\rangle\rangle) &= log\,\Delta(\sigma_2\langle z\rangle) + 12\,log(c_1(\sigma_2\langle z\rangle) + d_1) + 2\pi iN(\sigma_1) \\ &= log\,\Delta(z) + 12\{log(c_2 z + d_2) + log(c_1(\sigma_2\langle z\rangle) + d_1)\} \\ &\quad + 2\pi i\{N(\sigma_1) + N(\sigma_2)\}. \end{aligned}$$

Combining with (7), these imply the following:

$$12w(\sigma_1, \sigma_2) = N(\sigma_1\sigma_2) - N(\sigma_1) - N(\sigma_2).$$

In this way, we obtain the splitting theorem as follows:

THEOREM 3. *The factor set $w(\sigma_1, \sigma_2)$ does split in the unique way on $\Gamma = SL(2, \boldsymbol{Z})$, i.e. there exists only one mapping v from Γ into $\boldsymbol{R}$ such that*

$$w(\sigma_1, \sigma_2) = v(\sigma_1\sigma_2) - v(\sigma_1) - v(\sigma_2), \text{ for } \sigma_1, \sigma_2 \in \Gamma. \tag{9}$$

Furthermore, this v is rational-valued, in fact, $12v(\sigma) \in \boldsymbol{Z}$.

Proof. For the existence and the last assertion, it is enough that we put 1/12 of $N(\sigma)$ in (8) as $v(\sigma)$. Secondly, if there exist two v's satisfying the relation (9), their difference u satisfies the following property: $u(\sigma_1\sigma_2) = u(\sigma_1) + u(\sigma_2)$ for any σ_1, $\sigma_2 \in \Gamma$. On the other hand, since $S = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$ have relations $S^4 = (ST)^6 = I$, we have $4u(S) = 6(u(S) + u(T)) = u(I) = 0$ i.e. $u(S) = u(T) = 0$. It is also well-known that Γ is generated by these S and T, and so, $u(\sigma) = 0$ for arbitrary $\sigma \in \Gamma$. Thus the uniqueness was proved.

Remark 1. The factor set for the n-fold covering group $w(\sigma_1, \sigma_2)$ (mod. n) also splits on Γ, but not uniquely, in fact, it can be shown that there are 12 different ways of splitting for any case of $n \geq 2$.

Remark 2. In a certain sense, Γ is a maximal splitting subgroup for $w(\sigma_1, \sigma_2)$. In fact, we can prove that $w(\sigma_1, \sigma_2)$ never splits on any subgroup of $SL(2, \boldsymbol{Q})$ ($\boldsymbol{Q}$: the rational number field) which contains Γ properly.

Remark 3. For the existence proof of theorem 3, we used the property of modular forms, but essentially owing to Rademacher and others, it can be proved only by arithmetical means as we will be mentioned later.

2-2. Let us now investigate $v(\sigma)$ in theorem 3, in detail. This $v(\sigma)$ is represented by means of Dedekind sums, and we here show it by purely arithmetical steps. Hereafter, $v(\sigma)$ is always in the sense of theorem 3.

First, we need the values of $v(\sigma)$ for the special elements:

LEMMA 4. *$v(I) = 0$, $v(-I) = 1/2$, $v(S) = -1/4$, $v(T) = 1/12$, where $I = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$, $S = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$, and $T = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$ as before.*

Proof. The relations; $(-I)^2 = I$, $S^2 = (ST)^3 = -I$ are well-known. On the other hand, the value of $w(\sigma_1, \sigma_2)$ can be found by theorem 2 immediately. Thus we have the following equalities:

$$\begin{aligned}
&v(I) - 2v(I) = w(I, I) = 0,\\
&v(I) - 2v(-I) = w(-I, -I) = -1,\\
&v(-I) - 2v(S) = w(S, S) = 1,\\
&v(-I) - v(ST) - v((ST)^2) = w(ST, (ST)^2) = 1,\\
&v((ST)^2) - 2v(ST) = w(ST, ST) = 0,\\
&v(ST) - v(S) - v(T) = w(S, T) = 0.
\end{aligned}$$

If we solve the above as simultaneous equations, we get the assertion of the lemma. Moreover, if we solve them as congruence equations (*mod.* n), an equality

$$12v(T) \equiv 1 \pmod{n}$$

can be obtained, which was used in the proof of theorem 1.

LEMMA 5. *For* $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$,

$$v(-\sigma) = v(\sigma) + 1/2\, sgn(c(d)), \tag{10}$$

$$v(T^n\sigma) = v(\sigma T^n) = v(\sigma) + n/12, \tag{11}$$

$$v(\sigma S) = \begin{cases} v(\sigma) + 3/4, & \textit{if } c(d) > 0 \textit{ and } d(-c) < 0, \\ v(\sigma) - 1/4, & \textit{otherwise}, \end{cases} \tag{12}$$

$$v(\sigma^{-1}) = \begin{cases} -v(\sigma) + 1, & \textit{if } c = 0 \textit{ and } d < 0, \\ -v(\sigma), & \textit{otherwise}. \end{cases} \tag{13}$$

Proof. Because of theorem 2, its corollary and lemma 4, the lemma can be easily shown.

LEMMA 6. *For* $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, *such that* $c \neq 0$, *or* $c = 0$ *and* $d > 0$, *the equality*

$$v(\hat{\sigma}) = -v(\sigma) \textit{ holds, where } \hat{\sigma} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}.$$

Proof. Rather more complicated than lemma 5. But, the assertion is obvious for $\sigma = T$ or S, and the general case can be treated by induction, because Γ is generated by T and S, and $\hat{\sigma}_1 \hat{\sigma}_2 = \widehat{\sigma_1 \sigma_2}$.

Now, for $\sigma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ in the case of $c=0$, i.e. $\sigma=T^n$ or $-T^n$, $v(\sigma)$ is easily calculated by the use of (10), (11) of lemma 5:

$$v(\sigma)=\frac{b}{12d}+\frac{1-sgn(d)}{4}.$$

Next, let us consider the case of $c\neq 0$. By (11) of lemma 5 it is verified that

$$v(\sigma)-\frac{a+d}{12c} \text{ is dependent only on } c \text{ and } d(mod.\ c).$$

In consideration of (10) of lemma 5, it is natural to put as follows:

$$t(d,c)=v(\sigma)-\frac{a+d}{12c}+\frac{sgn(c)}{4}. \tag{14}$$

Then, particularly putting $S=\begin{pmatrix} & -1\\ 1 & \end{pmatrix}$ in σ, we have

$$t(0,1)=0. \tag{15}$$

In general, we have the following formulas on $t(d,c)$:

LEMMA 7. *For any $c\neq 0$, d such that $(c,d)=1$,*

$$t(d,c)=t(d',c), \text{ if } d\equiv d'(mod.\ c), \tag{16}$$

$$t(d,-c)=-t(d,c)=t(-d,c), \tag{17}$$

$$t(d,c)=t(a,c), \text{ if } ad\equiv 1\ (mod.\ c). \tag{18}$$

Proof. All assertion are got from lemma 5, lemma 6, and the definition (14).

Furthermore, $t(d,c)$ has the "reciprocity law":

LEMMA 8. *For any $c>0$, $d>0$ such that $(c,d)=1$,*

$$t(d,c)+t(c,d)=\frac{1}{4}-\frac{1}{12}\left(\frac{d}{c}+\frac{c}{d}+\frac{1}{cd}\right). \tag{19}$$

Proof. Let $\sigma_1=\begin{pmatrix}d & b\\ c & a\end{pmatrix}$ be an element of Γ, and $\sigma_2=\sigma_1^{-1}S=\begin{pmatrix}a & -b\\ -c & d\end{pmatrix}\times\begin{pmatrix} & -1\\ 1 & \end{pmatrix}=\begin{pmatrix}-b & -a\\ d & c\end{pmatrix}$. Then, $w(\sigma_1,\sigma_2)=0$ by theorem 2. So, $v(\sigma_1)+v(\sigma_2)=v(S)=-1/4$. Combining this with (14), we get:

$$t(a,c)+t(c,d)+\frac{1}{12}\left(\frac{a+d}{c}+\frac{c-b}{d}\right)-\frac{1}{2}=-\frac{1}{4}.$$

Because of (18) and $ad - bc = 1$, the proof of lemma 8 is completed.

The symbol $t(d, c)$ is characterized by the above relations; (15)$\sim$(19). In fact, for any $c \neq 0$, d such that $(c, d) = 1$, we can get the explicit value of $t(d, c)$ inductively in finite steps by (15)$\sim$(19).

On the other hand, the following lemma can be proved only by arithmetical steps:

LEMMA 9. *(Rademacher and others)*
$t'(d, c) = -sgn(c) \cdot s(d, |c|)$ *satisfies all relations* (15)$\sim$(19). *Where,* $s(d, |c|)$ *is so-called Dedekind sum, i.e.*

$$s(d, |c|) = \sum_{k=1}^{|c|} \left(\left(\frac{k}{c}\right)\right)\left(\left(\frac{kd}{c}\right)\right),$$

and here, the symbol $((x))$ *denotes* $x - [x] - \frac{1}{2}$ *or* 0, *for* x *is not an integer or an integer, respectively.*

Proof. Well-known. For instance, we can refer to [8].

Therefore, $t(d, c)$ must coincide exactly with $t'(d, c)$. Thus, we have obtained *arithmetically* the following theorem on the explicit formula for $v(\sigma)$ in (9):

THEOREM 4. *The value of* $v(\sigma)$ *in theorem* 3 *is given by*

$$v(\sigma) = \begin{cases} \dfrac{b}{12d} + \dfrac{1 - sgn(d)}{4}, & \text{if } c = 0, \\[2ex] \dfrac{a + d}{12c} - sgn(c)\left\{\dfrac{1}{4} + s(d, |c|)\right\}, & \text{if } c \neq 0. \end{cases} \tag{20}$$

Remark. Summarizing all results in **2–1** and **2–2**, it has become possible to represent $\log\eta$-multipliers by means of Dedekind sums only by purely arithmetical steps, although this was firstly done by function theoretical means (Dedekind, [1]).

2–3. Finally, let us make some reconsideration about the reciprocity law of Dedekind sums. As it was observed in the proof of lemma 8, the reciprocity (19) is nothing but stated in other words for the formula:

$$w(\sigma_1, \sigma_2) = v(S) - v(\sigma_1) - v(\sigma_2), \tag{A}$$

where $\sigma_1\sigma_2 = S = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$.

And (A) is a special case of the splitting formula (6):

(B) $$w(\sigma_1, \sigma_2) = v(\sigma_1\sigma_2) - v(\sigma_1) - v(\sigma_2).$$

From these stand points, the very (B) should be called the generalized reciprocity law. Furthermore, the following modified formula is more convenient than (B), by its symmetrical expression:

(C) $$v(\sigma_1) + v(\sigma_2) + v(\sigma_3) = -\{w(\sigma_1, \sigma_2) + w(\sigma_3^{-1}, \sigma_3)\},$$

where $\sigma_1\sigma_2\sigma_3 = I$.

In fact, from (C) and (14), we can get the following "generalized reciprocity law of Dedekind sums":

THEOREM 5. *For* $\sigma_\nu = \begin{pmatrix} a_\nu & b_\nu \\ c_\nu & d_\nu \end{pmatrix}$ *such that* $\sigma_1\sigma_2\sigma_3 = I$ *and* $c_1c_2c_3 \neq 0$,

(21) $$t(d_1, c_1) + t(d_2, c_2) + t(d_3, c_3) = -\frac{sgn(c_1c_2c_3)}{4} + \frac{1}{12}\left\{\frac{c_1}{c_2c_3} + \frac{c_2}{c_3c_1} + \frac{c_3}{c_1c_2}\right\}.$$

Also the formula (21) was previously proved essentially by Rademacher ([6], see also Dieter, [2]). Besides he proved it by purely arithmetical method. Therefore, we can conclude that even the existence proof of theorem 3 can be done only by arithmetical steps, since (21) is equivalent to (C) or (B).

Remark 1. Throughout this section, we considered about the only case of $\Gamma = SL(2, \boldsymbol{Z})$. But an almost similar discussion is possible in the case of other arthmetic subgroups. For example, for the ϑ-group, we may get the explicit values for $log\,\vartheta$-multipliers by the same method, and the formula corresponding to (C) may be interesting. On $log\ \vartheta$-multipliers, Rademacher observed it by different ways, too ([7]).

Remark 2. As Kubota already remarked shortly in [3], if we study on the case of the double covering in the place of the universal covering, we may reach to the "generalized reciprocity law of quadratic residue symbols" as (21). And the classical reciprocity is corresponding to such as (A). For instance, the generalized reciprocity contains the formula as follows:

For $\sigma_\nu = \begin{pmatrix} a_\nu & b_\nu \\ c_\nu & d_\nu \end{pmatrix}$ *such that* $\sigma_1\sigma_2\sigma_3 = I$ *and* c_1, c_2, c_3 *are all positive odd numbers,*

(22) $$\left(\frac{d_1}{c_1}\right)\left(\frac{d_2}{c_2}\right)\left(\frac{d_3}{c_3}\right) = -\rho^{\left(\sum_{\nu=1}^{3} c_\nu(a_\nu+d_\nu-3)\right)}, \quad \rho = e^{\frac{\pi i}{4}},$$

where the symbol in the left-hand-side is Jacobi symbol.

What appears in the splitting formula for the case of n-fold covering ($n \geq 3$) may be more interesting.

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Mathematical Institute
Nagoya University

*) Added in proof. A part of the paper [3] is reproduced in the following: T. Kubota, On a classical theta function, Nagoya Math. J., **37** (1969), pp. 183~189.