W.G. Vogt
M.M. Eisen
G.R. Buis
Nagoya Math. J.
Vol. 34 (1969), 149–151

CONTRACTION GROUPS AND EQUIVALENT NORMS*

WILLIAM G. VOGT MARTIN M. EISEN GABE R. BUIS†

Using the notation in [1], the Lumer-Phillips theorem (3. 1 of [2]) is refined to single parameter groups in real Banach space and real Hilbert space. The theory can be extended to complex spaces.

DEFINITION 1.

Let X be a B-space with norm $\|\cdot\|_1$ and let $[\cdot,\cdot]_1$ be a corresponding semi-scalar product on X. Then the semi-scalar product $[\cdot,\cdot]$ is said to be equivalent to $[\cdot,\cdot]_1$ on X iff $\|\cdot\|_1$ and $\|\cdot\|$ are equivalent norms on X.

THEOREM 1.

Let A be a linear operator with $D(A)$ and $R(A)$ both contained in a B-space $(X, \|\cdot\|_1)$ such that $D(A)$ is dense in X. Then A generates a group $\{T_t; -\infty < t < \infty\}$ in X such that $\{T_t;\ t > 0\}$ is a negative contractive semi-group with respect to an equivalent norm $\|\cdot\|$ iff

$$-\delta\|x\|^2 < [Ax, x] < -\gamma\|x\|^2 \qquad (x \in D(A)) \tag{1}$$

where $\infty > \delta > \gamma > 0$ and $[\cdot,\cdot]$ is an equivalent scalar product consistent with $\|\cdot\|$, and

$$R(I(1-\gamma) - A) = X \qquad R(I(1+\delta) + A) = X. \tag{2}$$

Proof.

The sufficiency of conditions (1) and (2) follows immediately from the results in Yosida [1], pp. 250–254.

Conversely suppose that A generates a group such that $\|T_t\| < e^{-\beta t}$ $(t \geqq 0)$ where $\beta > 0$. It is known that for a group $\|T_t^{-1}\| < Me^{\alpha t}$, where

Received June 10, 1968.
Revised July 15, 1968.
* This research was supported in part by the National Aeronautics and Space Administration under Grant No. NGR 39–011–039 with the University of Pittsburgh.

† Presently with TRW Systems Group, Redondo Beach, California, U.S.A.

$M>1$ and α can be chosen such that $\alpha>\beta$ [1]. Define $S_t=T_t^{-1}e^{-\alpha t}$ and define $\|\cdot\|_2$ by

$$\|x\|_2=\sup_{t>0}\|S_t x\|.$$

This yields an equivalent semi-scalar product and the left side of inequality (1) with $\delta=\alpha$. To show the right side is also valid consider

$$[T_s e^{\beta s}x-x,\,x]_2\leqq\|T_s e^{\beta s}x\|_2\|x\|_2-\|x\|_2^2. \tag{3}$$

Next estimate $\|T_s e^{\beta s}x\|_2$ as follows

$$\|T_s e^{\beta s}x\|_2=\sup_{t\geqq 0}\|T_{s-t}e^{\alpha(s-t)}x\|\leqq\max\left(\|x\|,\ e^{(\beta-\alpha)s}\|x\|_2\right)\leqq\|x\|_2.$$

Hence, (3) yields $[T_s e^{\beta s}x-x,\,x]_2\leqq 0$ which in turn implies the right side of (1) with $\gamma=\beta$.

Finally (2) follows from theorem 3. 1 of [2] applied to the contraction operators $T_{-t}e^{\alpha t}$ (with respect to $\|\cdot\|_2$) and $T_t e^{\beta t}$ (with respect to $\|\cdot\|_1$).

Remark.

Theorem 1 is valid for $(H,\ [\cdot,\cdot]_1)$ a Hilbert space and $[\cdot,\cdot]$ an equivalent scalar product.

Proof.

Using the results of theorem 1, it need only be shown that there exists a scalar product $[\cdot,\cdot]$ equivalent to the scalar product $[\cdot,\cdot]_1$ such that (1) holds. Define $[\cdot,\cdot]$, for any group $\{T_t;\ -\infty<t<\infty\}$ which is negative with respect to $\|\cdot\|_1$, by

$$[x,y]=\int_0^\infty [T_t x,\ T_t y]_1 dt. \tag{4}$$

By hypothesis, $\|T_t\|_1\leqq Me^{-\beta t}$ $(t\geqq 0)$, where $\beta>0$ and $M\geqq 1$; hence

$$[x,x]=\leqq(M^2/2\beta)\,\|x\|_1^2. \tag{5}$$

Since $\{T_t\}$ is a group, there exist constants $\alpha\geqq\beta$ and $1/k\geqq 1$ such that $\|T_t^{-1}\|_1\leqq(1/k)e^{\alpha t}$ for $\gamma\geqq 0$. By using the fact that $\|T_t x\|_1\geqq\|T_t^{-1}\|_1^{-1}\|x\|_1$ it follows from (4) that

$$[x,x]\geqq(k^2/2\alpha)\,\|x\|_1^2. \tag{6}$$

We leave it to the reader to verify that $[\cdot,\cdot]$ is a scalar product. The equivalence of the two scalar products follows from (5) and (6).

To show that an equation of the form (1) is valid we consider

$$[T_t x,\ T_t x] - [x, x] = \lim_{n\to\infty}\left\{\int_0^n [T_s T_t x,\ T_s T_t x]_1 ds - \int_0^n [T_s x,\ T_s x]_1 ds\right\}$$

$$= -\int_0^t [T_s x,\ T_s x]_1 ds, \qquad (t > 0).$$

Since $\lim_{t\to 0^+} t^{-1}([T_t x,\ T_t x] - [x, x]) = 2[Ax, x]$ the last equality implies that

$$2[Ax, x] = -\|x\|_1^2 \qquad (x \in D(A)). \tag{7}$$

Equations (5), (6), and (7) yield (1) with $\gamma = \beta/M^2$ and $\delta = \alpha/k^2$.

References

[1] K. Yosida, Functional Analysis, Springer-Verlag, Berlin (1965).
[2] G. Lumer and R.S. Phillips, "Dissipative operators in a Banach space," Pacific J. Math, **11,** (1961) 679–698.
[3] W. Feller, "On the generation of unbounded semi-groups of bounded Linear Operators," Ann. of Math, **58,** (1953) 166–174.

University of Pittsburgh
Pittsburgh, Pa., U.S.A.