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Nagoya Math. J.
Vol. 37 (1970), 1–4

NOTE ON THE BEHAVIOR OF SOLUTIONS OF PARABOLIC EQUATIONS WITH UNBOUNDED COEFFICIENTS*)

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To Professor Katuzi Ono on the occasion of his 60th birthday

1. Let

$$L = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + c - \frac{\partial}{\partial t}$$

be a parabolic differential operator defined in $\Omega = R^n \times (0, \infty)$, where R^n is the n-dimensional Euclidean space, the point $x \in R^n$ is represented by its coordinates $(x_1, \cdots, x_n)$ and $a_{ij}(=a_{ji})$, b_i and c are functions in $(x, t) \in \Omega$. We assume that there exist constants $k_1(>0)$, K_1, $K_2(\geqslant 0)$, $K_3(>0)$ and K_4 such that

$$(1) \qquad \begin{cases} k_1(|x|^2+1)^{1-\lambda}|\xi|^2 \leqslant \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \leqslant K_1(|x|^2+1)^{1-\lambda}|\xi|^2, \\ |b_i| \leqslant K_2(|x|^2+1)^{1/2}, \ (1 \leqslant i \leqslant n), \\ c \leqslant -K_3(|x|^2+1)^{\lambda} + K_4 \end{cases}$$

for some $\lambda \in (0, 1]$.

Consider the Cauchy problem

$$(2) \qquad \begin{cases} Lu = 0 \quad \text{in } \Omega, \\ u(x, 0) = f(x). \end{cases}$$

2. Throughout this note, we shall say that $u(x, t)$ is a solution of the problem (2) when $u(x, t)$ is continuous in $\bar{\Omega} = R^n \times [0, \infty)$, twice continuously differentiable in Ω and satisfies (2).

Received October 21, 1968

*) This research was supported by the National Science council in Taiwan.

In this note we shall prove the following which is a general form of Krzyżański's theorem [2].

THEOREM. *Let* $u(x,t)$ *be a solution of the Cauchy problem* (2) *and* $|u(x,t)| \leqslant K_5 e^{\mu(|x|^2+1)^\lambda}$ *for some constants* K_5 *and* μ. *Assume that the coefficients of* L *satisfy* (1). *If the Cauchy data* $f(x)$ *is bounded in* R^n *and if*

$$\frac{1}{2K_1}[2K_1(1-\lambda) - k_1 n](\sqrt{K_2^2 n^2 + 4K_1 K_3} - K_2 n) + K_4 < 0, \tag{3}$$

then $u(x,t)$ *tends to zero uniformly in* $x \in R^n$ *as* t *tends to infinity.*

In the case of the differential operator

$$L_0 = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + (-K_3|x|^2 + K_3') - \frac{\partial}{\partial t},$$

we may take $k_1 = K_1 = 1$, $K_2 = 0$, $K_4 = K_3' - K_3$ and $\lambda = 1$ in Theorem. So the solution $u(x,t)$ of the Cauchy problem

$$\begin{cases} L_0 u = 0 \quad \text{in } \Omega, \\ u(x,0) = f(x) \end{cases}$$

for a bounded Cauchy data $f(x)$ tends to zero uniformly in $x \in R^n$ as t tends to infinity, if

$$\sqrt{K_3}n > K_3' - K_3.$$

3. To prove our theorem, we use the following.

LEMMA. *Let* α *be a positive root of the quadratic equation* $AX^2 + BX + C = 0$, *where* $B \geqslant 0$ *and* $C < 0$. *Then the function*

$$\varphi(t) = \alpha \tanh A\alpha t$$

satisfies the inequality

$$\varphi'(t) + A\varphi^2(t) + B\varphi(t) + C \leqslant 0.$$

The proof is given by the direct calculation, so we may omit it here.

Now we shall give the proof of Theorem.

Let $\varphi(t)$ and $\psi(t)$ be functions twice continuously differentiable in $[0,\infty)$. Putting

$$H(x,t) = \exp[-\varphi(t)(|x|^2+1)^\lambda + \psi(t)], \tag{4}$$

we see from (1) that

$$
\begin{aligned}
\frac{LH}{H} &= 4\lambda^2\varphi^2(t)(|x|^2+1)^{2\lambda-2}\sum_{i,j=1}^{n} a_{ij}x_ix_j + 4\lambda(1-\lambda)\varphi(t)(|x|^2+1)^{\lambda-2}\sum_{i,j=1}^{n} a_{ij}x_ix_j \\
&\quad - 2\lambda\varphi(t)(|x|^2+1)^{\lambda-1}\sum_{i=1}^{n}(a_{ii}+b_ix_i) + c + \varphi'(t)(|x|^2+1)^{\lambda} - \psi'(t) \\
&\leqslant (|x|^2+1)^{\lambda}[\varphi'(t) + 4K_1\lambda^2\varphi^2(t) + 2K_2n\lambda\varphi(t) - K_3] \\
&\quad + [4\lambda(1-\lambda)K_1\varphi(t) - 2\lambda k_1 n\varphi(t) + K_4 - \psi'(t)].
\end{aligned}
$$

So, if

$$\varphi(t) = \alpha \tanh 4K_1\lambda^2\alpha t \tag{5}$$

for the positive root

$$\alpha = \frac{-K_2n+\sqrt{K_2^2n^2+4K_1K_3}}{4K_1\lambda}$$

of the quadratic equation $4K_1\lambda^2X^2 + 2K_2n\lambda X - K_3 = 0$, then we see from Lemma that

$$\varphi'(t) + 4K_1\lambda^2\varphi^2(t) + 2K_2n\lambda\varphi(t) - K_3 \leqslant 0.$$

Further, it is easy to see that

$$\psi(t) = \frac{2K_1(1-\lambda) - k_1n}{2K_1\lambda}\log[\cosh 4K_1\lambda^2\alpha t] + K_4t \tag{6}$$

satisfies

$$4\lambda(1-\lambda)K_1\varphi(t) - 2\lambda k_1n\varphi(t) + K_4 - \psi'(t) = 0$$

for $\varphi(t)$ given by (5). Thus $H(x,t)$ given by (4) for $\varphi(t)$ in (5) and $\psi(t)$ in (6) satisfies

$$LH \leqslant 0$$

in Ω. It is evident that $H(x,0) = 1$.

As the Cauchy data $f(x)$ in (2) is bounded, we may assume $|f(x)| < M$ in R^n. If we put

$$w_+(x,t) = MH(x,t) + u(x,t),$$

then $Lw_+ = MLH + Lu = MLH \leqslant 0$ in Ω and $w_+(x,0) = M + f(x) \geqslant 0$. Moreover, we have clearly $|w_+(x,t)| \leqslant K_4' e^{\mu'(|x|^2+1)^{\lambda}}$ in Ω for some constants K_4'

and μ'. The maximum principle due to Bodanko [1] implies that $w_+(x,t) \geqslant 0$ in Ω, that is,

$$-MH(x,t) \leqslant u(x,t)$$

in Ω. We apply the same argument to $w_-(x,t) = MH(x,t) - u(x,t)$ as the above, we get

$$MH(x,t) \geqslant u(x,t).$$

Thus we obtain

$$\begin{aligned} |u(x,t)| &\leqslant MH(x,t) \\ &\leqslant M\exp\{[4K_1\lambda(1-\lambda)\alpha - 2\lambda k_1 na + K_4]t\} \end{aligned}$$

for α in (5) throughout Ω. From the assumption (3), it is obvious that $u(x,t)$ tends to zero uniformly in $x \in R^n$ as t tends to infinity.

Remark. Krzyżański [2] considered the case $\lambda = 1$ in our Theorem and gave an analogous result.

REFERENCES

[1] W. Bodanko, Sur le problème de Cauchy et les problèmes de Fourier pour les equations paraboliques dans un domaine non borné, Ann. Polo. Math., **18** (1966), 79–94.
[2] M. Krzyżański, Evaluations des solutions de l'equation lineaire de type paraboliques à coefficients non borné, Ann. Polo. Math., **11** (1961–62), 253–260.

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