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ON THE UNIQUENESS IN CAUCHY'S PROBLEM FOR ELLIPTIC SYSTEMS WITH DOUBLE CHARACTERISTICS

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Dedicated to Professor Katuzi Ono on his 60th birthday

1. We consider in the 2 dimensional space with the coordinate (x, y). Let Γ be a segment of the y-axis containing the origin in its interior and let Ω be a domain whose boundary contains Γ. We treat the solutions u_p $(p = 1, \cdots, m)$ of the elliptic system

$$\frac{\partial u_p}{\partial x} + \sum_{q=1}^{m} a_{p,q}(x, y)\frac{\partial u_q}{\partial y} + \sum_{q=1}^{m} b_{p,q}(x, y)u_q = 0, \tag{1. 1}$$

where $a_{p,q} \in C^1(\bar{\Omega})$, $b_{p,q} \in L^\infty(\bar{\Omega})$ and $u_p \in C^1(\bar{\Omega})$. The system (1. 1) is written in the form

$$U_x + AU_y + BU = 0, \tag{1. 2}$$

where $U = (u_1, \cdots, u_p)$, $A = (a_{p,q})$ and $B = (b_{p,q})$. The characteristics of this system are said to have multiplicities not greater than two in $\bar{\Omega}$, if the following condition is satisfied: There is a non-singular matrix T whose elements belong to $C^1(\bar{\Omega})$ such that the matrix

$$A' = T^{-1}AT$$

has the direct sum

$$A' = \begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_s \end{pmatrix}$$

of one- or two-rowed square blocks of the type

$$\alpha_k = (\lambda_k)$$

or

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$$\alpha_k = \begin{pmatrix} \lambda_k & \mu_k \\ 0 & \lambda_k \end{pmatrix},$$

respectively.

Douglis [3] showed in 1960 that if the characteristics of the system (1. 2) are complex and of multiplicities not greater than two in $\bar{\Omega}$, then any solution of (1. 2), which is zero on Γ, is identically zero in Ω.

If the direct sum A' consists of only one-rowed blocks, that is, the characteristics are distinct, then this theorem was proved by Carleman [2]. On the other hand uniqueness for elliptic equations, in any number of dimensions, whose characteristics are at most double was shown by several mathematiciens (see c.f. [5], [6], [10], [11], [12], [13],).

In this note we shall try to prove uniqueness in Cauchy's problem for the elliptic system (1. 2) under weaker assumptions. That is the following

MAIN THEOREM. *Assume that the characteristics of the system* (1. 2) *are complex* (*elliptic*) *and of multiplicities not greater than two in* $\bar{\Omega}$. *Then there is a positive constant* δ *such that if the solutions* u_p *of* (1. 2) *are in* $C^1(\bar{\Omega})$ *and satisfy*

$$u_p = o(\exp(-y^{-2\delta})) \quad (y \to 0, \ p = 1, \cdots, m)$$

along Γ, *then* $u \equiv 0$ *in* Ω.

For single elliptic equations of second order with real coefficients this theorem was proved in any dimension by Mergelyan [9], Landis [7] and Lavrentév [8]. When characteristics of (1. 2) are distinct, this statement was shown by the author [4]. Thus we proceed as in [4]. The method used in this note consists in establishing an energy integral estimates developed by Calderón [1] and Mizohata [10].

2. In this section we assume that Ω is a domain which contains the origin 0. We consider in Ω the first order elliptic system

$$(2.1) \qquad \begin{aligned} u_{1x} + \lambda u_{1y} + \mu u_{2y} &= f_1, \\ u_{2x} + \lambda u_{2y} &= f_2, \end{aligned}$$

where $u, \lambda \in C^1(\bar{\Omega})$. We set

$$U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \text{ and } \Lambda = \begin{pmatrix} \lambda & \mu \\ 0 & \lambda \end{pmatrix}.$$

Then the system (2. 1) is written in the form

$$LU \equiv U_x + \Lambda U_y = F. \tag{2. 2}$$

We put $|U| = |u_1| + |u_2|$. Let us prepare a mean value property for solutions of (2. 2).

PROPOSITION 1. *For solutions of* (2. 2), *it holds that if* $1 < p < 2$ *and* $0 < R < 1$,

$$|U(0)| \leqq CR^{(2/p)-2}\left\{\left(\iint_{r \leqq R} |U|^2 dxdy\right)^{1-\frac{1}{p}} + \left(\iint_{r \leqq R} |F|^2 dxdy\right)^{1-\frac{1}{p}}\right\}, \tag{2. 3}$$

where C *is a constant independent of* R *and depends only on the values of* U, Λ *and* F *in* Ω.

Proof. We denote simply by C the constants independent of R. We take a C^∞ function such that

$$\phi(r) = \begin{cases} 1 & \text{in } r \leqq R/2 \\ 0 & \text{in } r > R \end{cases}$$

and $|\phi_x|$, $|\phi_y| \leqq CR^{-1}$. Set $V = \phi U$. Then we see

$$LV = L\phi U + \phi F, \tag{2. 4}$$

where

$$L\phi = \phi_x \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \phi_y \begin{pmatrix} \lambda & \mu \\ 0 & \lambda \end{pmatrix}.$$

Let $E(x, y)$ be the fundamental solutions of the following elliptic system with constant coefficients

$$L_0 = \frac{\partial}{\partial x} + \Lambda(0) \frac{\partial}{\partial y}.$$

It is well known that

$$E(x, y) = O(r^{-1}) \qquad (r \to 0, \ r = (x^2 + y^2)^{1/2}). \tag{2. 5}$$

Since V has compact carrier and $V_x + \Lambda(0)V_y = LV + (\Lambda(0) - \Lambda)\cdot V_y$, we have from the property of E

$$U(0) = \iint_{r \leqq R} E(x, y) \{LV + (\Lambda(0) - \Lambda)V_y\} dxdy. \tag{2. 6}$$

We see by (2. 4) and (2. 5)

$$\left|\iint_{r\leqq R} E(x, y)LV\ dxdy\right| \leqq CR^{-1}\iint_{r\leqq R} r^{-1}(|U|+|F|)dxdy. \tag{2. 7}$$

By Green's formula we have

$$\iint_{r\leqq R} E(x, y)(\Lambda(0)-\Lambda)V_y\ dxdy = -\iint_{r\leqq R} E(x, y)(\Lambda(0)-\Lambda)_y V dxdy - \iint_{r\leqq R} E_y(x, y)(\Lambda(0)-\Lambda)V dxdy.$$

Hence we get from (2. 5)

$$\left|\iint_{r\leqq R} E(x, y)(\Lambda(0)-\Lambda)V_y\, dxdy\right| \leqq C\iint_{r\leqq R} r^{-1}|U|\,dxdy. \tag{2. 8}$$

Combining (2. 6), (2. 7) and (2. 8), we obtain

$$|U(0)| \leqq C\ R^{-1}\iint_{r\leqq R} r^{-1}(|U|+|F|)dxdy. \tag{2. 9}$$

Put $m=\max_{\Omega}|U|$. Then we see by Hölder's inequality

$$\iint_{r\leqq R} r^{-1}m^{-1}|U|\,dxdy \leqq \left(\iint_{r\leqq R} r^{-p}dxdy\right)^{1/p}\cdot \left(\iint_{r\leqq R}(m^{-1}|U|)^q dxdy\right)^{1/q},$$

where $p^{-1}+q^{-1}=1$. Since $q>2$ and $m^{-1}|U|\leqq 1$, it holds

$$\iint_{r\leqq R} r^{-1}m^{-1}|U|\,dxdy \leqq C\ R^{(2-p)/p}\left(\iint_{r\leqq R}(m^{-1}|U|)^2 dxdy\right)^{1/q}.$$

Thus we get

$$\iint_{r\leqq R} r^{-1}|U|\,dxdy \leqq C\ R^{\frac{2}{p}-1}\left(\iint_{r\leqq R}|U|^2\,dxdy\right)^{1-\frac{1}{p}}. \tag{2. 10}$$

Similarly we have

$$\iint_{r\leqq R} r^{-1}|F|\,dxdy$$
$$\leqq C\, R^{\frac{2}{p}-1}\left(\iint_{r\leqq R}|F|^2 dxdy\right)^{1-\frac{1}{p}}.$$

Combining (2. 9), (2. 10) and (2. 11) we have obtained the inequality (2. 3).

3. Let us denote by S_d an open disk with the center $(d/2, 0)$ and with the radius $d/2$. We put $\Omega_h = \{0 < x < h\} \cap S_1$, $\Gamma_h = \{0 \leqq x \leqq h\} \cap \partial S_1$, $l_h = \{x = h\} \cap S_1$ and $\|U(x, \cdot)\|^2 = \int_{l_h} |U(x, y)|^2 dy$. In this section we see how the local behavior of the solutions of (1. 2) are controled by the Cauchy data. We shall apply the method developped by Mizohata [10].

LEMMA 1 ([4]). *Let* $u \in C^1(\bar{\Omega}_a)$ *and* $u = o(\exp(-r^{-2\delta-\varepsilon}))$ $(r \to 0)$ *along* Γ_a *for some positive numbers* δ, ε. *Then there is a function* v *such that*

(3. 1) $\quad v \in C^0(\bar{\Omega}_a) \cap C^1(\bar{\Omega}_a - \{0\})$ *and* $v = u$ *on* Γ_a,

(3. 2) $\quad \|v(h)\|^2$, $\|v_x(h)\|^2$ *and* $\|v_y(h)\|^2$
$$= o(\exp(-h^{-\delta})) \quad (h \to 0).$$

The details of the proof are omitted (see [4]). Here we show only how the function is constructed. Let φ be a C^∞ function on real line such that

$$\int_{-\infty}^{\infty} \varphi(x)dx = 1 \text{ and the carrier of } \varphi \subset \{|x| \leqq 1\}.$$

Then we define v in the form

$$v(x, y) = \begin{cases} u(x, \sqrt{x-x^2})\, f_{k(x)/3}(y - \sqrt{x-x^2}) & \text{for } y \geqq 0 \\ u(x, -\sqrt{x-x^2})\, f_{k(x)/3}(y + \sqrt{x-x^2}) & \text{for } y < 0, \end{cases}$$

where $k(x) = \exp(-x^{-\delta-(\varepsilon/3)})$ and $f_s(x) = \int_{-2s}^{2s} \varphi((x-y)/s)s^{-1}ds$.

LEMMA 2 (*Mizohata* [10]). *Let* $\lambda \in C^1(\bar{\Omega}_a)$ *and* $w \in C^0(\bar{\Omega}_a) \cap C^1(\bar{\Omega}_a - \{0\})$. *We assume that the imaginary part of* $\lambda \neq 0$ *in* $\bar{\Omega}_a$ *and* $w = 0$ *on* Γ_a *and* $\|w(\varepsilon)\|$, $\|w_x(\varepsilon)\|$, $\|w_y(\varepsilon)\| \to 0$ $(\varepsilon \to 0)$. *Then there are positive constants* h_0, n_0 *and* c *depending only on* λ *and* w *such that if* $0 < h < h_0$, $n > n_0$, *it holds*

$$\int_0^h \varphi_n^2 \|w_x + \lambda w_y\|^2 dx + c\varphi_n^2(h)$$
$$\geqq \frac{1}{4n}\left(\int_0^h \varphi_n'^2 \|w\|^2 dx + \int_0^h \varphi_n^2 \|\lambda_2 w_y\|^2 dx\right),$$

where $\varphi_n(x) = (x + n^{-1})^{-n}$ *and* $\lambda = \lambda_1 + i\lambda_2$.

The proof is omitted (see [10]).

Now we consider in Ω_a the nonlinear elliptic system

$$u_{1x} + \lambda u_{1y} + \mu u_{2y} = F_1(x, y, u_1, u_2), \quad u_{2x} + \lambda u_{2y} = F_2(x, y, u_1, u_2), \tag{3. 3}$$

where λ, $\mu \in C^1(\bar{\Omega}_a)$ and the imaginary part of $\lambda \neq 0$ in $\bar{\Omega}_a$. We assume that

$$|F_i(x, y, u_1, u_2)| \leqq C(|u_1| + |u_2|) \quad (i = 1, 2). \tag{3. 4}$$

We prepare the following

PROPOSITION 2. *Let* u_1, u_2 *be in* $C^1(\bar{\Omega}_a)$ *and solutions of* (3. 3) *in* Ω_a. *If for some* $\varepsilon > 0$, $\delta > 1$,

$$u_1,\ u_2 = o(\exp(-r^{-2\delta-\varepsilon})) \quad (r \to 0) \quad \text{on } \Gamma_a,$$

then we have

$$\int_0^h \|u_i\|^2 dx = o(exp(-h^{-\delta})) \quad (h \to 0,\ i = 1, 2).$$

Proof. From Lemma 1 there are functions v_i $(i = 1, 2)$ such that

$$v_i \in C^0(\bar{\Omega}_a) \cap C^1(\bar{\Omega}_a - \{0\}) \text{ and } v_i = u_i \text{ on } \Gamma_a, \tag{3. 5}$$

$$\|v_i(h)\|^2,\ \|v_{ix}(h)\|^2 \text{ and } \|v_{iy}(h)\|^2 = o(\exp(-h^{-h-\frac{\varepsilon}{3}})) \quad (h \to 0). \tag{3. 6}$$

We put $w_i = u_i - v_i$. Then the equations (3. 3) are reduced to

$$w_{1x} + \lambda w_{1y} + \mu w_{2y} = G_1(x, y, u_1, u_2, v_{1x}, v_{2x}, v_{2y}),$$
$$w_{2x} + \lambda w_{2y} = G_2(x, y, u_1, u_2, v_{2x}, v_{2y}).$$

We easily see

$$|G_i| \leqq C(|u_1| + |u_2| + |v_{1x}| + |v_{2x}| + |v_{2y}|) \quad i = 1, 2. \tag{3. 7}$$

From now on we denote simply by c the constants independent of n and h. We have by Lemma 2 for $h < h_0$, $n > n_0$,

$$\int_0^h \varphi_n^2 \|G_1 - \mu w_{2y}\|^2 dx + c\varphi_n^2(h) \geqq \frac{1}{4n} \int_0^h \varphi_n'^2 \|w_1\|^2 dx \tag{3. 8}$$

and

$$(3.\ 9) \qquad \int_0^h \varphi_n^2 \|G_2\|^2 dx + c\varphi_n^2(h)$$
$$\geqq \frac{1}{4n}\left(\int_0^h \varphi_n'^2 \|w_2\|^2 dx + \int_0^h \varphi_n^2 \|\lambda_2 w_{2y}\|^2 dx\right).$$

Multiplying both sides of (3. 9) by $4nM$ for large M, we add (3. 9) to (3. 8). Then it holds

$$\int_0^h \varphi_n^2(2\|G_1\|^2 + 4nM\|G_2\|^2)dx + nc\varphi_n^2(h)$$
$$\geqq M\int_0^h \varphi_n'^2 \|w_2\|^2 dx + \frac{1}{4n}\int_0^h \varphi_n'^2 \|w_1\|^2 dx$$
$$+ (M-c)\int_0^h \varphi_n^2 \|\lambda_2 w_{2y}\|^2 dx.$$

Let us fix M such that $M - c > 0$. Then we obtain

$$nc\left\{\int_0^h \varphi_n^2(\|G_1\|^2 + \|G_2\|^2)\, dx + \varphi_n^2(h)\right\}$$
$$\geqq \frac{1}{n}\int_0^h \varphi_n'^2(\|w_1\|^2 + \|w_2\|^2)dx.$$

We substitute $w_i = u_i - v_i$ and (3. 7) into this inequality. Then we have

$$nc\left\{\int_0^h \varphi_n^2(\|u_1\|^2 + \|u_2\|^2 + \|v_{1x}\|^2 + \|v_{2x}\|^2 + \|v_{2y}\|^2)dx + \varphi_n^2(h)\right\}$$
$$+ \frac{c}{n}\int_0^h \varphi_n'^2(\|v_1\|^2 + \|v_2\|^2)dx$$
$$\geqq \frac{1}{n}\int_0^h \varphi_n'^2(\|u_1\|^2 + \|u_2\|^2)dx.$$

If $h + \frac{1}{n}$ is sufficiently small, we see

$$nc\left\{\int_0^h \varphi_n^2\, (\|v_{1x}\|^2 + \|v_{2x}\|^2 + \|v_{2y}\|^2)\, dx + \varphi_n^2(h)\right\}$$
$$+ \frac{c}{n}\int_0^h \varphi_n'^2\, (\|v_1\|^2 + \|v_2\|^2)dx$$
$$\geqq \frac{1}{n}\int_0^h \varphi_n'^2\, (\|u_1\|^2 + \|u_2\|^2)dx.$$

Combining (3. 6) and (3. 10), we obtain

$$n^2c\left(\frac{h}{2}+\frac{1}{n}\right)^{2n+2}\left\{n^{2n}\exp(-h^{-\delta-(\varepsilon/3)})+\left(h+\frac{1}{n}\right)^{-2n}\right\}$$
$$\geqq\int_0^{h/2}(\|u_1\|^2+\|u_2\|^2)dx.$$

Let us take $h+\frac{1}{n}$ sufficiently small and nh sufficiently large. As an easy computation shows, in order to prove

$$\int_0^{h/2}\|u_i\|^2dx=o\left(\exp\left(-\left(\frac{h}{2}\right)^{-\delta}\right)\right)\quad(h\to 0), \tag{3. 11}$$

it is sufficient to show that we can choose n in such a way that

$$n^{2n+2}\leqq\exp\left(\left(\frac{h}{2}\right)^{-\delta-\varepsilon_1}\right) \tag{3. 12}$$

and

$$n^2\left(\frac{hn+2}{2hn+2}\right)^{2n}\leqq\exp\left(-\left(\frac{h}{2}\right)^{-\delta-\varepsilon'}\right)\qquad \varepsilon_1,\ \varepsilon'>0 \tag{3. 13}$$

where ε_1 is a given number and ε' will be determined later. If there is a positive number $\bar{\varepsilon}$ such that

$$n^{1+\bar{\varepsilon}}\leqq\left(\frac{h}{2}\right)^{-\delta-\varepsilon_1}, \tag{3. 14}$$

then (3. 12) holds. Since nh is sufficiently large, if we show that

$$\left(\frac{3}{5}\right)^{2n}\leqq\exp\left(-\left(\frac{h}{2}\right)^{-\delta-\varepsilon'}\right), \tag{3. 15}$$

then (3. 13) holds. Let us take positive numbers ε', $\bar{\varepsilon}$ such that

$$\delta+\varepsilon'<(\delta+\varepsilon_1)/1+\bar{\varepsilon}.$$

Noting that $1<2\log\left(\frac{5}{3}\right)$, we can take n in such a way that

$$\frac{1}{2\log\frac{5}{3}}\left(\frac{2}{h}\right)^{\delta+\varepsilon}<n<\left(\frac{2}{h}\right)^{(\delta+\varepsilon_1)/(1+\bar{\varepsilon})} \tag{3. 16}$$

It is easily seen that the inequality (3. 16) implies (3. 14), (3. 15) and that $nh\to\infty$. Thus we have proved (3. 11).

We consider the following elliptic system in Ω_a.

$$U_x+AU_y=F(x,y,U), \tag{3. 17}$$

where $U=(u_1, \cdots, u_m)$, $F=(F_1, \cdots, F_m)$ and

$$\Lambda = \begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_s \end{pmatrix}.$$

We assume that $\alpha_k \in C^1(\bar{\Omega}_a)$ and α_k is one- or two-rowed square blocks of the type

$$\alpha_k = (\lambda_k)$$

or

$$\alpha_k = \begin{pmatrix} \lambda_k & \mu_k \\ 0 & \lambda_k \end{pmatrix},$$

respectively. Further let us assume that

$$|F_k(x, y, u_1, \ldots, u_m)| \leqq C(|u_1| + \cdots + |u_m|). \tag{3. 18}$$

Then we can prove the following in a quite similar manner as in Proposition 2.

COROLLARY 1. *Let U be in $C^1(\bar{\Omega}_a)$ and a solution of* (3. 17) *in Ω_a. If for some $\varepsilon > 0$, $\delta > 1$,*

$$U = o(\exp(-r^{-2\delta-\varepsilon})) \quad (r \to 0) \quad on \ \Gamma_a,$$

then we have

$$\int_0^h \|u_i\|^2 dx = o(\exp(-h^{-\delta})) \quad (h \to 0, \ i = 1, \cdots, m).$$

Now we can prove the following

THEOREM 1. *Let U be a solution of the elliptic system* (3. 17) *in Ω_a and U be in $C^1(\bar{\Omega}_a)$. Then if for some $\varepsilon > 0$, $\delta > 1$,*

$$U = o(\exp(-r^{-2\delta-\varepsilon})) \quad (r \to 0) \quad on \ \Gamma_a,$$

we have

$$U = o(\exp(-r^{-\delta})) \quad (r \to 0) \quad in \ S_{1/2} \cap \Omega_a.$$

Proof. We set $-2\delta - \varepsilon = -2\left(\delta + \frac{\varepsilon}{3}\right) - \frac{\varepsilon}{3}$. We regard $\delta + \frac{\varepsilon}{3}$ as new δ and $\frac{\varepsilon}{3}$ as ε in Theorem 1. Then by Corollary 1 we have

$$(3.19)\qquad \int_0^h \|u_i\|^2 dx = o\Big(\exp\Big(-h^{-\delta-\frac{\varepsilon}{3}}\Big)\Big) \quad (h \to 0,\ i = 1, \cdots, m)$$

For the point $(x^{(0)}, y^{(0)})$ in $S_{1/2}$ we denote by $r_1(x^{(0)}, y^{(0)})$ the radius of a circle tangent to S whose center is $(x^{(0)}, y^{(0)})$. It is easily seen that

$$(3.20)\qquad r_1(x^{(0)}, y^{(0)}) \sim x^{(0)} \quad (x^{(0)} \to 0).$$

Let us apply Proposition 1 for the disk with center $(x^{(0)}, y^{(0)})$ and with radius $r_1(x^{(0)}, y^{(0)})$. Putting $p = 3/2$ in (2. 3) we have for $(x^{(0)}, y^{(0)}) \in S_{1/2}$

$$\begin{aligned} &|u_i(x^{(0)}, y^{(0)})| \\ &\leqq C r_1(x^{(0)}, y^{(0)})^{-2/3}\Big\{\Big(\iint_{r_0 \leqq r_1(x^{(0)}, y^{(0)})} \\ &(\sum_{i=1}^m |u_i|^2) dx dy)^{1/3} \\ &+ \Big(\iint_{r_0 \leqq r_1(x^{(0)}, y^{(0)})} (\sum_{i=1}^m |F_i|^2) dx dy)^{1/3}\Big\}, \end{aligned}$$

where $r_0 = \sqrt{(x - x^{(0)})^2 + (y - y^{(0)})^2}$ and C is a constant independent of $(x^{(0)}, y^{(0)})$. Combining (3. 18) and (3. 21), we see

$$\begin{aligned} &|u_i(x^{(0)}, y^{(0)})| \\ &\leqq C\ r_1(x^{(0)}, y^{(0)})^{-2/3}\Big\{\int_0^{x_0 + r_1(x^{(0)}, y^{(0)})} (\sum_{i=1}^m |u_i|^2) dx dy\Big\}^{1/3}. \end{aligned}$$

By Corollary 1 and (3. 20), we obtain

$$|u_i(x^{(0)}, y^{(0)})| \leqq C\ x_0^{-2/3} \exp\Big(-x_0^{-\delta-\frac{\varepsilon}{3}}\Big).$$

Thus we have proved the theorem.

4. We consider the next transformation from (x, y)-plane to (θ, ρ)-plane as in [4].

$$(4.1)\qquad \rho = r^2/x, \quad \theta = \tan^{-1}(y/x).$$

Put $R_{1/2} = \{(\theta, \rho) \mid |\theta| < \pi/2,\ 0 < \rho < 1/2\}$. We eliminate the part $\rho = 1/2$ from the boundary of $R_{1/2}$ and denote the remainder by $\partial R_{1/2}$. And let us put $\bar{R}_a = R_a + \partial R_a$. Then the transformation (4. 1) maps $S_{1/2}$ onto $R_{1/2}$ in one-to-one way. And we see that this transformation and its inverse are C^∞. we have

$$\begin{pmatrix} \theta_x & \theta_y \\ \rho_x & \rho_y \end{pmatrix} = \begin{pmatrix} -\dfrac{1}{\rho}\tan\theta & 1/\rho \\ 1-\tan^2\theta & 2\tan\theta \end{pmatrix}.$$

For the function $u(x, y)$ in $\bar{S}_{1/2}$ we define a function $\tilde{u}(\theta, \rho)$ in $\bar{R}_{1/2}$ by

$$\tilde{u}(\theta, \rho) = \begin{cases} u(\theta, \rho) & \text{for } (\theta, \rho) \in R_{1/2} \\ u(0, 0) & \text{for } (\theta, \rho) \in \partial R_{1/2}. \end{cases}$$

From now on we denote $\tilde{u}(\theta, \rho)$ simply by u. It is easily seen that if $u(x, y) \in C^1(\bar{S}_{1/2})$ then $u(\theta, \rho) \in C^1(\bar{R}_{1/2})$.

We consider the next equation in $S_{1/2}$

$$u_x + \lambda u_y \equiv H, \tag{4. 3}$$

where $u \in C^1(\bar{S}_{1/2})$, $\lambda \in C^1(\bar{S}_{1/2})$ and the imaginary part of $\lambda \neq 0$ in $\bar{S}_{1/2}$. We set $f(\theta, \rho) = |\rho_x + \lambda\rho_y|^2\cos^4\theta$ and $\lambda = \lambda_1 + i\lambda_2$. Then by (4. 2) the equation (4. 3) is transformed into

$$u_\rho + \frac{1}{\rho}(Q + iP)u_\theta \equiv \tilde{H}, \tag{4. 4}$$

where

$$\tilde{H} = f^{-1}\cos^4\theta(\rho_x + \bar{\lambda}\rho_y)H,$$
$$P(\theta, \rho) = \lambda_2 f^{-1}\cos^2\theta,$$

and

$$Q(\theta, \rho) = f^{-1}\cos\theta\{\sin\theta(\sin^2\theta - \cos^2\theta) + \lambda_1(\cos^3\theta - 3\cos\theta\sin^2\theta) + 2|\lambda|^2\cos^2\theta\sin^2\theta\}.$$

LEMMA 3 ([4]). *The function $f(\theta, \rho)$ is in $C^1(\bar{R}_{1/2})$ and there is a positive constant m such that $f > m$ in $\bar{R}_{1/2}$. And QP_θ/P is continuous in $\bar{R}_{1/2}$.*

The proof is omitted (see [4]).

From now on we denote by $\| \ \|$ a L^2 norm with respect to θ ($|\theta| < \pi/2$). And we put

$$\phi_n(\rho) = \exp(n\rho^{-\delta}) \quad (\delta > 0, \ n > 0).$$

We denote ϕ_n simply by ϕ. Then we have

PROPOSITION 4 ([4]). *If* $u \in C^1(\bar{S}_{1/2})$ *and satisfies for some positive numbers* δ, $\varepsilon\,(\delta > 1)$,

$$u = o(\exp(-r^{-\delta-\varepsilon})) \quad (r \to o) \ \textit{in} \ S_{1/2},$$

then we have

$$\begin{aligned}
&\int_0^h \phi^2 \|u_\rho + \frac{1}{\rho}(Q + iP)\, u_\theta\|^2 d\rho \\
&\geqq n\delta(\delta + 1 - M) \int_0^h \frac{\phi^2}{\rho^{\delta+2}} \|u\|^2 d\rho \\
&\quad - cn\delta \int_0^h \frac{\phi^2}{\rho^{\delta+1}} \|u\|^2 d\rho - \frac{c}{h}\phi^2(h) \\
&\quad + \frac{1}{2}\int_0^h \left\| \frac{1}{\rho}\phi i P u_\theta - \phi' u \right\|^2 d\rho,
\end{aligned} \tag{4. 6}$$

where $M = \max_{R_{1/2}} \left| Q_\theta + 1 - \frac{P_\theta Q}{P} \right|$ *and* c *is a constant independent of* n, h *and* δ.

Since the proposition was shown in detail in [4], we omit the proof.

We consider the elliptic system (3. 3) in $S_{1/2}$. Then we have the following

PROPOSITION 5. *Let* u_1, u_2 *be in* $C^1(\bar{S}_{1/2})$ *and solutions of the elliptic system* (3. 3) *in* $S_{1/2}$. *If it holds for* $\delta > \max(2, M-1)$ (M *is the constant in* (4. 6))

$$u_i = o(\exp(-r^{-\delta})) \quad (r \to 0,\ i = 1, 2) \quad \textit{in} \ S_{1/2},$$

then u_i *vanish identically in a neighborhood of the origin.*

Proof. We denote simply by c the positive constant independent of n. We assume $u \not\equiv 0$ in $\rho < h$. And we shall show that $u = 0$ in $\rho < h/2$. Then we see by (4. 6)

$$\begin{aligned}
&c\left\{ \frac{\phi^2(h)}{h} + \int_0^h \phi^2 \left\| u + \frac{1}{\rho}(Q + iP) u_\theta \right\|^2 d\rho \right\} \\
&\geqq n \int_0^h \frac{\phi^2}{\rho^{\delta+2}} \|u\|^2 d\rho \\
&\quad + \int_0^h \left\| \frac{1}{\rho}\phi i P u_\theta - \phi' u \right\|^2 d\rho \\
&\geqq n \int_0^h \frac{\phi^2}{\rho^{\delta+2}} \|u\|^2 d\rho
\end{aligned} \tag{4. 8}$$

$$+\int_0^h \|\rho^{\frac{\delta}{2}-1}\phi i P u_\theta - \rho^{\frac{\delta}{2}}\phi' u\|^2 d\rho.$$

We set

$$I_n^2 = \int_0^h \|\rho^{\frac{\delta}{2}}\phi' u\|^2 d\rho$$

$$p_n^2 I_n^2 = \int_0^h \|\rho^{\frac{\delta}{2}-1}\phi i P u_\theta\|^2 d\rho.$$

Then (4. 8) becomes

$$c\left\{\frac{\phi^2(h)}{h} + \int_0^h \phi^2 \left\| u_\rho + \frac{1}{\rho}(Q+iP)u_\theta \right\|^2 d\rho \right\}$$
$$\geqq (p_n - 1)^2 I_n^2 + \frac{1}{n} I_n^2.$$

Hence we have

$$cn\left\{\frac{\phi^2(h)}{h} + \int_0^h \phi^2 \left\| u_\rho + \frac{1}{\rho}(Q+iP)u_\theta \right\|^2 d\rho \right\} \geqq p_n^2 I_n^2 + I_n^2 . \tag{4. 9}$$

We consider the first equation of the elliptic system. That is

$$u_{1x} + \lambda u_{1y} + \mu u_{2y} = F_1 \tag{4. 10}$$

Then this equation is transformed into

$$|\rho_x + \lambda\rho_y|^2 (\rho^{\frac{\delta}{2}} u_1)_\rho + (\theta_x + \lambda\theta_y)(\rho_x + \bar{\lambda}\rho_y)(\rho^{\frac{\delta}{2}} u_1)_\theta$$
$$= (\rho_x + \bar{\lambda}\rho_y)\rho^{\frac{\delta}{2}} F_1 - \mu(\rho_x + \bar{\lambda}\rho_y)\cdot$$
$$\rho^{\frac{\delta}{2}}(u_{2\rho}\rho_y + u_{2\theta}\theta_y) + \frac{\delta}{2}|\rho_x + \lambda\rho_y|^2 \cdot \rho^{\frac{\delta}{2}-1} u_1.$$

By Lemma 3 we see

$$|\rho_x + \lambda\rho_y|^2 = \cos^{-4}\theta f(\theta, \rho) \quad (f > m > 0 \text{ in } R_{1/2})$$

and

$$|\rho_x + \lambda\rho_y| \leqq \text{const.} \cos^{-2}\theta.$$

Thus (4. 10) becomes

$$
\begin{aligned}
&|(\rho^{\frac{\delta}{2}}u_1)_\rho + \frac{1}{\rho}(Q+iP)(\rho^{\frac{\delta}{2}}u_1)_\theta| \\
(4.\ 11) \qquad &\leqq c\{|F_1|+|u_1|+\rho^{\frac{\delta}{2}}\cos^2\theta\cdot \\
&\qquad \Big(\frac{1}{\rho}|u_{2\theta}|+|\tan\theta||u_{2\rho}|\Big)\Big\},
\end{aligned}
$$

where P and Q are of the type (4. 5). Let us regard $\rho^{\frac{\delta}{2}}u_1$ in (4. 11) as u in (4. 9). Then we have from (4. 9)

$$
\begin{aligned}
&c\Big\{\frac{1}{h}\phi^2(h)+\int_0^h \phi^2\|F_1\|^2\,d\rho+\int_0^h \phi^2\|u_1\|^2 d\rho \\
&+\int_0^h \phi^2\|\rho^{\frac{\delta}{2}-1}\cos^2\theta u_{2\theta}\|^2 d\rho \\
(4.\ 12) \qquad &+\int_0^h \phi^2\|\rho^{\frac{\delta}{2}}\cos\theta\ \sin\theta\ u_{2\rho}\|^2 d\rho\Big\} \\
&\geqq n\int_0^h \frac{\phi^2}{\rho^{\delta+2}}\|\rho^{\frac{\delta}{2}}u_1\|^2 d\rho.
\end{aligned}
$$

Now we consider the second equation of the elliptic system. Then we have from (4. 4) and (4. 5)

$$
(4.\ 13) \qquad |u_{2\rho}| \leqq |F_2| + c\rho^{-1}\cos\theta|u_{2\theta}|.
$$

By (4. 12) and (4. 13), we see

$$
\begin{aligned}
&c\Big\{\frac{1}{h}\phi^2(h)+\int_0^h \phi^2(\|F_1\|^2+\|F_2\|^2)d\rho \\
(4.\ 14) \qquad &+\int_0^h \phi^2\|\rho^{\frac{\delta}{2}-1}\cos^2\theta\ u_{2\theta}\|^2 d\rho \\
&\geqq n\int_0^h \frac{\phi^2}{\rho^{\delta+2}}\|\rho^{\frac{\delta}{2}}u_1\|^2 d\rho.
\end{aligned}
$$

On the other hand we have from (3. 3), (4. 5) and (4. 9)

$$
\begin{aligned}
&cn\Big\{\frac{1}{h}\phi^2(h)+\int_0^h \phi^2\|F_2\|^2 d\rho\Big\} \\
(4.\ 15) \qquad &\geqq \int_0^h \|\rho^{\frac{\delta}{2}-1}\phi iPu_{2\theta}\|^2 d\rho+\int_0^h \|\rho^{\frac{\delta}{2}}\phi' u_2\|^2 d\rho.
\end{aligned}
$$

Let us note that $|P|\geqq c\cos^2\theta (c>0)$. Then (4. 15) becomes

$$
(4.\ 16) \qquad cn\Big\{\frac{1}{h}\phi^2(h)+\int_0^h \phi^2\|F_2\|^2 d\rho\Big\}
$$

$$\geqq \int_0^h \phi^2 \| \rho^{\frac{\delta}{2}-1} \cos^2\theta \, u_{2\theta} \|^2 d\rho + n^2 \int_0^h \frac{\phi^2}{\rho^{\delta+2}} \|u_2\|^2 d\rho.$$

Multiplying both sides of (4. 16) by a large constant, we add (4. 16) to (4. 14). Then we obtain

$$c\left\{\frac{1}{h}\phi^2(h) + \int_0^h \phi^2(\|F_1\|^2 + \|F_2\|^2)d\rho\right\} \geqq \int_0^h \frac{\phi^2}{\rho^2}(\|u_1\|^2 + \|u_2\|^2)d\rho. \tag{4. 17}$$

Combining (3. 4) and (4. 17), we have for sufficiently small h

$$\frac{c}{h}\phi^2(h) \geqq \int_0^h \phi^2(\|u_1\|^2 + \|u_2\|^2)d\rho.$$

Hence

$$\frac{c}{h}\phi^2(h)\phi^{-2}\left(\frac{h}{2}\right) \geqq \int_0^{h/2}(\|u_1\|^2 + \|u_2\|^2)d\rho.$$

Let n tend to zero. Then $u_1 = u_2 = 0$ in $\rho < h/2$. Thus we have completed the proof.

We consider the elliptic system (3. 17) in $S_{1/2}$. Then we can prove the following in a quite similar manner as in Proposition 5

COROLLARY 2. *Let U be in $C^1(\bar{S}_{1/2})$ and a solution of the elliptic system* (3. 17) *in $S_{1/2}$. If it holds for $\delta > \max(2, M-1)$*

$$U = o(\exp(-r^{-\delta})) \quad (r \to 0) \ \textit{in} \ S_{1/2},$$

then U vanish identically.

Combining Theorem 1 and Corollary 2, we obtain

THEOREM 2. *Let U be a solution of the elliptic system* (3. 17) *in Ω_a and U be in $C^1(\bar{\Omega}_a)$. Then there is a positive number δ such that if for $\delta' > \delta$*

$$U = o(\exp(-r^{-\delta'})) \quad (r \to 0) \ \textit{on} \ \Gamma_a,$$

then $U = 0$ in a neighborhood of the origin.

Theorem 2 means our Main Theorem by an adequate coordinate transformation.

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