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ABSOLUTE CONTINUITY OF MARKOV PROCESSES AND GENERATORS

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Introduction.

Let $(x_t, \zeta, \mathfrak{B}_t, P_x)$ be a (standard) Markov process with state space S defined on the abstract space Ω. Here, x_t is the sample path, ζ is the terminal time and $\mathfrak{B}_t$ is the smallest σ-field of Ω in which x_s, $s \leqq t$ are measurable. Let P'_x, $x \in S$ be another family of Markovian measures defined on $(\mathfrak{B}_t, \Omega)$. It is a known fact that $(\mathfrak{B}_t[t<\zeta], P'_x)$ is absolutely continuous with respect to $(\mathfrak{B}_t[t<\zeta], P_x)$ for any $t>0$ and $x \in S$, if and only if there exists a positive right continuous multiplicative functional (MF) M_t with $P_x(M_t) \leqq 1$, $x \in S$, $t \geqq 0$, such that it is the Radon-Nikodym derivative of $(\mathfrak{B}_t[t<\zeta], P'_x)$ with respect to $(\mathfrak{B}_t[t<\zeta], P_x)$, where $\mathfrak{B}_t[t<\zeta]$ is the σ-field in $[t<\zeta]$ formed by all $B \cap [t<\zeta]$, $B \in \mathfrak{B}_t$. Then there arises naturally the following problem; How can we characterize the class of all the Markov process which is absolutely continuous with respect to a given Markov process or, equivalently, the class of all the Markov process which is transformed through MF of a given Markov process? In particular can we characterize this class in terms of the generator of Markov process?

In case of Brownian motion, this problem is solved through the works of Maruyama [6], Motoo [8], Dynkin [1] and Wentzell [15]. It is roughly the following; the conservative Markov process which is absolutely continuous with respect to Brownian motion has the generator expressed as $\frac{1}{2}\Delta + \sum f_i \frac{\partial}{\partial x_i}$: Hence the transformation by MF is so-called that of "drift". On the other hand the same problem has been solved in case of Markov chain by Kunita-Watanabe [4]; two (minimal) Markov chains x_t and x'_t with the same state space S are mutually absolutely continuous if and only if $q_{x,y}=0$ implies $q'_{x,y}=0$ and vice versa, where $q_{x,y}=\lim_{t\downarrow 0}\frac{P_t(x,y)}{t}$ and $P_t(x,y)$ is the transition function of x_t ($q'_{x,y}$ is defined similarly from x'_t).

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From these two special cases, it is expected that the transformation of continuous Markov process through MF would be that by drift and that the transformation of purely discontinuous process would be that of Lévy measure. Further the transformation would be, in general, a suitable combination of the above two. To prove this conjecture, the author had to extend the generator of the Markov process in a specific way. The relation between the two Markov processes is then stated as that of the corresponding generators in the extended sense.

§1 and §2 are rather introductory parts. We will state several results on additive functional and stochastic integral by additive martingale. These are the reformulation and the extension of the works by Motoo-Watanabe [9] and Watanabe [16].

We will extends in §3 the domain of the generator in a specific way and then express the extended generator in the form

$$Au = \sum a_{ij} B_{\eta_i} B_{\eta_j} u + \sum b^i B_{\eta_i} u + \int [u(y) - u(\cdot) - \sum B_{\eta_i} u(\cdot)(\eta_i(y) - \eta_i(\cdot))] n(\cdot, dy),$$

where $\{a_{ij}\}$ is positive definite and symmetric, B_{η_i} is an operator of derivation and $n(x, \cdot)$ is a σ-finite measure (Theorem 3. 1).[1)]

In §4, we shall show how the extended generator A may be changed through the transformation by MF. Roughly, the extended generator A' of the transformed process becomes $A' = A + B$, where

$$Bu = \sum f^i B_{\eta_i} u + \int (u(y) - u(\cdot))(e^{f(\cdot, y)} - 1) n(\cdot, dy).$$

We will further obtain the conditions (stated (B) in Theorem 4. 1) concerning f^i and f, under which $A + B$ becomes conversely the extended generator of the transformed process. These conditions are complicated but it depends on the ellipticity of $\{a_{ij}\}$. For instance, it turns out that if $\{a_{ij}\}$ are uniformly elliptic, any bounded functions $\{f^i\}$ satisfy the condition (B). Conversely, if $\{a_{ij}\}$ degenerate on a neighborhood of a point, we can not choose $\{f^i\}$ to be linearly independent on the neighborhood.

As an application of §4, we shall discuss, in §5, how the boundary condition of diffusion process can be changed through the transformation by MF. The possibility of changing the boundary condition depends also on the ellipticity of the boundary operator. Suppose we are given a diffusion

1) Similar expression of the extended generator has been obtained by Skorohod [13].

process on a compact C^∞-manifold with the boundary condition $Lu=0$, where L is of the form

$$Lu=\sum\alpha^{ij}\frac{\partial^2 u}{\partial\eta_i\partial\eta_j}+\sum\beta^i\frac{\partial u}{\partial\eta_i}+\int\Big[u(y)-u(\cdot)-\sum\frac{\partial u}{\partial\eta_i}(\eta_i(y)-\eta_i(\cdot))\Big]n(\cdot,dy).$$

The possibility of changing the boundary condition depends on the ellipticity of $\{\alpha^{ij}\}$ and follows a similar rule as that of the extended generator.

§1. Stochastic integral.

Suppose we are given a standard Markov process $\boldsymbol{M}=(x_t,\zeta,\mathfrak{F}_t,P_x)$ with the state space S defined on the basic space Ω. Here, $x_t=x_t(\omega)$, $\omega\in\Omega$ is the sample path, ζ is the terminal time, $\mathfrak{F}_t$, $t\geqq 0$ is the increasing family of σ-fields of Ω and P_x, $x\in S$, is the family of Markovian measures on $(\Omega,\mathfrak{F}_t)$ starting from x. (We use the same notation as [3]). A stopping time T is called a quasi-hitting time (QHT) if $T(\theta_t)+t=T$ for $t<T$ and $\lim_{t\downarrow 0}T(\theta_t)+t=T$ holds a.e. P_x, $\forall x\in S$. We assume, throughout this paper, that the process $\boldsymbol{M}$ satisfies Meyer's Hypothesis (L) and that $\boldsymbol{M}$ is conservative, i.e., $P_x(\zeta=\infty)=1$ for all $x\in S$ or locally conservative, i.e., there exists an increasing sequence of QHT $\{T_n\}$ such that $T_n<\zeta$ and $\lim_{n\to\infty}T_n=\zeta$ holds a.e. P_x, $\forall x\in S$. By the latter assumption, each stopped process $\boldsymbol{M}_n=(x_{t\wedge T_n},+\infty,\mathfrak{F}_{t\wedge T_n},P_x)$ becomes a Hunt process.

A real valued stochastic process $X_t=X_t(\omega)$ is called a functional if it is $\mathfrak{F}_t$-measurable for each $t\geqq 0$ and, if there exists a set N of $\mathfrak{F}=\mathfrak{F}_\infty$ with $P_x(N)=0$, $\forall x\in S$ (N is called a null set) such that for $\omega\notin N$, $X_t(\omega)$ is right continuous and has left hand limits for $t<\zeta$ and $X_t(\omega)=X_\zeta^-(\omega)$ holds. Here $X_\zeta^-=\lim_{\varepsilon\downarrow 0}X_{\zeta-\varepsilon}$. A functional X_t is p-th integrable (integrable if $p=1$) if $E_x(X_t{}^p)<\infty$ (or P_x-ess sup $|X_t|<\infty$ if $p=\infty$) for each $x\in S$ and $0\leqq t<\infty$. Further, if the p-th integrable functional X_t is a martingale $(\mathfrak{F}_t,P_x)$ for all $x\in S$, X_t is called p-th integrable martingale. A functional X_t is called locally p-th integrable if there exists an increasing sequence of stopping times $\{T_n\}$ with the limit ζ such that each $X_{t\wedge T_n}$, $n\leqq 1$ is p-th integrable. Further, if each $X_{t\wedge T_n}$ is a martingale $(\mathfrak{F}_{t\wedge T_n},P_x)$ for all $x\in S$, X_t is called a locally p-th integrable martingale. In particular, if we can choose such $\{T_n\}$ as a sequence of QHT, X_t is called locally p-th integrable (martingale) relative to QHT. Such $\{T_n\}$ are called the associated stopping times (or QHT) of X_t. It should be noted that any local martingale X_t is quasi-left continuous, i.e., for any increasing sequence of stopping times $\{T_n\}$ with the limit T, $\lim_{n\to\infty} X_{T_n} = X_T$ holds on the set $[T<\infty]$, a.e. P_x, $\forall x \in S$.

A functional X_t is called an AF (additive functional) if, except for ω of a null set, $X_t + X_s(\theta_t) = X_{t+s}$ holds for all $0 \leqq t,\ s<\infty$. We denote by $\mathfrak{M}^p$ (or $\mathfrak{M}^{p,\mathrm{loc}}$) the set of all AF which are (locally) p-th integrable martingale. When $p=2$, we write $\mathfrak{M}^2$ or $\mathfrak{M}^{2,\mathrm{loc}}$ as $\mathfrak{M}$ or $\mathfrak{M}^{\mathrm{loc}}$ respectively. We also denote by $\mathfrak{A}^+$ or $\mathfrak{A}^{+\mathrm{loc}}$ the set of all integrable (or locally integrable) increasing AF. We put $\mathfrak{A}$ (or $\mathfrak{A}^{\mathrm{loc}}$) $= \{A = A^1 - A^2;\ A^i \in \mathfrak{A}^+\}$ (or $A^i \in \mathfrak{A}^{+\mathrm{loc}}$).

The following Tanaka's lemma plays a fundamental role in our later discussion (private communication).

LEMMA 1. 1. *Let X_t be an AF whose absolute values of jumps are dominated by a positive constant (independent of ω), then X_t is locally p-th integrable relative to QHT for every $1 \leqq p < \infty$.*

PROPOSITION 1. 1. *If X_t is of $\mathfrak{M}^{1,\mathrm{loc}}$, X_t is a locally integrable martingale relative to QHT.*

Proof. We assume first that X_t has bounded jumps. Then X_t is locally square integrable relative to QHT by Lemma 1. 1. Let $\{Q_n\}$ be the associated QHT of X_t[2] and $\{T_n\}$ be the associated stopping times of the local martingale X_t. Then $X_{t\wedge Q_n \wedge S_p}$ is a martingale by optional sampling theorem. Since

$$E(X^2_{t\wedge Q_n \wedge T_p}) \leqq E(X^2_{t\wedge Q_n}) < \infty,$$

$\{X_{t\wedge Q_n\wedge T_p}\ ;\ p=1,2,\cdots\}$ is P_x-uniformly integrable for each x and n. Then $X_{t\wedge Q_n} = \lim_{p\to\infty} X_{t\wedge Q_n \wedge T_p}$ is a square integrable martingale.

We shall next consider the general case. Let $\{T_n\}$ be the associated stopping times of the local martingale X_t. For a fixed $c>0$, we define R_n by induction as follows; $T_0 = 0$,

$$R_n = T_n \wedge \inf\{t > R_{n-1};\ |X_t - X_{R_{n-1}}| > c\}$$

Then, clearly $X_{t\wedge R_n}$ is integrable and $|X_{t\wedge R_n}^- - X_{t\wedge R_n}| < c$ holds a.e. for each fixed t. Hence $\Delta X_{t\wedge R_n} = X_{t\wedge R_n} - X^-_{t\wedge R_n}$ is integrable. Put $A^+_t = \sum_{n=1}^{\infty} \Delta X_{R_n}$

2) In case of locally conservative process, we may and do assume, through the later discussions, that each associated QHT T_n of a local martingale is strictly less than ζ, a.e. P_x, $\forall x \in S$.

$I(\Delta X_{R_n}>0$ and $R_n \leqq t)$ and $A_t^- = \sum_{n=1}^{\infty} \Delta X_{R_n} I(\Delta X_{R_n}<0$ and $R_n \leqq t)$[3]. Then, clearly A_t^+ and A_t^- are locally integrable and the absolute values of jumps of $X_t-(A_t^+ + A_t^-)$ are dominated by $2c$. This fact shows that

$$Y_t^+ = \sum_{\substack{\Delta X_s > 2c \\ s \leqq t}} \Delta X \quad , \quad Y_t^- = -\sum_{\substack{\Delta X_s < -2c \\ s \leqq t}} \Delta X_s$$

are locally integrable, purely discontinuous increasing AF. Then, Lemma 1. 2 described below concludes that there exists a continuous and increasing AF $\tilde{Y}_t^+$ and $\tilde{Y}_t^-$ such that $Y_t^+ - \tilde{Y}_t^+$ and $Y_t^- - \tilde{Y}_t^-$ are locally integrable martingale relative to QHT. Put $Z_t = X_t - (Y_t^+ - \tilde{Y}_t^+) + (Y_t^- - \tilde{Y}_t^-)$. Then Z_t has bounded jumps and hence it is a local martingale relative to QHT. It is now clear that X_t is a local martingale relative to QHT.

LEMMA 1. 2. *Let X_t be a purely discontinuous and increasing quasi-left continuous AF. If X_t is locally integrable, there exists a unique continuous increasing AF $\tilde{X}_t$ such that $X_t - \tilde{X}_t$ is a local martingale relative to QHT.*

Proof. By the Meyer decomposition, there exists a unique and continuous increasing process $\tilde{X}_t$ such that $X_t - \tilde{X}_t$ is a local martingale. We will show that $\tilde{X}_t$ is an AF. It is known that there exists an integrable sequence of AF X_t^n which is purely discontinuous, quasi-left continuous and increases to X_t. For each X_t^n, there exists a continuous and increasing AF $\tilde{X}_t^n$ such that $X_t^n - \tilde{X}_t^n$ is a martingale (See [16]). It is now easy to see that $\tilde{X}_t^n$ increases to $\tilde{X}_t$ and hence X_t is an AF.

It remains to prove that $X_t - \tilde{X}_t$ is a local martingale relative to QHT. By Lemma 1. 1 due to Tanaka, $\tilde{X}_t$ is locally integrable relative to QHT. Let $\{T_n\}$ be the associated QHT. Then $X_{t\wedge T_n} - \tilde{X}_{t\wedge T_n}$ is a martingale as is easily seen.

Two local martingales X_t and Y_t are called orthogonal if X_tY_t is a local martingale. We denote by $\mathfrak{M}_c^{1,\mathrm{loc}}$ the set of all $X_t \in \mathfrak{M}^{1,\mathrm{loc}}$ which is continuous in t. Then clearly $\mathfrak{M}_c^{1,\mathrm{loc}} \subset \mathfrak{M}^{\infty,\mathrm{loc}} \subset \mathfrak{M}^{2,\mathrm{loc}}$. We denote by $\mathfrak{M}_d^{1,\mathrm{loc}}$ the set of all $X_t \in \mathfrak{M}^{1,\mathrm{loc}}$ which is orthogonal to every elements of $\mathfrak{M}_c^{1,\mathrm{loc}}$. We have

PROPOSITION 1. 2. *Every $X_t \in \mathfrak{M}^{1,\mathrm{loc}}$ has a unique decomposition $X_t = X_t^c + X_t^d$, where $X_t^c \in \mathfrak{M}_c^{1,\mathrm{loc}}$ and $X_t^d \in \mathfrak{M}_d^{1,\mathrm{loc}}$.*

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3) $I(\Gamma)$ is the indicator function of the set Γ.

Proof. By the proof of Proposition 1. 1, X_t has a decomposition $X_t = Y_t + Z_t$, where Y_t has bounded jumps, Z_t is of finite variation, and Y_t and Z_t have no common jumps. Then Y_t and Z_t are orthogonal (See [7]). Therefore, it suffices to prove the proposition in case X_t has bounded jumps. Let $\{T_n\}$ be the associated QHT of $X_t \in \mathfrak{M}^{2,\mathrm{loc}}$. Then $X_{t\wedge T_n}$ is a square integrable martingale and is an AF of the stopped process $\boldsymbol{M}_n = (x_{t\wedge T_n}, \infty, \mathfrak{F}_{t\wedge T_n}, P_x)$. Hence there exists a continuous AF $X_t^{n,c}$ and discontinuous (orthogonal to $X_t^{n,c}$) $X_t^{n,d}$, both of which are square integrable martingales. Uniqueness of such decomposition implies $X_{t\wedge T_n}^{m,c} = X_t^{n,d}$ and $X_{t\wedge T_n}^{m,d} = X_t^{n,d}$ for each $m > n$. Hence there exists $X_t^c \in \mathfrak{M}_c^{1,\mathrm{loc}}$ and $X_t^d \in \mathfrak{M}_d^{1,\mathrm{loc}}$ such that $X_{t\wedge T_n}^c = X_t^{n,c}$ and $X_{t\wedge T_n}^d = X_t^{n,d}$. Uniqueness of the decomposition is clear.

PROPOSITION 1. 3. *For each $X_t \in \mathfrak{M}^{\mathrm{loc}}$, there exists a unique continuous increasing AF $\langle X\rangle_t$ such that $X_t^2 - \langle X\rangle_t$ is a local martingale.*

Proof. If X_t is square integrable, the proposition is known [9]. The reduction of the general case to this is made similarly as the preceding proposition.

Now, let X_t be an element of $\mathfrak{M}^{1,\mathrm{loc}}$ and $X_t = X_t^c + X_t^d$ be the decomposition of Proposition 1. 2. We define an increasing AF $[X]_t$ by

$$[X]_t = \sum_{s\leqq t} (\varDelta X_s)^2 + \langle X^c\rangle_t.$$

Then $X_t^2 - [X]_t$ is a local martingale ([7]). We define $[X, Y]$ for $X, Y \in \mathfrak{M}^{1,\mathrm{loc}}$ by $\frac{1}{4}\{[X+Y] - [X-Y]\}$. The following form of stochastic integral is due to Meyer [7].

THEOREM 1. 1. *Let $X \in \mathfrak{M}^{1,\mathrm{loc}}$ and $\varPhi$ be a very well measurable function such that $\int_0^t \varPhi_s^2 d[X]_s < \infty$ for each $t < \infty$ and $\sum_{s\leqq t} \varPhi_s \varDelta X_s I(\varPhi_s \varDelta X_s, \varDelta X_s \geqq 1)$ is locally integrable. Then there exists a unique local martingale Y_t satisfying*

$$\int_0^t \varPhi_s d[X, Z]_s = [Y, Z]_t \quad \forall Z \in \mathfrak{M}^{1,\mathrm{loc}}. \tag{1. 1}$$

In particular, if $\varPhi$ is of the form $\varPhi_t(\omega) = f(x_{t-})$, where f is a Borel function, then the above Y is an AF.

DEFINITION. Y of the above theorem is written as $\int \varPhi dX$ and is called the stochastic integral of $\varPhi$ by X.

Remark. Let $X_t \in \mathfrak{M}^{p,\mathrm{loc}}$ and $Y_t \in \mathfrak{M}^{q,\mathrm{loc}}$, where q is the conjugate of p. Then $[X, Y]_t$ is locally integrable and there exists a unique continuous AF $\langle X, Y\rangle_t$ which is the difference of two increasing AF, such that $[X, Y]_t - \langle X, Y\rangle_t$ is a local martingale. In particular if $p = q = 2$, the relation (1. 1) is equivalent to

$$\int \Phi d\langle X, Z\rangle = \langle Y, Z\rangle \quad \forall Z \in \mathfrak{M}^{\mathrm{loc}} \tag{1. 2}$$

and the stochastic integral $\int \Phi dX$ is the same one defined in [4]. We omit the details. (See [7]).

§2. Radon-Nikodym derivative of continuous and increasing *AF*.

Let φ be of $\mathfrak{A}_c^{+,\mathrm{loc}}$. A universal measurable function f on S is called locally p-th φ-integrable (locally φ-integrable if $p = 1$), if $\int_0^t |f(x_s)|^p d\varphi_s$ is locally integrable. Clearly, $\int_0^t f(x_s) d\varphi_s$ is of $\mathfrak{A}_c^{\mathrm{loc}}$ if f is locally φ-integrable. For $\varphi \in \mathfrak{A}_c^{+,\mathrm{loc}}$, there exists a σ-finite measure μ on S such that $\int f d\varphi = 0$ a.e. P_x, $\forall x \in S$ if and only if $f = 0$ a.e. μ. Furthermore, if μ and μ' have the above property, μ and μ' are mutually absolutely continuous. (See [9]). Such μ is called a canonical measure of φ.

Let φ be of $\mathfrak{A}_c^{\mathrm{loc}}$ and $\psi \in \mathfrak{A}_c^{+,\mathrm{loc}}$. φ is called aboslutely continuous with respect to ψ (denoted by $\varphi \prec \psi$) if any universal measurable set E of S with the property $\int_0^t I_E d\psi = 0$, satisfies $\int I_E d\varphi = 0$, where I_E is the indicator function of the set E. Similarly, φ is singular to ψ if there exists a universal measurable set E such that $\int I_E d\varphi = \varphi$ and $\int I_E d\psi = 0$ holds.

PROPOSITION 2. 1. *Let φ be of $\mathfrak{A}_c^{\mathrm{loc}}$ and ψ be of $\mathfrak{A}_c^{+,\mathrm{loc}}$. Then, φ is uniquely decomposed into the sum of two continuous and increasing AF φ^1 and φ^2, where $\varphi^1 \prec \psi$ and φ^2 is singular to ψ. Furthermore, there is a universal measurable function f on S such that $\varphi^1 = \int f d\psi$. f is unique up to measure 0 relative to a canonical measure of ψ.*

The above proposition is a trivial modification of a result obtained by Motoo-Watanabe [9]. Actually they have proved that for $A = \varphi + \psi$, there are nonnegative universal measurable functions g and h such that $\varphi = \int g dA$

and $\psi = \int h dA$ hold. We define $\varphi^1 = \int gh^{-1} I_E d\psi$ and $\varphi^2 = \varphi - \varphi^1$, where $E = \{x: h(x) \neq 0\}$. It is easy to see that these φ^1 and φ^2 are what we want.

DEFINITION. The above f is called the Radon-Nikodym derivative of φ^1 with respect to ψ.

Let φ be of $\mathfrak{A}_c^+$ such that (a) $\langle X, Y\rangle \prec \varphi$ is satisfied for every X and Y of $\mathfrak{M}^{\text{loc}}$ and, (b) φ_t has the decomposition $t \wedge \xi + \psi_t$, where ψ_t is singular to $t \wedge \xi$. The above φ is said to be canonical AF of the given standard process. Let μ be a canonical measure of the above φ. The pair (φ, μ) is called a canonical system of the standard process. A canonical system exists certainly. For, Motoo-Watanabe [9] showed that there exists a countable family $\{X^n\}$ of $\mathfrak{M}$ such that $\langle X^1\rangle \succ \langle X^2\rangle \succ \cdots$, each of them is orthogonal and every X of $\mathfrak{M}$ is expressed as $\sum \int f_n dX^n$. Then $\varphi_t^1 = t \wedge \xi + \langle X^1\rangle_t$ satisfies the condition (a) and $t \wedge \xi \prec \varphi^1$. Let f be the Radon-Nikodym derivative of φ^1 with respect to $t \wedge \xi$. Then $f \geqq 1$. Define $\varphi = \int f^{-1} d\varphi^1$. This φ satisfies the conditions (a) and (b).

We shall denote by $(X, Y)_\varphi$ the Radon-Nikodym derivative of $\langle X, Y\rangle$ with respect to a canonical AF φ. In particular $(X, X)_\varphi$ is denoted by $(X)_\varphi$.

PROPOSITION 2. 3. *Let φ be a canonical AF and μ, a canonical measure of φ. Then for every X, Y, Z of $\mathfrak{M}^{\text{loc}}$ we have*

(1) $(X, Y)_\varphi = (Y, X)_\varphi, \quad (X)_\varphi \geqq 0$ a.e. μ;

(2) $(X, Y + Z)_\varphi = (X, Y)_\varphi + (X, Z)_\varphi$ a.e. μ;

(3) $|(X, Y)_\varphi| \leqq (X)_\varphi^{\frac{1}{2}} (Y)_\varphi^{\frac{1}{2}}$ a.e. μ;

(4) *If* $Z = \int f dX$, *then* $(Z, Y)_\varphi = f(X, Y)_\varphi$ a.e. μ.

(1), (2) and (4) are immediate consequence of the definition, (3) follows from Lemma 10. 1 and its proof of [9].

A sequence $\{X^n\}$ of $\mathfrak{M}^{\text{loc}}$ is a Cauchy sequence if there exists a sequence of stopping times $\{T_p\}$ with the limit ζ such that, for each p, $E_x((X^n_{t\wedge T_p} - X^m_{t\wedge T_p})^2) \to 0$ as $n, m \to \infty$. Let $\{X^n\}$ be a Cauchy sequence. There exists a unique X of $\mathfrak{M}^{\text{loc}}$ such that for each p, $E_x((X^n_{t\wedge T_p} - X_{t\wedge T_p})^2) \to 0$ as $n \to \infty$ (See [4]). A subset $\mathfrak{N}$ of $\mathfrak{M}^{\text{loc}}$ is closed if any Cauchy sequence of $\mathfrak{N}$ has the limit in $\mathfrak{N}$. A subset $\mathfrak{N}$ of $\mathfrak{M}^{\text{loc}}$ is called a subspace if (i) $X, Y \in \mathfrak{N} \Longrightarrow X + Y \in \mathfrak{N}$, (ii) $X \in \mathfrak{N} \Longrightarrow \int f dX \in \mathfrak{N}$, where f is a locally square

$\langle X\rangle$-integrable function and (iii) $\mathfrak{N}$ is closed. Let $\mathfrak{N}$ be a subspace. The set of all Y which is orthogonal to every element of $\mathfrak{N}$ is denoted by $\mathfrak{N}^{\perp}$.

PROPOSITION 2. 4. *Let $\mathfrak{N}$ be a subspace of $\mathfrak{M}^{\text{loc}}$ and let F be a mapping from $\mathfrak{N}$ to the space of all locally φ-integrable function satisfying $F(X+Y)=F(X)+F(Y)$ and $F\left(\int gdX\right)=gF(X)$. There is a unique Z of $\mathfrak{N}$ satisfying $F(Y)=(Z,Y)_{\varphi}$, $\forall Y\in\mathfrak{N}$ if and only if there is a locally φ-integrable function f such that $|F(Y)|\leqq f^{\frac{1}{2}}(Y)_{\varphi}^{\frac{1}{2}}$ is satisfied for each Y of $\mathfrak{N}$.*

The above proposition is an analogue of the Riesz theorem of Hilbert space. "Only if" part is clear from Proposition 2. 3 (3). Set $\mathfrak{N}'=\{X\in\mathfrak{N};\ F(X)=0 \text{ a.e. } \mu\}$. Then $\mathfrak{N}'$ is a subspace. Indeed, the linear property (i) and (ii) is clear; we shall show $\mathfrak{N}'$ is closed. Let $\{X^n\}$ be a Cauchy sequence of $\mathfrak{N}$ and let X be its limit. Then

$$\left|\int_0^t F(X-X^n)\,d\varphi\right|\leqq\left(\int_0^t f^2 d\varphi\right)^{\frac{1}{2}}\langle X-X^n\rangle_t^{\frac{1}{2}}\to 0 \quad \text{in } L^{2,\text{loc}}.^{4)}$$

Since $F(X^n)=0$, $F(X)=0$. Thus $\mathfrak{N}'$ is closed.

We can choose Y from $(\mathfrak{N}')^{\perp}\cap\mathfrak{N}$ such that $(Y)_{\varphi}$ has the maximal support, i.e., $\{x:(Y)_{\varphi}(x)\neq 0\}$ contains $\{x:(Y')_{\varphi}(x)\neq 0\}$ a.e. μ for every Y' of $(\mathfrak{N}')^{\perp}\cap\mathfrak{N}$. Note the relation $F(Y)^2(Y)_{\varphi}^{-2}d\langle Y\rangle\leqq f\,d\varphi$. Then $Z=\int F(Y)(Y)_{\varphi}^{-1}dY$ is well defined as an element of $\mathfrak{N}$. It is easy to see that $(Z,X)_{\varphi}=F(X)$ holds for every X of $\mathfrak{N}'\cup\mathfrak{L}(Y)$ (linear sum), where $\mathfrak{L}(Y)=\left\{\int hdY;\ h \text{ is locally square } \langle Y\rangle\text{-integrable}\right\}$.

We shall finally prove $\mathfrak{N}=\mathfrak{N}'\cup\mathfrak{L}(Y)$. Let U be any element of $\mathfrak{N}$. Set $U^1=\int F(U)(Z)_{\varphi}^{-1}dz$. Then $U^1\in\mathfrak{L}(Y)$ and $F(U^1)=F(U)(Z)_{\varphi}^{-1}F(Z)=F(U)$. Therefore $F(U^2)=0$, where $U^2=U-U^1$. Consequently $U^2\in\mathfrak{N}'$. We can now conclude that $\mathfrak{N}'\cup\mathfrak{L}(Y)\supset\mathfrak{N}$. The converse relation is clear.

Following S. Watanabe [15], we shall define a Lévy measure. Let $f(x,y)$ be a $S\times S$-measurable function. Set $\sum_{s\leqq t,\ x_s\neq x_{s-}} f(x_{s-},x_s)$ by $P_f(t)$ if it is well defined. A kernel $n(x,dy)$ on S such that $n(x,\{x\})=0$ is called a Lévy measure if, for any f such that $P_f(t)$ is integrable,

$$P_f(t)-\int_0^t nf\,d\varphi^{5)} \tag{2. 4}$$

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4) There exists an increasing sequence of stopping times $\{T_p\}$ with the limit ζ such that each stopped process $\langle X-X^n\rangle_{t\wedge T_p}$ converges to 0 in L^2-sense.

5) $nf(x)=\int_S n(x,dy)f(y)$.

becomes a martingale. The existence of Lévy measure is proved in [15]. Let $\mathfrak{M}_d^{loc}$ be the orthogonal complement of $\mathfrak{M}_c^{loc}$. Then the set of all $P_f(t)-\int nfd\varphi$ which are locally square integrable is dense in $\mathfrak{M}_d^{loc}$. Let F_Q be the set of all $S\times S$-measurable functions $f(x,y)$ such that nf^2 is locally φ-integrable. Then for each $f\in F_Q$ we can associate Q_f of $\mathfrak{M}_d^{loc}$ in such a way that if $P_f(t)$ is locally integrable, Q_f agrees with (2. 4) and satisfies

$$(Q_f)_\varphi = nf^2. \tag{2. 5}$$

We shall write sometimes Q_f as $\int f(x_s, y)q(ds, dy)$.

§3. Extension of generator.

Let φ be of $\mathfrak{A}_c^+$ and $\mathfrak{D}(A_\varphi)$ be the set of all bounded measurable function u such that there exists a locally φ-integrable function f on S satisfying

$$X_t^u = u(x_t) - u(x_0) + \int_0^t fd\varphi \tag{3. 1}$$

is of $\mathfrak{M}^{loc}$. We define the operator A_φ for $\mathfrak{D}(A_\varphi)$ by $A_\varphi u = -f$. It is uniquely determined up to measure 0 relative to a canonical measure of φ. In particular $\mathfrak{D}(G)$ is the set of all bounded function u such that (3. 1) holds for bounded f and φ of the form $t\wedge\xi$. We define Gu for $u\in\mathfrak{D}(G)$ by $-f$. G with its domain $\mathfrak{D}(G)$ is the *generator* of the standard process.

The following proposition is immediate.

PROPOSITION 3. 1. *Let (φ, μ) be a canonical system. Then $\mathfrak{D}(G)\subset\mathfrak{D}(A_\varphi)$ and $A_\varphi u = Gu$ holds for $u\in\mathfrak{D}(G)$. Furthermore $\mathfrak{D}(G)$ coincides with $\{u\in\mathfrak{D}(A_\varphi);$ $A_\varphi u$ is bounded and agrees with* 0 a.e. $\nu\}$, *where ν is the canonical measure of the singular part ψ of φ relative to $t\wedge\xi$.*

Let $\{\eta_n\}$ be (at most) countable family of $\mathfrak{D}(A_\varphi)$. If S is a manifold, it is natural to take such $\{\eta_n\}$ as its coordinate system. We shall call a bounded measurable function u on S is of the class $C^2(S)$ if for each x_0 of S there exists a C^2-class function $U(y_1, \cdots, y_N)$ on $R^N (N=1,2,\cdots)$ such that $u(x) = U(\eta_1(x), \cdots, \eta_N(x))$ holds in a neighborhood of x_0. We define differential operators B_{η_i} for such $u\in C^2(S)$ by

$$
(3.2) \qquad \begin{cases} B_{\eta_i} u(x) = \dfrac{\partial U}{\partial y_i}(\eta(x)), & i = 1, \cdots, N, \\ \qquad\quad = 0, & i \geqq N. \end{cases}
$$

THEOREM 3.1. *Let (φ, μ) be a canonical system. Then $C^2(S) \subset \mathfrak{D}(A_\varphi)$ and every u of $C^2(S)$ has the following expression.*

$$
(3.3) \qquad A_\varphi u(x) = \frac{1}{2} \sum a^{ij}(x) B_{\eta_i} B_{\eta_j} u(x) + \sum b^i(x) B_{\eta_i} u(x)
$$

$$
+ \int [u(y) - u(x) - \sum B_{\eta_i} u(x)(\eta_i(y) - \eta_i(x))] n(x, dy), \quad \text{a.e. } \mu.
$$

Here a^{ij} are positive definite, symmetric, locally φ-integrable functions, and b^i are locally φ-integrable functions.

Proof. Let $X^{\eta_i} = X^i + Y^i$, $X^i \in \mathfrak{M}_c^{\text{loc}}$ and $Y^i \in \mathfrak{M}_d^{\text{loc}}$ be the decomposition of Proposition 1. 2. Then

$$
Y^i = \int (\eta_i(y) - \eta_i(x_s)) q(ds, dy)
$$

by [16]. Set $\phi^i = \int A_\varphi \eta_i d\varphi$. Let $U(y_1, \cdots, y_N)$ be a C^2-class function on R^N. Since $\eta_i(x_t) = X_t^i + Y_t^i + \phi^i - \eta_i(x_0)$, formula on stochastic integral [4] is applicable and we obtain

$$
U(\eta_1(x_t), \cdots, \eta_N(x_t)) - U(\eta_1(x_0), \cdots, \eta_N(x_0)) = X'_t + \phi'_t,
$$

where

$$
(3.4) \qquad X'_t = \sum \int_0^t \frac{\partial U}{\partial y_i} dX_s^i + \int_0^t [U(\eta_i(y)) - U(\eta_i(x_s))] q(ds, dy)
$$

$$
(3.5) \qquad \phi'_t = \frac{1}{2} \sum \int_0^t \frac{\partial^2 U}{\partial y_i \partial y_j} d\langle X^i, X^j \rangle + \sum \int_0^t \frac{\partial U}{\partial y_i} d\phi_i
$$

$$
+ \int_0^t \int_S n(x_s, dy) [U(\eta_i(y)) - U(\eta_i(x_s)) - \sum \frac{\partial U}{\partial y_i}(\eta_i(x_s))(\eta_i(y) - \eta_i(x_s))] d\varphi.
$$

Suppose u of $C^2(S)$ coincides with $U(\eta_1(x), \cdots, \eta_N(x))$ on a neighborhood V of a point x_0. Then we can conclude from (3. 4) and (3. 5) that $u(x_{t\wedge T}) - u(x_0) = X_{t\wedge T} + \phi_{t\wedge T}$, where $X_t \in \mathfrak{M}^{\text{loc}}$ and ϕ_t agrees with ϕ'_t replacing $U(\eta_i(x), \cdots, \eta_N(x))$ by $u(x)$ in the expression (3. 5). Here T is the hitting time for V^c. Set

(3. 6) $$a^{ij} = (X^i, X^j)_\varphi, \quad b^i = A_\varphi \eta_i.$$

Then we obtain the expression (3. 3) for x of V, provided $u \in \mathfrak{D}(A_\varphi)$. Since such $\{V\}$ covers S, we obtain (3. 3) for all x provided $u \in \mathfrak{D}(A_\varphi)$.

We shall show that $C^2(S) \subset \mathfrak{D}(A_\varphi)$. Let $u \in C^2(S)$ and, $\{V_m\}$ and $\{W_m\}$ be open coverings of S such that $\bar{V}_m \subset W_m$ and that for each m there exists $U_m(y_1, \cdots, y_{N_m})$ of C^2-class function on R^{N_m} satisfying $u(x) = U_m(\eta_1(x), \cdots, \eta_{N_m}(x))$. Set $V'_m = V_m - \bigcup_{k=1}^{m-1} V_k$ and define $T_1(\omega) = T_{W_m{}^c}(\omega)$ if $x_0(\omega) \in V'_m$, where $T_{W_m^c}$ is the hitting time for the set W_m^c. Define T_m by induction as $T_{m-1} + T_1(\theta_{T_{m-1}})$. A similar argument as the preceding paragraph concludes that $u(x_{(T_{m-1} \vee t) \wedge T_m}) - u(x_{T_{m-1}})$ is the sum of two elements belonging to $\mathfrak{M}^{\text{loc}}$ and $\mathfrak{A}_c^{\text{loc}}$; the latter is absolutely continuous with respect to φ. Therefore $u(x_{t \wedge T_m}) - u(x_0)$ has a similar decomposition. Note that T_m increases to $+\infty$. Then we can conclude that $u \in \mathfrak{D}(A_\varphi)$.

COROLLARY. *If u and v are of $\mathfrak{D}(A_\varphi)$, uv is of $\mathfrak{D}(A_\varphi)$ and*

(3. 7) $$(X^u, X^v)_\varphi = A_\varphi uv - uA_\varphi v - vA_\varphi u$$

holds. In particular if u and v are of $C^2(S)$, then $(X^u, X^v)_\varphi$ is expressed as

(3. 8) $$\sum a^{ij} B_{\eta_i} u B_{\eta_j} v + \int (u(x) - u(y))(v(x) - v(y)) n(x, dy).$$

Proof. It suffices to prove the case $u = v$. We may apply the theorem by setting $\eta_1 = u$. Then $u^2 \in \mathfrak{D}(A_\varphi)$ and from (3. 5) and (3. 6) we obtain

(3. 9) $$A_\varphi u^2 = (X_c^u)_\varphi + 2uA_\varphi u + \int (u(y) - u(x))^2 n(x, dy),$$

where X_c^u is the projection of X^u to $\mathfrak{M}_c^{\text{loc}}$. Note that

(3. 10) $$(X_d^u)_\varphi = \int (u(y) - u(x))^2 n(x, dy),$$

where X_d^u is the projection of X^u to $\mathfrak{M}_d^{\text{loc}}$. Therefore $(X^u)_\varphi = (X_c^u)_\varphi + (X_d^u)_\varphi = A_\varphi u^2 - 2uA_\varphi u$. The expression (3. 8) follows from (3. 7) by calculating the right hand of (3. 7) using (3. 3).

§4. Transformation by MF.

A functional M_t is a *MF* (*multiplicative functional*) if $M_t M_s(\theta_t) = M_{t+s}$ is satisfied for $t + s < \zeta$. We shall assume the following (M. 1) $\sim$ (M. 2) throughout this paper.

(M. 1) M_t is strictly positive for $t<\zeta$ and $M_t=0$ for $t\geqq\zeta$.

(M. 2) M_t is a local martingale.

It is known that there exists a standard process $(x_t^M, \zeta^M, \mathfrak{F}_t^M, P_x^M)$ having $P_t^M(x, E)=E_x(M_t;\ x_t\in E)$ as its transition function. We shall call such (x_t^M, P_x^M) an M_t-process of (x_t, P_x). The operator and generator etc. of (x_t^M, P_x^M) are denoted by A_φ^M, G^M etc. We define the M_t-process on the same space $(x_t, \mathfrak{B}_t, \Omega)$ as that of (x_t, P_x). Then we have

(4. 1) $$E_x(M_T;\ B,\ T<\zeta)=P_x^M(B;\ T<\zeta)\quad B\in\mathfrak{B}_T$$

for any stopping time T (see [5]). Thus if $P_x(T<\zeta)=1$, $(\mathfrak{B}_T, P_x)$ and $(\mathfrak{B}_T, P_x^M)$ are mutually absolutely continuous. Hence continuous functional Z_t with respect to (x_t, P_x) may be considered as a continuous functional with respect to (x_t, P_x^M). Furthermore, we obtain

(4. 2) $$E_x\Big(\int_0^{T\wedge t} M_s dZ_s\Big)=E_x(M_{t\wedge T}Z_{t\wedge T})=E_x^M(Z_{t\wedge T})$$

if Z_t is of $\mathfrak{A}_c$ and if $P_x(T<\zeta)=1$.

THEOREM 4. 1. *Let $(x_t, \zeta, \mathfrak{F}_t, P_x)$ be a standard process with a canonical system (φ, μ).*

(I) *Let (x_t, P_x^M) be a M_t-process. Then (φ, μ) is also a canonical system of (x_t, P_x^M) and $\mathfrak{D}(A_\varphi)=\mathfrak{D}(A_\varphi^M)$ holds. Moreover, $B=A_\varphi^M-A_\varphi$ is decomposed into the following two linear mappings B_1 and B_2 from $\mathfrak{D}(A_\varphi)$ to locally φ-integrable functions;*

(B_1) *$B_1uv=uB_1v+vB_1u$ holds for any u and v of $\mathfrak{D}(A_\varphi)$. There exists a locally φ-integrable function h such that*

(4, 3) $$|\sum f_n B_1 u_n|\leqq h^{\frac{1}{2}}\{\sum f_n f_m (X_c^{u_n}, X_c^{u_m})_\varphi\}^{\frac{1}{2}}$$

holds for any measurable $\{f_1, \cdots, f_N\}$ and $\{u_1, \cdots, u_N\}$ of $\mathfrak{D}(A_\varphi)$.

(B_2) *There exists a $S\times S$-measurable function $f(x, y)$ such that $n|e^f-1|$ is locally φ-integrable and B_2 is expressed as*

(4. 4) $$B_2u(x)=\int(u(y)-u(x))(e^{f(x,y)}-1)n(x, dy).$$

(II) *Conversely if A' is a linear operator with domain $\mathfrak{D}(A_\varphi)$ such that $B=A'-A_\varphi$ satisfies the conditions of* (I), *there exists a unique M_t-process such that $A_\varphi^M=A'$.*

Proof. We divide the proof into several steps.

1. Let M_t be a MF satisfying (M. 1) — (M. 2). Then it has the following expression

$$M_t=\exp\Big\{X_t+Q_{f_1}(t)-\frac{1}{2}\langle X\rangle_t-\int_0^t n(e^{f_1}-1-f_1)d\varphi+P_{f_2}(t)-\int_0^t n(e^{f_2}-1)d\varphi\Big\}. \tag{4. 5}$$

Here, $X\in\mathfrak{M}_c^{\text{loc}}$ and $f=f(x,y)$ is a $S\times S$-measurable function such that $n(f_1)^2$ and $n|e^{f_2}-1|$ are locally φ-integrable, where $f_1=fI_{\{f<1\}}$ and $f_2=f-f_1$. Conversely if X and f satisfy the above conditions, the MF defined by the right hand of (4. 5) satisfies (M. 1) and (M. 2). Furthermore, we have

$$M_t-1=\int_0^t M_s dX_s+\int_0^t M_s dQ_{(e^f-1)}(s). \tag{4. 6}$$

(See [4]).

2. Let $u\in\mathfrak{D}(A_\varphi)$. Set

$$Z_t=X_t+Q_{(e^f-1)}(t) \text{ and } Bu=(Z,X^u)_\varphi. \tag{4. 7}$$

We shall show that every $u\in\mathfrak{D}(A_\varphi)$ belongs to $\mathfrak{D}(A_\varphi^M)$ and satisfies $A_\varphi^M u=(A_\varphi+B)u$. Set $Y_t=\int_0^t MdZ$. Then it is a local martingale and further $X_t^u Y_t-\int_0^t MBud\varphi$ is also a local martingale. On the other hand, since

$$X_t^u Y_t=M_t(u(x_t)-u(x_0))-M_t\int_0^t A_\varphi ud\varphi-X_t^u, \tag{4. 8}$$

$$M_t(u(x_t)-u(x_0))-M_t\int_0^t A_\varphi ud\varphi-\int_0^t MBud\varphi$$

is a local martingale. Let $\{S_p\}$ be the associated stopping times of the above local martingale. We may assume, without loss of generality, that each $M_{t\wedge S_p}$ is a martingale. Then,

$$E_x(M_{t\wedge S_p}u(x_{t\wedge S_p}))-u(x)-E_x(M_{t\wedge S_p}\int_0^{t\wedge S_p}(A_\varphi+B)ud\varphi)=0 \tag{4. 9}$$

or equivalently,

$$E_x^M(u(x_{t\wedge S_p}))-u(x)-E_x^M\Big(\int_0^{t\wedge S_p}(A_\varphi+B)ud\varphi\Big)=0. \tag{4. 10}$$

Clearly, $u(x_t)-u(x_0)-\int_0^t(A_\varphi+B)ud\varphi$ is a locally square integrable AF of the M_t-process. Hence u belongs to $\mathfrak{D}(A_\varphi^M)$ and we get

$$A_\varphi^M u = (A_\varphi + B)u. \tag{4. 11}$$

3. We shall next prove $\mathfrak{D}(A_\varphi^M) = \mathfrak{D}(A_\varphi)$. Define M_t^1 by M_t^{-1} if $t < \zeta$ and by 0 if $t > \zeta$. Then M_t^1 is a MF satisfying $(M. 1) \sim (M. 2)$ relative to (x_t, P_x^M) and M_t^1-process of (x_t, P_x^M) coincides with (x_t, P_x). Let φ^1 be a canonical AF of (x_t^M, P_φ^M). Then $\mathfrak{D}(A_{\varphi^1}^M) \subseteqq \mathfrak{D}(A_{\varphi^1})$ by the preceding paragraph. Let $u \in \mathfrak{D}(A_{\varphi^1}^M)$ and write AF defined by (3. 1) relative to (x_t, P_x) and (x_t, P_x^M) by X_t^u and $X_t^{u,M}$ respectively. Then Lemma 4. 1 given after the proof of this theorem concludes that $\langle X_c^u \rangle = \langle X_c^{u,M} \rangle^M$. Note that $(e^{f(x,y)}n(x, dy), \varphi)$ is a Lévy system of (x_t, P_x^M) (see [4]). Then we obtain

$$\langle X^{u,M} \rangle^M = \langle X_c^u \rangle + \iint (u(y) - u(x))^2 e^{f(x,y)}\, n(x, dy) d\varphi\,. \tag{4. 12}$$

which implies $\langle X^{u,M} \rangle^M \prec \varphi$. This show that we can choose φ as a canonical AF of (x_t, P_x^M). Thus we obtains $\mathfrak{D}(A_\varphi) \subset \mathfrak{D}(A_\varphi^M) \subset \mathfrak{D}(A_\varphi)$.

4. It remains to prove the conditions (B_1) and (B_2). Set $B_1 u = (X, X_c^u)_\varphi$ and $B_2 u = (Q_{(e^f - 1)}, X_d^u)_\varphi$, where X and $Q_{e^f - 1}$ are the ones appeared in (4. 7). Then $B_2 u$ satisfies (B_2) by (2. 5). (B_1) follows from

$$\begin{aligned} |\textstyle\sum f_n B u_n| &= |(X, \textstyle\sum \int f_n dX_c^{u_n})_\varphi| \\ &\leqq (X)_\varphi^{\frac{1}{2}} \left(\textstyle\sum \int f_n dX_c^{u_n}\right)_\varphi^{\frac{1}{2}} \\ &\leqq (X)_\varphi^{\frac{1}{2}} \{\textstyle\sum f_n f_m (X_c^{u_n}, X_c^{u_m})_\varphi\}^{\frac{1}{2}}\,. \end{aligned}$$

Thus we have proved (I) of Theorem 4. 1.

5. Conversely let A' be the operator of (II). We define $F(Y) = \sum f_n B u_n$ if $Y = \sum \int f_n dX_c^{u_n}$. Then $|F(Y)| \leqq h^{\frac{1}{2}}(Y)_\varphi^{\frac{1}{2}}$ holds. Since $\mathfrak{M}_c^{\mathrm{loc}}$ coincides with the closure of $\left\{\sum \int f_n dX_c^{u_n};\ n = 1, 2, \cdots\right\}$, F can be extended to $\mathfrak{M}_c^{\mathrm{loc}}$ uniquely in such a way that $F(Y) \leqq h^{\frac{1}{2}}(Y)_\varphi^{\frac{1}{2}}$ holds for every Y of $\mathfrak{M}_c^{\mathrm{loc}}$. Then there exists a unique X of $\mathfrak{M}_c^{\mathrm{loc}}$ satisfying $F(Y) = (X, Y)_\varphi$ for all $Y \in \mathfrak{M}_c^{\mathrm{loc}}$ by Proposition 2. 4. Define a MF by (4. 5) using this X and f. Then M_t-process is exactly what we want by the first part of this theorem.

LEMMA 4. 1. *Let* $u \in \mathfrak{D}(A_\varphi)$ *and let* $0 = t_0 < t_1 < \cdots < t_n \to +\infty$. *Then* $\sum_{t_n \leqq t} (u(x_{t_n}) - u(x_{t_{n-1}}))^2$ *converges to* $[X^u]_t$, i.e.,

$$\langle X^{u,c}\rangle_t + \sum_{s \leqq t} (u(x_s) - u(x_{s-}))^2 \tag{4. 13}$$

in the sense of $L^{2,\mathrm{loc}}$, *if* $\sup_n |t_n - t_{n-1}| \to 0$.

Proof. We may assume that X_t^u is square integrable. Meyer [7] has shown that $\sum_{t_n \leqq t} (X_{t_n}^u - X_{t_{n-1}}^u)^2$ converges to (4. 13), while the limit of the above coincides with the limit of $\sum_{t_n \leqq t} (u(x_{t_n}) - u(x_{t_{n-1}}))^2$ as is easily seen. Hence the lemma holds.

Condition (B_1) of Theorem 4. 1 is not clear. But for $u \in C^2(S) \subset \mathfrak{D}(A_\varphi)$, it can be rewritten in a clear form.

COROLLARY 1. *If* $u \in C^2(S)$, *then it holds that*

$$B_1 u = \sum f^i B_{\eta_i} u. \tag{4. 14}$$

Here, $\{f^i\}$ *are measurable functions such that there exists a locally* φ*-integrable function* h *satisfying*

$$|\sum f^i g_i| \leqq h^{\frac{1}{2}} (\sum a^{ij} g_i g_j)^{\frac{1}{2}} \tag{4. 15}$$

for any family of functions $\{g_i\}$. *In particular, if the range of* $C^2(S)$ *by* A_φ *is dense in the space of locally* φ*-integrable functions, the expression* (4. 1) *togetther with* (4. 1) *are equivalent to Condition* (B_1).

COROLLARY 2. *Suppose an operator* B_1 *satisfies the condition of Corollary* 1. *Then there exists an* M_t*-process such that* $A_\varphi^M u = (A_\varphi + B_1 + B_2)u$ *holds for every* $u \in C^2(S)$. *In particular, if the range of* $C^2(S)$ *by* A_φ *is dense in the space of locally* φ*-integrable functions, such* M_t*-process is unique.*

Proof of Corollary 1. Let V be an open set such that $u(x) = U(\eta_1(x), \cdots, \eta_N(x))$ for $x \in V$, where $U \in C^2(R^n)$. Then $X_t^{u,c}$ (= projection of X^u to $\mathfrak{M}_c^{\mathrm{loc}}$) is expressed as $\sum \int_0^t B_{\eta_i} u dX^i$ for $t < T = \inf\{t > 0;\ x_t \in V^c\}$ by (3. 4). Therefore, $B_1 u = \sum (X, X^i)_\varphi B_{\eta_i} u$ holds a.e. μ on V. Set $f^i = (X, X^i)_\varphi$. It is easy to see that $\{f^i\}$ satisfy (4. 15) by applying (4. 3).

The proof of Corollary 2 is similar to that of Theorem 4. 1 (II).

Remark. The condition (4. 15) is closely related to the ellipticity of a^{ij}. We shall discuss this problem in the next section.

§5. Transformation of diffusion process with a boundary condition.

As an application of Theorem 4. 1, we shall discuss how the boundary condition of diffusion can be changed through the transformation by *MF*.

Let D be a connected domain in N-dimensional manifold of class C^∞ and have compact closure $\bar{D}$. The boundary $\partial D = \bar{D} - D$ is assumed to be $N-1$-dimensional hypersurface of class C^3. Let A be an elliptic operator given on D by

$$(5.\ 1)\qquad Au = \sum_{i,j=1}^{N} a^{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b^i \frac{\partial u}{\partial x_i},$$

where a^{ij} are contravariant tenser of order 2 of class C^2 which is symmetric and positive definite, and b^i are vectors of C^1.

A conservative standard process whose semigroup maps C[6] into C and is strongly continuous is called a diffusion process if its infinitesimal generator G with the domain $\mathfrak{D}'(G)$ is a closed extension of A. Throughout this section,we shall consider diffusion process satisfying the following hypothesises.

HYPOTHESIS. I. *$(\alpha - A)u = f$ has a solution $u \in \mathfrak{D}'(G) \cap C^2$ for f of a dense subset of C.*

HYPOTHESIS. II. *For each x_0 of $\bar{D}$, there exists $(\eta_1, \cdots, \eta_N)$ of C^2-class functions which is a coordinate system on a neighborhood of x_0 and each η_i coincides with the difference of two bounded regular excessive functions.*

HYPOTHESIS. III. *The resolvent kernel of the process does not have mass on the boundary, i.e. $G_\alpha(x, \partial D) = 0$ holds for every $\alpha > 0$, $x \in \bar{D}$.*

Let $(\eta_1^x, \cdots, \eta_N^x)$ be a class of C^2-class functions satisfying the Hypothesis II and U_x be an open neighborhood of x in which $(\eta_1^x, \cdots, \eta_N^x)$ is a coordinate system. Let $\{U_{x_n}\}$ be a finite open covering of $\bar{D}$. We shall fix such $\{(\eta_1^{x_n}, \cdots, \eta_N^{x_n})\}$.

LEMMA 5. 1. *There exists a canonical AF φ such that each $\eta_i^{x_n}$ is of $\mathfrak{D}(A_\varphi)$ and $\int I_D d\varphi = t \wedge \zeta$.*

DEFINITION. We shall call $\int I_{\partial D}\, d\varphi$ a local time on the boundary.

---

6) The space of all continuous functions on $\bar{D}$.

Proof. By Hypothesis II, there exists a unique $\varphi^{i,n}$ of $\mathfrak{A}_c$ such that $\eta_i^{x_n}(x) = E_x\left(\int_0^\infty e^{-\alpha t} d\varphi_t^{i,n}\right)$. Then

$$X_t^{\eta_i^{x_n}} = \eta_i^{x_n}(x_t) - \eta_i^{x_n}(x_0) + \varphi_t^{i,n} - \alpha \int_0^t \eta_i^{x_n}(x_s) ds$$

is of $\mathfrak{M}$. We shall denote by $\psi_t^{i,n}$ the singular part of $\varphi_t^{i,n}$ relative to $t \wedge \zeta$. Let φ^1 be a canonical AF. Set $\varphi = \varphi^1 + \sum_{i,n} \psi^{i,n} \cdot 2^{-(i+n)}$. Then φ is a canonical AF and $\varphi_t^{i,n} - \alpha \int_0^t \eta_i^{x_n} ds \prec \varphi$, which implies $\eta_i^{x_n} \in \mathfrak{D}(A_\varphi)$.

By Theorem 3. 1, any u of C^2 is of $\mathfrak{D}(A_\varphi)$. Set $h = u - G_\alpha(\alpha - A)u$. Then we obtain

$$h(x) = E_x\left(\int_0^\infty e^{-\alpha t} A_\varphi u(x_s) d\varphi_s\right) - E_x\left(\int_0^\infty e^{-\alpha t} Au(x_s) ds\right).$$

Note that $(\alpha - A)h = 0$ on D. Then h coincides with $h' = E(e^{-\alpha t} h(x_{T-}))$ everywhere, where T is the hitting time for the set ∂D. Indeed we know that $h(x) = h'(x)$ except for irregular points of ∂D. Therefore $\beta G_{\alpha+\beta} h = \beta G_{\alpha+\beta} h'$ holds everywhere. Letting $\beta \to \infty$ we obtain $h = h'$ everywhere. Let R be the first time of t such that $\int_0^t A_\varphi u d\varphi - \int_0^t Au ds = 0$. Then $R \geqq T$ a. e. $P_x(\forall x)$. Consequently, the fine support of the above is included in ∂D by Getoor [2]. Thus we have

$$\int_0^t I_D A_\varphi u d\varphi = \int Au ds. \tag{5. 2}$$

Then by Corollary to Theorem 3. 1, we obtain that $\int I_D d\langle X^u, X^v\rangle \prec t \wedge \zeta$ for u, v of C^2. Since $\{X^u;\ u = G_\alpha f,\ \{f\}$ is dense in $C\}$ generates $\mathfrak{M}$([10]), we can conclude that $\int_0^t I_D d\langle X, Y\rangle \prec t \wedge \zeta$ for any X and Y of $\mathfrak{M}^{\text{loc}}$, using Hypothesis I. Hence we have proved the lemma.

COROLLARY. *Let φ be a canonical AF with the property of Lemma* 5. 1. *Then $A_\varphi = A$ if A_φ is restricted to D.*

Proof. By (5. 2), $A_\varphi u = Au$ holds for any u. Set $u = \eta_i$, then the coefficients b^i of A_φ and A coincide. Noting the formula (3. 8), it is easy to see that the coefficients a^{ij} of A_φ and A coincide each other.

By Hypothesis I, the infinitesimal generator is the closure of A restricted to the domain $\mathfrak{D}'(G) \cap C^2$, i.e., the process is characterized by the operator

A and the domain $\mathfrak{D}'(G) \cap C^2$. We shall characterize $\mathfrak{D}'(G) \cap C^2$ in terms of boundary condition.

THEOREM 5. 1. *Let* $(x_t, \zeta, \mathfrak{F}_t, P_x)$ *be a diffusion process with* ϕ *as a local time on the boundary. Let* ν *be a canonical measure of* ϕ. *There exists an operator* L *from* C^2 *to the space of locally* ϕ*-integrable functions expressed as*

$$Lu = \sum_{i,j=1}^{N-1} \alpha^{ij} \frac{\partial^2 u}{\partial \eta_i \partial \eta_j} + \sum_{i=1}^{N} \beta^i \frac{\partial u}{\partial \eta_i} + \int \Big[u(y) - u(x) - \sum \frac{\partial u}{\partial \eta_i}(x)(\eta_i(y) - \eta_i(x)) \Big] n(x, dy) \text{ a.e.} \tag{5. 3}$$

and $u \in C^2$ *is of* $\mathfrak{D}'(G)$ *if and only if* $Lu = 0$ a.e. ν. *Here* $(\eta_1, \cdots, \eta_N)$ *is a canonical coordinate system,*[7] a^{ij} *are positive definite and symmetric locally* ϕ*-integrable functions on* ∂D, β^i *are locally* ϕ*-integrable functions and* $n(x, dy)$ *is the Lévy measure.*

Such diffusion is called (A, L)-diffusion.

Remark. Wentzell [14] has obtained a similar expression of the boundary operator L without our Hypothesises I~III. But our result is sharper than his. In fact, it is not clear in [14] whether or not any C^2-class function satisfying Wentzell's boundary condition belongs to $\mathfrak{D}'(G)$.

Proof. Let L be the restriction of operator A_φ to ∂D. Then $u \in C^2$ is of $\mathfrak{D}'(G)$ if and only if $Lu = 0$ a.e. ν by Proposition 3. 1 and Hypothesis III. We may and do assume that canonical neighborhoods belonging to $\{U_{x_n}\}$ (defined at the paragraph after Hypothesis II). Then $(\eta_i, \cdots, \eta_N)$ of expression (3. 3) may be considered as a canonical coordinate system on a neighborhood of a boundary point. Let α^{ij} and β^i be the restrictions of coefficients a^{ij} and b^i of (3. 3) to ∂D, respectively. We have to prove $\alpha_{iN} = 0$ for $i = 1, \cdots, N$.

Let r be a nonnegative constant. By formula on stochastic integral [4], $\eta_N(x_t)^{2+r} - \eta_N(x_0)^{2+r}$ is the sum of the following X_t^r, Y_t^r and Z_t^r.

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7) For any x_0 of ∂D there exists a neighborhood U of x_0 and a co-ordinate system $(\eta_1, \cdots, \eta_N)$ such that $\partial D \cap U$ and $D \cap U$ are characterized by $\eta^N = 0$ and $\eta_N > 0$ respectively. We call such $(\eta_1, \cdots, \eta_N)$ a canonical co-ordinate system. The neighborhood U is called a canonical neighborhood. Let $\{U_1, \cdots, U_p\}$ be an open covering of canonical neighborhoods. The expression (5.3) means that Lu coincides with the right hand a.e. ν if $x \in U_i$ and $(\eta_1, \cdots, \eta_N)$ is a canonical co-ordinate of U_i.

$$X_t^r = (2+r)\int_0^t |\eta_N|^{1+r} F dX^N + \int \{|\eta_N(y)|^{2+r} - |\eta_N(x_s)|^{2+r}\} q(ds, dy)$$

$$Y_t^r = \frac{(2+r)(1+r)}{2}\int_0^t |\eta_N|^r d\langle X^N\rangle$$

$$Z_t^r = \int_0^t n\{|\eta_N|^{2+r} - |\eta_N|^{2+r} - (2+r)|\eta_N|^{1+r} F\} d\varphi$$

$$+ (2+r)\int_0^t |\eta_N|^{1+r} F A_\varphi \eta_N d\varphi,$$

where F is the function on R taking the value 1 on $x > 0$ and 0 on $x \leqq 0$. X_t^r and Z_t^r converge to X_t^0 and Z_t^0 $(r=0)$, while Y_t^r converges to $\int I_D d\langle X^N\rangle$ as r tends to zero. Consequently we obtain $\langle X^N\rangle = \int I_D d\langle X^N\rangle$, which implies $\alpha_{NN} = 0$. Now the inequality $|\alpha_{iN}| \leqq \alpha_{ii}^{\frac{1}{2}} \alpha_{NN}^{\frac{1}{2}}$ concludes that $\alpha_{iN} = 0$ for every i.

Remark. The boundary operator L depends on the choice of local time on the boundary, obviously. Let ψ' be another canonical AF and L' be the boundary operator relative to ψ'. Suppose $\psi \prec \psi'$ and let $f = \frac{d\psi'}{d\psi}$. Then $L = fL'$ as is easily seen.

Let $\{f^i, i = 1, \cdots, N\}$ be a contravariant tensor on the manifold D. We shall call that $\{f^i\}$ is associated with locally φ-integrable functions relative to a^{ij}, if there exists a set of measurable functions $\{f_i, i=1, \cdots, N\}$ on D such that $\sum a^{ij} f_i f_j$ is locally φ-integrable and $\sum a^{ij} f_j = f^i$ holds for each i. If the determinant $|a|$ of the matrix (a^{ij}) is not zero everywhere, such $\{f_i\}$ is unique and is equal to $\{\sum_j a_{ij} f^j\}$ where $a_{ij} = A^{ij}/|a|$ and A^{ij} is the cofactor of a^{ij}. Hence a contravariant vector $\{f^i\}$ is associated with locally-φ-integrable functions if and only if each f^i is measurable and $\sum a_{ij} f^i f^j$ is locally φ-integrable.

THEOREM 5. 2. *Let (x_t, P_x) be an (A, L)-diffusion on $\bar{D}$ with a canonical system (φ, μ). Let (x_t, P_x) be an (A', L')-diffusion on $\bar{D}$ with the same canonical measure. Then (A', L')-diffusion is an M_t-process of (A, L)-diffusion if and only if the following conditions* (1) *and* (2) *are satisfied.*

(1). (a) $a^{ij} = a^{ij\prime}$, (b) $b^i - b^{i\prime}$ *is a contravariant vector on D associated with locally $t \wedge \zeta$-integrable functions relative to a^{ij}.*

(2). *There exists a φ-integrable function f on ∂D such that $L'' = fL'$ satisfies* (a$'$) $\alpha^{ij} = \alpha^{ij\prime\prime}$, (b$'$) $\{\beta^i - \beta^{i\prime\prime}\}$ *is a contravariant vector on ∂D associated with locally φ-integrable functions relative to α^{ij}, and* (c$'$) *there exists a bounded $\bar{D}$-measurable* *function* $f(x, y)$ *such that* nf^2 *is locally* φ*-integrable and* $n'(x, dy) = e^{f(x,y)}n(x, dy)$ a.e. μ_2. *Here* μ_2 *is the restriction of* μ *to* ∂D.

Proof. Suppose that (A', L')-diffusion is an M_t-process of (A, L)-diffusion. Then there exists a φ-integrable function f on ∂D such that B of (4. 3) agrees with $A - A'$ on D and with $L - L''$ on ∂D by Theorem 5. 1 and the remark after that. Hence (a) and (a′) hold by Theorem 4. 1. Note that the operator B is invariant under the choice of co-ordinate system. Then so is $\sum (b^i - b^{i'}) \frac{\partial u}{\partial \eta_i}$. Hence $b^i - b^{i'}$ is a contravariant vector. We can easily conclude from (4. 4) that $b^i - b^{i'}$ is associated with a locally φ-integrable functions relative to a^{ij}. The proof of (2) is similar. "If" part of this theorem follows from the second half of Theorem 4. 1 and Hypothesis I.

Condition (b) and (b′) are closely related to the ellipticity of a^{ij} and $\alpha^{i,j}$.

Corollary 1. *If* α^{ij} *is uniformly elliptic, the condition* (b′) *is equivalent to that* $\{\beta^i - \beta^{i''}\}$ *is a contravariant vector such that each component is locally* φ*-integrable.*

Corollary 2. *If* $\alpha^{ij} \equiv 0$, (b′) *is equivalent to* $\beta^i = \beta^{i''}$. *In addition if* (A, L)*-diffusion has continuous path, then* $L = L''$.

The above corollary shows that the boundary condition can not be changed through M_t-transformation if the sample path is continuous and if $\alpha_{ij} \equiv 0$.

§6. Appendix. Diffusion process with the given boundary condition.

Suppose the boundary condition of the diffusion process is already known such as Wentzell [14] and Sato-Ueno [11]. We are interested in the relation between such boundary conditions and the one obtained in the preceding section probabilistically. Our main result in this section is that the diffusion process discussed in [11] satisfies Hypothesises I~III and the boundary operator defined in the preceding section coincides with the given one.

Let L be a boundary operator defined by

$$Lu = \sum_{i,j=1}^{N-1} \alpha^{ij} \frac{\partial^2 u}{\partial \eta_i \partial \eta_j} + \sum_{i=1}^{N} \beta^i \frac{\partial u}{\partial \eta_i} + \int \Big[u(y) - u(x)) - \sum \frac{\partial u}{\partial \eta_i} (\eta_i(y) - \eta_i(x)) \Big] n(x, dy). \tag{6. 1}$$

Here $(\eta_i, \cdots, \eta_N)$ is a canonical coordinate system. We assume that $n(x, dy)$ is concentrated on ∂D and if $u \in C^2$, Lu is continuous. We shall call the diffusion process on $\bar{D}$ as $(\hat{A}_L)$-diffusion if the infinitesimal generator of the diffusion coincides with the closure of A restricted to the domain $\{u \in C^2; Lu = 0\}$.

Suppose $\hat{A}_{L-\lambda}$-diffusion exists for each $\lambda \geqq 0$. Sato-Ueno [11] have proved the following. For each $\alpha > 0$, there exists a Markov process on the boundary with the resolvent K_λ^α, $\lambda \geqq 0$, whose infinitesimal generator is the closure of LH_α; the resolvent G_α of $(\hat{A}_L)$-diffusion is expressed as

$$G_\alpha f = G_\alpha^{\min} f + H_\alpha K_0^\alpha \overline{LG_\alpha^{\min}} f. \tag{6. 2}$$

Here, $G_\alpha^{\min}$ is the resolvent of $(\hat{A}_L)$-diffusion absorbed at the boundary, $H_\alpha u$ is a continuous function on $\bar{D}$ taking the value u on ∂D and satisfying $(\alpha - A)H_\alpha u = 0$ on D, and $\overline{LG_\alpha^{\min}}$ is the extension of the operator $LG_\alpha^{\min}$ $((LG_\alpha^{\min})f = L(G_\alpha^{\min} f))$.

THEOREM 6. 1. *Suppose $(\hat{A}_{L-\lambda})$-diffusion exists for each $\lambda \geqq 0$. Then $(\hat{A}_L)$-diffusion satisfies Hypothesis* **I~III**. *Moreover we can choose local time ϕ on the boundary in such a way that*

$$E_x\left(\int_0^\infty e^{-\alpha t} f(x_t) d\phi_t\right) = H_\alpha K_0^\alpha f(x) \tag{6. 3}$$

holds for every $x \in \bar{D}$ and f.

COROLLARY. *L coincides with the probabilistic boundary operator.*

For the proof, we prepare several lemmas.

LEMMA 6. 1. *A nonnegative measurable function on ∂D is K^α-excessive (excessive relative to K_λ^α, $\lambda \geqq 0$) if and only if it is the restriction of an α-excessive function on ∂D.*

Proof. Let $f \geqq 0$. Since $LG_\alpha^{\min} f$ is positive, $H_\alpha K_0^\alpha (LG_\alpha^{\min}) f$ is α-excessive. Note that $LG_\alpha^{\min} f = \frac{\partial}{\partial \eta_N} G_\alpha^{\min} f$ is dense in $C(\partial D)$. Then $H_\alpha K_0^\alpha f$ $(f \geqq 0)$ is also α-excessive. Let u be a K^α-excessive function. Then it is an increasing limit of potentials $K_0^\alpha f_n (f_n \geqq 0)$. Therefore $H_\alpha u$ is α-excessive. Conversely suppose u is an α-excessive function on $\bar{D}$. Then u is an increasing limit of

$G_\alpha f_n$ $(f_n \geqq 0)$. Since $H_\alpha G_\alpha f_n = H_\alpha K_0^\alpha (LG_\alpha^{\min}) f_n$, the restriction of $G_\alpha f_n$ on ∂D is K^α-excessive. Consequently, the restriction of u on ∂D is also K^α-excessive.

COROLLARY. *Every point of ∂D is regular to itself.*

Proof. Let f be a positive continuous function on ∂D. Then $u = H_\alpha K_0^\alpha f$ is a continuous and α-excessive function. Since $(\alpha - A)u = 0$ on D, u satisfies $u(x) = E_x(e^{-\alpha T_{\partial D}} u(x_{T_{\partial D}}))$. Note that $K_0^\alpha f$ is dense in ∂D, we obtain that $f(x) = E_x(e^{-\alpha T_{\partial D}} f(x_{T_{\partial D}}))$ for every x of ∂D and f. Therefore $P_x(T_{\partial D} = 0) = 1$ for every x of ∂D.

LEMMA 6. 2. *Let (x_t^α, P_x^α) be the Markov process on the boundary with K_λ^α, $\lambda \geqq 0$, as its resolvent. Then*

$$E_x(e^{-\alpha T_E};\ x_{T_E} \in F) = E_x^\alpha(x_{T_E}^\alpha \in F) \quad \forall x \in \partial D$$

is satisfied for every open set E of ∂D and Borel set $F \subset E$.

Proof is similar to [10]. Note that each point of E is regular to itself relative to (x_t, P_x) by Corollary to Lemma 6. 1. Then we obtain from Lemma 6. 1

$$E_x(e^{-\alpha T_E} K^\alpha f(x_{T_E})) = E_x^\alpha(K_0^\alpha f(x_{T_E}^\alpha)), \quad \forall x \in \partial D,$$

because the left hand is α-order balayage of $H_\alpha K_0^\alpha f$ to the set E relative to (x_t, P_x) while the other is the balayage relative to $(x_t^\alpha,\ P_x^\alpha)$. Since $K_0^\alpha f$ is dense in $C(\partial D)$, we obtain the lemma.

LEMMA 6. 3. *Let $(x_t,\ P_x)$ be a standard process on S and let $U_\alpha(x, dy)$ be a nonnegative kernel satisfying*

(1) *$U_\alpha f(x)$ is a bounded regular α-excessive function if f is a bounded positive function,*

(2) *$H_E^\alpha U_\alpha f = U_\alpha f$ is satisfied if the support of f is included in the open set E, where $H_E^\alpha(x, dy) = E_x(e^{-\alpha T_E};\ x_{T_E} \in dy)$.*

Then there exists a unique continuous and increasing AF A_t such that

$$U_\alpha f(x) = E_x\left(\int_0^\infty e^{-\alpha t} f(x_t)\, dA_t\right). \tag{6. 4}$$

Proof. It is well known ([6], [9]) that, for each bounded $f \geqq 0$, there exists a nonnegative and increasing AF A_t^f such that $U_\alpha f(x) = E_x\left(\int_0^\infty e^{-\alpha t} dA_t^f\right)$. We have to prove $A_t^f = \int_0^t f(x_s) dA_s$, where $A_t = A_t^1$.

Let K be a closed set and $\{G_n\}$, a sequence of open sets decreasing to K. Then we obtain

$$U_\alpha(x, K) = \lim_{n\to\infty} H_{G_n}^\alpha U_\alpha(x, K) = H_K^\alpha U_\alpha(x, K),$$

because $U_\alpha(x, K)$ is regular. The above relation implies

$$E_x\left(\int_0^\infty e^{-\alpha t} dA_t^K\right) = E_x\left(\int_{T_K}^\infty e^{-\alpha t} dA_t^K\right),$$

where $A_t^K = A_t^{I_K}$ and T_K is the hitting time for the set K. Hence $A_{T_K}^K = 0$, which concludes

$$T_K \leqq \inf\{t > 0;\ A_t^K > 0\}.$$

Consequently the fine support of A_t^K is included in K. Then by Getoor [2], we obtain

$$A_t^K = \int_0^t I_K dA^K. \tag{6. 5}$$

This formula also holds for open set K. Indeed, let $\{F_n\}$ be a sequence of closed sets increasing to the open set K. It is easy to see that $A_t^{F_n}$ increases with n and $A_t^K = \lim_{n\to\infty} A_t^{F_n}$. Since $A_t^{F_n} = \int_0^t I_D dA^{F_n}$ is satisfied for each n, we obtain $A_t^K = \int_0^t I_D dA^K$.

Now let K be a closed set. Note that $A_t = A_t^K + A_t^{K^c}$ and that both of A_t^K and $A_t^{K^c}$ satisfy (6. 5). Then we obtain

$$\int_0^t I_K dA_s = \int_0^t I_K dA_s^K = A_t^K.$$

The Lemma is now clear.

Proof of Theorem 6. 1. Set $U_\alpha f = H_\alpha K_0^\alpha f$. Then $U_\alpha f$ is α-excessive if $f \geqq 0$ by Lemma 6. 1. Since $U_\alpha f$ is continuous if f is continuous, $U_\alpha f$ satisfies the condition (1) of Lemma 6. 3. Let E be an open set of ∂D containing the support of f. Then

$$\begin{aligned}H_E^\alpha U_\alpha f(x) &= E_x(e^{-\alpha T_{\partial D}} E_{x_{T_{\partial D}}}(e^{-\alpha T_E} K_0^\alpha f(x_{T_E}))) \\ &= E_x(e^{-\alpha T_{\partial D}} E^\alpha_{x_{T_{\partial D}}}(K_0^\alpha f(x^\alpha_{T_E}))) \\ &= E_x(e^{-\alpha T_{\partial D}} K_0^\alpha f(x_{T_{\partial D}})) \\ &= H_\alpha K_0^\alpha f = U_\alpha f,\end{aligned}$$

by Lemma 6. 2. Therefore there exists a unique $\varphi_t^{(\alpha)}$ of $\mathfrak{A}_c^+$ such that

$$H_\alpha K_0^\alpha f(x) = E_x\left(\int_0^\infty e^{-\alpha t} f(x_t)\, d\varphi_t^{(\alpha)}\right)$$

It is clear that the support of $A_t^{(\alpha)}$ is included in ∂D.

We shall next prove $\varphi_t^{(\alpha)}$ does not depend on α. First we notice the following relation

$$H_\alpha K_0^\alpha f - H_\beta K_0^\alpha f + (\alpha-\beta)\, G_\beta H_\alpha K_0^\alpha f = 0.$$

Indeed, put the left hand of the above by u. It is easy to see that $(\beta - A)u = 0$ on D and $Lu = 0$ on ∂D. Therefore $u = 0$. We have, on the other hand,

$$E.\left(\int_0^\infty e^{-\beta t} d\varphi_t^{(\alpha)}\right) - E.\left(\int_0^\infty e^{-\alpha t} d\varphi_t^{(\alpha)}\right) + (\beta-\alpha)\, G_\beta E.\left(\int_0^\infty e^{-\alpha t} d\varphi_t^{(\alpha)}\right) = 0$$

as is easily shown. Hence

$$H_\beta K_0^\beta 1 = E.\left(\int_0^\infty e^{-\beta t} d\varphi_t^{(\alpha)}\right),$$

which implies $\varphi_t^{(\alpha)} = \varphi_t^{(\beta)}$. Thus we have proved (6. 3).

To show Hypothesis II, it suffices to prove any $u \in C^2$ is written as the difference of two bounded regular excessive functions. Set $h = u - G_\alpha (\alpha - A)u$. Then $(\alpha - A)h = 0$. Therefore $LH_\alpha h = Lh = Lu$. Hence we have $h = (LH_\alpha)^{-1}Lu$. Consequently,

(6. 6) $$u = G_\alpha(\alpha - A)u + H_\alpha K_0^\alpha Lu.$$

Therefore any u of C^2 is written as the difference of two α-excessive function. Hypothesises I and III follow immediately from [11] and the proof is now complete.

Proof of Corollary to Theorem 6. 1. It suffices to show that $C^2 \subset \mathfrak{D}(A_\varphi)$ and $A_\varphi u$ coincides with Lu on ∂D. Let u be of C^2. Then (6. 6) implies

$$u(x) = E_x\left(\int_0^\infty e^{-\alpha t}(\alpha - A)\, u(x_t)dt\right) - E_x\left(\int_0^\infty e^{-\alpha t} Lu(x_t)d\phi_t\right).$$

Therefore

$$u(x_t) - u(x_0) - \int_0^t Au(x_s)ds - \int_0^t Lu(x_s)d\phi_s$$

is of $\mathfrak{M}_c^{\mathrm{loc}}$. Hence $u \in \mathfrak{D}(A_\varphi)$ and $A_\varphi u = Lu$ on ∂D.

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