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ON THE EISENSTEIN SERIES FOR THE PRINCIPAL CONGRUENCE SUBGROUPS

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Let Γ be a Fuchsian group (of finite type) acting on the upper half plane. To each parabolic cusp κ_i $(i = 1, \cdots, h)$, corresponds a Eisenstein serie

$$E_i(\tau, s) = \sum_{\Gamma_i \backslash \Gamma} y(\sigma_i^{-1}\sigma\tau)^s$$

where Γ_i is the stationary subgroup of Γ with respect to κ_i and σ_i is an element of $SL(2, \boldsymbol{R})$, such that $\sigma_i\infty = \kappa_i$. (Here we denote by $y(\tau)$ the imaginary part of τ.)
Then,

$$\mathrm{E}(\tau, s) = \begin{pmatrix} E_1(\tau, s) \\ \vdots \\ E_h(\tau, s) \end{pmatrix}$$ satisfies the functional

equation:

$$\mathrm{E}(\tau, s) = \Phi(s)\, \mathrm{E}(\tau, 1 - s). \qquad (*)$$

(For details, see Kubota [1].)

In this paper, we shall give an elementary proof of the functional equation (*) in case $\Gamma = \Gamma_N$ (the principal congruence subgroup of Stufe N). For the explicit form of $\Phi(s)$, see Proposition 1, 2 in §2 (the case $N = p^n$) and Theorem in §3 (general case).

§1

For a positive integer $N > 1$ and a pair of integers $a = \{a_1, a_2\}$ we put

$$\Theta(t\,; a_1, a_2) = \sum_{\{m,n\} \equiv \{a_1, a_2\}\ (N)} e^{-\pi t |m\tau + n|^2 / y}$$

where $\tau = x + iy$, $y > 0$.

LEMMA 1.

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$$\Theta(t\,;a_1,a_2)=\frac{1}{tN^2}\sum_{\{b_1,b_2\}\bmod N} e^{\frac{2\pi i}{N}\begin{vmatrix}a_1,a_2\\b_1,b_2\end{vmatrix}}\Theta\Big(\frac{1}{tN^2}\,;b_1,b_2\Big)\,. \tag{1}$$

Proof is omitted.

To a pair $\{a_1,a_2\}$ such that $(a_1,a_2,N)=1$, there corresponds a Eisenstein series for Γ_N

$$E(\tau,s\,;a_1,a_2)=\sum_{\substack{\{m,n\}\equiv\{a_1,a_2\}\,(N)\\(m,n)=1}}\frac{y^s}{|m\tau+n|^{2s}}\,.$$

Since $\{a_1,a_2\}$ and $\{-a_1,-a_2\}$ give rise to the same Eisenstein series, there are $\frac{1}{2}N^2\prod_{p|N}\Big(1-\frac{1}{p^2}\Big)$ distinct series for $N>2$. (For $N=2$, there are three such series.)

Moreover, we put

$$E^*(\tau,s\,;a_1,a_2)=\sum_{\{m,n\}\equiv\{a_1,a_2\}\,(N)}\frac{y^s}{|m\tau+n|^{2s}}\,.$$

These series converge uniformly on compact sets in the upper half plane, if Re $s>1$.

From the definition, we have

$$\int_0^\infty\Theta(t\,;a_1,a_2)t^{s-1}dt=\pi^{-s}\Gamma(s)E^*(\tau,s\,;a_1,a_2)\,. \tag{2}$$

For a character mod N, such that $\chi(-1)=1$, we put

$$\Theta(t\,;a,\chi)=\sum_{\substack{(u,N)=1\\u \bmod N}}\overline{\chi(u)}\,\Theta(t\,;ua_1,ua_2).$$

From (1), it follows that

$$\Theta(t\,;a,\chi)=\frac{1}{tN^2}\sum_{b\bmod N} e^{\frac{2\pi i}{N}\begin{vmatrix}a_1,a_2\\b_1,b_2\end{vmatrix}}\Theta\Big(\frac{1}{tN^2}\,;b,\bar\chi\Big)\,. \tag{1'}$$

$E(\tau,s\,;a,\chi)$ and $E^*(\tau,s\,;a,\chi)$ are defined in the same way.

LEMMA 2.

$$\begin{aligned}E^*(\tau,s\,;a,\chi)&=\prod_{p|N}(1-p^{-2s})\zeta(2s)E(\tau,s\,;a,\chi_0),\ \text{if }\chi=\chi_0\equiv1\\&=L(2s,\bar\chi)E(\tau,s\,;a,\chi)\qquad\qquad,\ \text{if }\chi\neq\chi_0\,.\end{aligned} \tag{3}$$

Proof. (1) If $\chi\neq\chi_0$, we have

$$E(\tau,s;a,\chi)=\sum_{\substack{(u,N)=1\\ u \bmod N}}\overline{\chi(u)}\Big\{\sum_{\substack{(d,N)=1\\ d \bmod N}}\Big(\sum_{dq\equiv 1\,(N)}q^{-2s}\Big)E(\tau,s;dua_1,dua_2)$$

$$=\sum_{\substack{(d,N)=1\\ d \bmod N}}\chi(d)\sum_{dq\equiv 1\,(N)}q^{-2s}E(\tau,s;a,\chi)$$

$$=\sum_{(q,N)=1}\overline{\chi(q)}\;q^{-2s}\;E(\tau,s;a,\chi).$$

$$=L(2s,\bar{\chi})E(\tau,s;a,\chi)$$

(2) Let $\chi=\chi_0$. If $N=p_1^{k_1}\cdots p_r^{k_r}$ is a factorization into prime factors, then we have

$$\sum_{(q,N)=1}q^{-2s}=\sum_{i_1,\cdots,i_j}(-1)^j\sum_{p_{i_1}\cdots p_{i_j}\mid q}q^{-2s}$$

$$=\zeta(2s)\sum_{i_1,\cdots,i_j}(-1)^j(p_{i_1}\cdots p_{i_j})^{-2s}.$$

Since, as in (1), we have

$$E^*(\tau,s;a,\chi_0)=\Big(\sum_{(q,N)=1}q^{-2s}\Big)E(\tau,s;a,\chi_0)$$

we obtain the desired result.

Remark. As is seen from the definition,

$$\sum_{\chi}E(\tau,s;a,\chi)=E(\tau,s;a_1,a_2).$$

Therefore, the functional equation of $E(\tau,s;a_1,a_2)$ can be obtained from that of $E(\tau,s;a,\chi)$.

§2

In this section, we shall prove the functional equation of $E(\tau,s;a,\chi)$ in case $N=p^n$.

Since $E(\tau,s;a,\chi)$ is a χ-homogeneous function, i.e.

$$E(\tau,s;ua,\chi)=\chi(u)E(\tau,s;a,\chi),\quad (ua=\{ua_1,ua_2\},\;(u,p)=1)$$

we may restrict ourselves to the case $a\in I$, where

$$I=\{(a_1,a_2);\,a_1=1\text{ or }a_2=1,\;a_1\equiv 0\;(p)\}.$$

It is easy to see that, for $a,b\in I$,

$$\langle a,b\rangle=\begin{vmatrix}a_1,a_2\\ b_1,b_2\end{vmatrix}\equiv 0\;(p^k)\text{ if and only if }a\equiv b\;(p^k)\;(1\leqq k\leqq n).$$

1) The case $\chi=\chi_0$

(a) Let $n\geqq 2$. Then, for $a\in I$, we have from (1)′

$$\Theta(t;a,\chi_0)-\frac{1}{p}\sum_{a'\in I,\,a'\equiv a(p^{n-1})}\Theta(t;a',\chi_0)$$
$$=\frac{1}{tp^{2n}}\sum_b c(b)\Theta\Bigl(\frac{1}{tp^{2n}};b,\chi_0\Bigr).$$

If $b\equiv 0\ (p)$, $e^{\frac{2\pi i}{p^n}<a,b>}-\frac{1}{p}\sum_{a'\in I,\,a'\equiv a(p^{n-1})}e^{\frac{2\pi i}{p^n}<a,b>}=0.$ Therefore, $c(b)=0$.

For $b\in I$, we have

$$c(b)=\sum_{(t,p)=1,\ t \bmod p^n}\Bigl\{e^{\frac{2\pi i}{p^n}<a,b>t}-\frac{1}{p}\sum_{\substack{a'\equiv a(p^{n-1})\\ a'\in I}}e^{\frac{2\pi i}{p^n}<a',b>t}\Bigr\}.$$

If $b\not\equiv a\ (p^{n-1})$, then, as we noted above,

$$\langle a',b\rangle=p^k u\qquad (k<n-1,\ (u,p)=1).$$

Therefore we have

$$\sum_{(t,p)=1,\ t \bmod p^n}e^{\frac{2\pi i}{p^n}<a',b>t}=\sum_{(t,p)=1,\ t \bmod p^n}e^{\frac{2\pi i}{p^r}t}$$
$$=\#\{t;\,t\equiv 1\ (p^r)\}\sum_{(t,p)=1,\ t \bmod p^r}e^{\frac{2\pi i}{p^r}t}=0,$$

because $\sum_{(t,N)=1,\ t \bmod N}e^{\frac{2\pi i}{N}t}=\mu(N)$ (Möbius function) and $r=n-k\geqq 2$.
Hence, $c(b)=0$.
If $b\equiv a\ (p^{n-1})$, we have

$$\sum_{a'\equiv a\ (p^{n-1}),\ a'\in I}e^{\frac{2\pi i}{p^n}<a',b>t}=\sum_{a'\equiv b\ (p^{n-1}),\ a'\in I}e^{\frac{2\pi i}{p^n}<a',b>t}=\sum_{v \bmod p}e^{\frac{2\pi i}{p}v}=0.$$

Therefore

$$c(b)=\sum_{(t,p)=1,\ t \bmod p^n}e^{\frac{2\pi i}{p^n}<a,b>t}=p^n-p^{n-1}\quad\text{if}\quad a=b$$
$$=\#\{t;\,t\equiv 1\ (p)\}\sum_{(t,p)=1,\ t \bmod p}e^{\frac{2\pi i}{p}t}=-p^{n-1}\quad\text{if}\quad a\neq b.$$

Thus we have proved the following formula:

$$(1'') \qquad \Theta(t; a, \chi_0) - \frac{1}{p} \sum_{a' \in I,\ a' \equiv a\ (p^{n-1})} \Theta(t; a', \chi_0)$$
$$= \frac{1}{tp^n} \left\{ \Theta\left(\frac{1}{tp^{2n}}; a, \chi_0\right) - \frac{1}{p} \sum_{a' \in I,\ a' \equiv a\ (p^{n-1})} \Theta\left(\frac{1}{tp^{2n}}; a', \chi_0\right) \right\}.$$

Now we denote by $E_n(\tau, s; a, \chi)$ the Eisenstein series for Γ_{p^n}. Then, it is easy to see that

$$\sum_{a' \in I,\ a' \equiv a\ (p^{n-1})} E_n^*(\tau, s; a', \chi_0) = E_{n-1}^*(\tau, s; a, \chi_0).$$

We put

$$G(s) = \pi^{-s}\Gamma(s)\zeta(2s)(1 - p^{-2s}) \left\{ E_n(\tau, s; a, \chi_0) - \frac{1}{p} E_{n-1}(\tau, s; a, \chi_0) \right\}$$

In view of (2), (3) and (1''), we have

$$G(s) = \int_{1/p^n}^{\infty} \left\{ \Theta(t; a, \chi_0) - \frac{1}{p} \sum_{a' \in I,\ a' \equiv a\ (p^{n-1})} \Theta(t; a', \chi_0) \right\} t^{s-1} dt$$
$$+ p^{n(1-2s)} \int_{1/p^n}^{\infty} \left\{ \Theta(t; a, \chi_0) - \frac{1}{p} \sum_{a'} \Theta(t; a, \chi_0) \right\} t^{-s} dt.$$

From this immediately follow the analytic continuation of $G(s)$ into the whole s-plane and the functional equation

$$(4) \qquad G(s) = p^{n(1-2s)} G(1-s).$$

(b) In case $n = 1$, a similar argument shows that

$$\Theta(t; a, \chi_0) - \frac{1}{p+1} \sum_{a' \in I} \Theta(t; a', \chi_0) = \frac{1}{tp} \left\{ \Theta\left(\frac{1}{tp^2}; a, \chi_0\right) - \frac{1}{p+1} \sum_{a' \in I} \Theta\left(\frac{1}{tp^2}; a', \chi_0\right) \right..$$

Therefore, as in (a), we can prove that

$$G(s) = \pi^{-s}\Gamma(s)\zeta(2s)(1 - p^{-2s}) \left\{ E_1(\tau, s; a, \chi_0) - \frac{1}{p+1} E(\tau, s) \right\}$$

is an entire function and satisfies the functional equation

$$(5) \qquad G(s) = p^{1-2s} G(1-s)$$

where
$$E(\tau, s) = \sum_{(m,n)=1} \frac{y^s}{|m\tau + n|^{2s}}$$

is the Eisenstein series for the full modular group.

As is well known, $E(\tau, s)$ is meromorphic in the whole s-plane, and satisfies the functional equation

$$E(\tau, s) = \frac{\sqrt{\pi}\,\Gamma\left(s - \frac{1}{2}\right)}{\Gamma(s)} \frac{\zeta(2s-1)}{\zeta(2s)} E(\tau, 1-s). \tag{6}$$

From (4), (5) and (6), we can obtain the following result.

PROPOSITION 1. *Let $\boldsymbol{E}_n(\tau, s; \chi_0)$ be the column of the $p^n + p^{n-1}$ functions $E_n(\tau, s; a, \chi_0)$, $(a \in I)$.*

Then, $\boldsymbol{E}_n(\tau, s; \chi_0)$ is a meromorphic function in the whole s-plane and satisfies the following functional equation:

$$\boldsymbol{E}_n(\tau, s; \chi_0) = \Phi_n(s)\boldsymbol{E}_n(\tau, 1-s; \chi_0)$$

where $\Phi_n(s) = \varphi(s) \langle c^{(n)}(a, b) \rangle$

(matrix of degree $p^n + p^{n-1}$)

$$\varphi(s) = \frac{\sqrt{\pi}\,\Gamma\left(s - \frac{1}{2}\right)}{\Gamma(s)} \frac{\zeta(2s-1)}{\zeta(2s)}$$

$$c^{(n)}(a, b) = \frac{p-1}{p^{2s}-1} p^{(n-1)(1-2s)} \qquad \text{if } a = b$$

$$= p^{-n+r+1} \frac{p^{2s-1}-1}{p^{2s}-1} p^{r(1-2s)} \qquad \text{if } a \equiv b \ (p^r)$$

$$(0 \leqq r \leqq n-1).$$

Proof. As is easily seen,

$$c^{(n)}(a, b) = \begin{cases} p^{n(1-2s)} \dfrac{1-p^{2s-2}}{1-p^{-2s}}\left(1 - \dfrac{1}{p}\right) + \dfrac{1}{p} c^{(n-1)}(a, b) & (n \geqq 2) \\ p^{1-2s} \dfrac{1-p^{2s-2}}{1-p^{-2s}}\left(1 - \dfrac{1}{p+1}\right) + \dfrac{1}{p+1} & (n = 1) \end{cases}$$

$$\text{if } a = b$$

$$= \begin{cases} -p^{n(1-2s)-1} \dfrac{1-p^{2s-2}}{1-p^{-2s}} + \dfrac{1}{p} c^{(n-1)}(a, b) & (n = 2) \\ -\dfrac{p^{1-2s}}{p+1} \dfrac{1-p^{2s-2}}{1-p^{-2s}} + \dfrac{1}{p+1} & (n = 1) \end{cases}$$

$$\text{if } p^{n-1} \| a - b$$

$$= \frac{1}{p} c^{(n-1)}(a, b) \qquad \text{if } p^k \| a - b \ (0 \leqq k \leqq n-2).$$

Hence, by induction on n, follows the desired result.

2) The case $\chi \neq \chi_0$

a) Let χ be a primitive character.

For $a=\{a_1, a_2\}$, such that $(a_1, a_2)=p^k u$ $(k \geqq 1,\ (u, p)=1)$, we have

$$\Theta(t; a, \chi)=\sum_{\substack{(u, p)=1 \\ u \bmod p^r}} \chi(u)\Big(\sum_{t \equiv 1\ (p^r)} \chi(t)\Big) \Theta(t; u a_1, u a_2)=0 .$$

$(r=n-k<n)$

Therefore, from (1), it follows that

$$\Theta(t; a, \chi)=\frac{1}{t p^{2n}} \sum_{b \in I} \sum_{\substack{(u, p)=1 \\ u \bmod p^n}} e^{\frac{2\pi i}{p^n}<a, b>u}\, \overline{\chi(u)}\, \Theta\Big(\frac{1}{t p^{2n}} ; b, \bar{\chi}\Big) .$$

We put $S_\chi=\sum_{u \bmod p^n} e^{\frac{2\pi i}{p^n} u} \chi(u)$ (Gauss sum). Then,

$$\Theta(t; a, \chi)=\frac{1}{t p^{2n}} \sum_{b \in I} S_{\bar{\chi}} \chi(\langle a, b\rangle)\, \Theta\Big(\frac{1}{t p^{2n}} ; b, \bar{\chi}\Big) . \tag{7}$$

By (2), (3) and (7), we obtain

$$\begin{aligned} \pi^{-s} \Gamma(s) L(2s, \bar{\chi}) E(\tau, s; a, \chi) &= \int_{1/p^n}^{\infty} \Theta(t; a, \chi) t^{s-1} d t \\ &+ \frac{S_{\bar{\chi}}}{p^{2ns}} \sum_{b \in I} \chi(\langle a, b\rangle) \int_{1/p^n}^{\infty} \Theta(t; b, \bar{\chi}) t^{-s} d t . \end{aligned} \tag{8}$$

As is easily seen, we have

$$\sum_{b \in I} \chi(\langle a, b\rangle) \overline{\chi(\langle b, a'\rangle)}=p^n \delta_{a, a'} \qquad (a, a' \in I).$$

Moreover, $\quad |S_\chi|^2=p^n$ and $\bar{S}_\chi=S_{\bar{\chi}}$ if $\chi(-1)=1$.

Therefore, from (8), immediately follows the functional equation

$$G(s, a, \chi)=p^{-2ns} S_{\bar{\chi}} \sum_{b \in I} \chi(\langle a, b\rangle)\, G(1-s, b, \bar{\chi}) \tag{9}$$

where $\quad G(s, a, \chi)=\pi^{-s} \Gamma(s)\, L(2s, \bar{\chi})\, E(\tau, s; a, \chi)$.

By the functional equation of Dirichlet L-function

$$H(s, \chi)=\pi^{-\frac{s}{2}}\, \Gamma\Big(\frac{s}{2}\Big) L(s, \chi)=S_\chi p^{-ns} H(1-s, \bar{\chi}),$$

(9) can be written as follows.

Let $\boldsymbol{E}(\tau, s; \chi)$ be the column of the p^n+p^{n-1} functions $E(\tau, s; a, \chi)$.

Then,

$$\boldsymbol{E}(\tau, s; \chi) = \Phi(s, \chi)\boldsymbol{E}(\tau, 1-s, \bar{\chi})$$

where

$$\Phi(s, \chi) = p^{-\frac{n}{2}} \frac{\sqrt{\pi}\,\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} \frac{L(2s-1, \bar{\chi})}{L(2s, \bar{\chi})} \left\langle p^{-\frac{n}{2}} \chi(\langle a, b \rangle)\right\rangle$$

b) We denote by r the integer $\min\{m; \chi(v)=1, \text{ if } v \equiv 1 \bmod p^m\}$. In a) we considered the case $r=n$. Let $r \leqq n-1$.
First we note

$$\sum_{t \bmod p^n} \chi(t)\, e^{\frac{2\pi i}{p^n}ct} = \overline{\chi(c')} S_\chi p^{n-r} \quad \text{if} \quad c = c'p^{n-r},\ (c', p)=1$$
$$= 0 \quad \text{otherwise.}$$

Since $\displaystyle\sum_{a' \equiv a\ (p^{n-1}),\ a' \in I} e^{\frac{2\pi i}{p^n}<a',\, b>u} = e^{\frac{2\pi i}{p^n}<a, b>u} \sum_{v \bmod p} e^{\frac{2\pi i}{p}v} = 0$, if $a \equiv b\ (p)$,

$$\text{(10)} \qquad \Theta(t; a, \chi) - \frac{1}{p} \sum_{\substack{a' \equiv a\ (p^{n-1}) \\ a' \in I}} \Theta(t; a', \chi)$$
$$= \frac{1}{t p^{2n}} p^{n-r} S_{\bar{\chi}} \sum_{\substack{b \equiv a\ (p^{n-r}) \\ b \in I}} \chi\left(\frac{\langle a, b\rangle}{p^{n-r}}\right) \Theta\left(\frac{1}{t p^{2n}}; b, \bar{\chi}\right).$$

On the other hand, we have

$$\sum_{\substack{b \equiv a\ (p^{n-r}) \\ b \in I}} \chi\left(\frac{\langle a, b\rangle}{p^{n-r}}\right) \sum_{\substack{b' \equiv b\ (p^{n-1}) \\ b' \in I}} \Theta(t; b', \chi)$$
$$= \sum_{\substack{b \equiv a\ (p^{n-r}) \\ b \in I}} \left\{ \sum_{\substack{b' \equiv b\ (p^{n-1}) \\ b' \in I}} \chi\left(\frac{\langle a, b'\rangle}{p^{n-r}}\right)\right\} \Theta(t; a, \chi) = 0.$$

Therefore, (10) is unchanged, if $\Theta(t; b, \chi)$ is replaced by

$$\Theta(t; b, \chi) - \frac{1}{p} \sum_{\substack{b' \equiv b\ (p^{n-1}) \\ b' \in I}} \Theta(t; b', \chi).$$

We put

$$G(s, a, \chi) = \pi^{-s}\Gamma(s) L(2s, \bar{\chi}) \left\{ E_n(\tau, s; a, \chi) - \frac{1}{p} E_{n-1}(\tau, s; a, \chi)\right\}.$$

Then, since $\sum_{\substack{a' \equiv a \ (p^{n-1}) \\ a' \in I}} E_n(\tau, s; a, \chi) = E_{n-1}(\tau, s; a, \chi)$

(we note that $r \leqq n-1$), we have

$$G(s, a, \chi) = \int_{1/p^n}^{\infty} \Big\{ \Theta(t; a, \chi) - \frac{1}{p} \sum_{\substack{a' \equiv a \ (p^{n-1}) \\ a' \in I}} \Theta(t; a', \chi) \Big\} t^{s-1} dt$$

$$+ \int_{1/p^n}^{\infty} p^{-2ns} p^{n-r} S_{\bar{\chi}} \sum_{\substack{b \equiv a \ (p^{n-1}) \\ b \in I}} \chi\Big(\frac{\langle a, b \rangle}{p^{n-r}}\Big) \Big\{ \Theta(t; b, \bar{\chi}) - \frac{1}{p} \sum_{\substack{b' \equiv b \ (p^{n-1}) \\ b' \in I}} \Theta(t; b', \bar{\chi}) \Big\} t^{-s} dt$$

$$= p^{-2ns} p^{n-r} S_{\bar{\chi}} \sum_{\substack{b \equiv a \ (p^{n-r}) \\ b \in I}} \chi\Big(\frac{\langle a, b \rangle}{p^{n-r}}\Big) G(1-s, b, \bar{\chi}).$$

(see the remark below)

From this and the result in *a*), we obtain the following

PROPOSITION 2.

$$\boldsymbol{E}_n(\tau, s; \chi) = \Phi_n(s, \chi) \boldsymbol{E}_n(\tau, 1-s; \bar{\chi})$$

where

$$\Phi_n(s, \chi) = \varphi(s, \chi) \langle c^{(n)}(a, b) \rangle$$

$$\varphi(s, \chi) = p^{-r} \frac{\sqrt{\pi}\,\Gamma\big(s - \frac{1}{2}\big)}{\Gamma(s)} \frac{L(2s-1, \bar{\chi})}{L(2s, \bar{\chi})}$$

$$c^{(n)}(a, b) = p^{(1-2s)(n-r-k)-k} \chi\Big(\frac{\langle a, b \rangle}{p^{n-r-k}}\Big)$$

$$\text{if } p^{n-r-k} \| \langle a, b \rangle \ (0 \leqq k \leqq n-r)$$

$$= 0 \text{ otherwise.}$$

Remark. Let $a \equiv b \ (p^{n-r})$. Then,

$$\sum_{\substack{c \equiv a \ (p^{n-r}) \\ c \in I}} \chi\Big(\frac{\langle a, c \rangle}{p^{n-r}}\Big) \overline{\chi\Big(\frac{\langle b, c \rangle}{p^{n-r}}\Big)} = p^r - p^{r-1} \quad \text{if} \quad a = b$$

$$= -p^{r-1} \quad \text{if} \quad p^{n-1} \| a - b$$

$$= 0 \quad \text{otherwise.}$$

§3

Let $N = p_1^{n_1} \cdots p_\lambda^{n_\lambda}$ be a factorization into distinct primes.

We put $N = N_i p_i^{n_i} \quad (1 \leqq i \leqq \lambda)$.

Let us choose a set of integers $\{d_1, \cdots, d_\lambda\}$, such that

$$d_i \equiv 0 \ (N_i), \ \equiv 1 \ (p_i^{n_i}) \quad (1 \leqq i \leqq \lambda).$$

Then, the mapping

$$Z^{\lambda} \ni \{a_1, \cdots, a_\lambda\} \longrightarrow a = \sum_{i=1}^{\lambda} d_i a_i \in Z$$

induces a ring-isomorphism of $Z/(N)$ onto $\prod_{i=1}^{\lambda} Z/(p_i^{n_i})$.

It is obvious that

(1) $(a^{(1)}, a^{(2)}, N) = 1$ if and only if $(a_i^{(1)}, a_i^{(2)}, p_i) = 1$

(2) $(u, N) = 1$ if and only if $(u_i, p_i) = 1$ $(1 \leqq i \leqq \lambda)$

Let $I = I_1 \times \cdots \times I_\lambda$, where $I_i = \{(a_1, a_2) \bmod p_i^{n_i}\,;\, a_1 = 1 \text{ or } a_2 = 1,\ a_1 \equiv 0\ (p_i)\}$. We denote by $V(I)$ the space of functions on I.

Then $$V(I) = V(I_1) \otimes \cdots \otimes V(I_\lambda).$$

For a character χ mod N, there exist characters χ_i mod $p_i^{n_i}$ such that

$$\chi(a) = \prod_{i=1}^{\lambda} \chi_i(a_i).$$

Let
$$\begin{aligned} r_i = r(\chi_i) &\neq 0 \qquad (1 \leqq i \leqq \mu) \\ &= 0 \qquad (\mu + 1 \leqq i \leqq \lambda). \end{aligned}$$

Then there exists a primitive character $\tilde{\chi}$ mod $\tilde{N} = \prod_{i=1}^{\mu} p_i^{r_i}$, such that

$$\chi(a) = \tilde{\chi}(a) \quad \text{if} \quad (a, N) = 1.$$

We put

$$T = T_1 \otimes \cdots \otimes T_\lambda$$

where T_i is a linear transformation in $V_i = V(I_i)$, defined by

$$\begin{aligned} T_i f(a) &= f(a) - \frac{1}{p_i} \sum_{a' \in I_i,\ a' \equiv a\ (p_i^{n_i})} f(a') \quad \text{if } n_i > 1 \text{ and } n_i > r_i \\ &= f(a) - \frac{1}{p_i + 1} \sum_{a' \in I_i} f(a') \qquad \text{if } n_i = 1 \text{ and } r_i = 0 \\ &= f(a) \qquad \text{if } n_i = r_i. \end{aligned}$$

Moreover, we define an endomorphism of $V = V(I)$ by

$$A f(a) = \sum_{b \in I} A(a, b) f(b)$$

$$A(a, b) = \sum_{\substack{(u, N)=1 \\ u \bmod N}} \overline{\chi(u)}\, e^{\frac{2\pi i}{N}\langle a, b\rangle u}$$

We have

$$\boldsymbol{A} = c\boldsymbol{A}_1 \otimes \cdots \otimes \boldsymbol{A}_\lambda \quad (c = \bar{S}_{\tilde{\chi}} N \tilde{N}^{-1})$$

where

$$A_i(a, b) = \begin{cases} \chi_i\left(\dfrac{\langle a, b\rangle}{p_i^{n_i - r_i}}\right) & \text{if } \ a \equiv b \ (p_i^{n_i - r_i}) \\ 0 & \text{otherwise} \qquad (1 \leqq i \leqq \mu) \end{cases}$$

$$= \begin{cases} \left(1 - \dfrac{1}{p_i}\right)\overline{\tilde{\chi}(p_i^{n_i})} & \text{if } \ a = b \\ -\dfrac{1}{p_i}\,\overline{\tilde{\chi}(p_i^{n_i})} & \text{if } \ a \neq b, \ a \equiv b \ (p_i^{n_i - 1}) \qquad (\mu < i \leqq \lambda) \\ 0 & \text{otherwise} \end{cases}$$

For, we have

$$A(a, b) = \prod_{i=1}^{\lambda} \sum_{\substack{(u_i, p_i)=1 \\ u_i \bmod p_i^{n_i}}} \overline{\chi_i(u_i)}\, e^{\frac{2\pi i}{p_i^{n_i}}\langle a_i, b_i\rangle u_i c_i} \quad \left(c_i = \frac{d_i}{N_i}\right)$$

$$= \prod_{i=1}^{\mu} p_i^{n_i - r_i} \chi_i(c_i) S_{\bar{\chi}_i} \chi_i\left(\frac{\langle a_i, b_i\rangle}{p_i^{n_i - r_i}}\right)$$

$$\times \prod_{i=\mu+1}^{\lambda} \sum e^{\frac{2\pi i}{p_i^{n_i}}\langle a_i, b_i\rangle u_i} \quad \text{if} \quad p_i^{n_i - r_i} \| a - b$$

$$= 0 \qquad \text{otherwise}$$

and

$$\prod_{i=1}^{\mu} \chi_i(c_i) S_{\bar{\chi}_i} = \prod_{i=\mu+1}^{\lambda} \overline{\tilde{\chi}(p_i^{n_i})} \bar{S}_{\tilde{\chi}}\,.$$

From the results obtained in §2, we have

$$\begin{aligned} \boldsymbol{T}_i \boldsymbol{A}_i = \boldsymbol{A}_i \boldsymbol{T}_i, \quad \boldsymbol{A}_i \boldsymbol{A}_i^* \boldsymbol{T}_i &= p_i^{r_i} \boldsymbol{T}_i \qquad (1 \leqq i \leqq \lambda) \\ &= \tilde{\chi}(p_i^{n_i}) \boldsymbol{T}_i, \quad \text{for} \quad i > \mu \\ &= \boldsymbol{A}_i \quad , \quad \text{for} \quad i \leqq \mu. \end{aligned} \tag{11}$$

Therefore we have

$$\boldsymbol{T}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{T} \quad \text{and} \quad \boldsymbol{A}\boldsymbol{A}^*\boldsymbol{T} = N^2 \boldsymbol{T}\,.$$

LEMMA. *For* $f(a,s,\chi)=\pi^{-s}\Gamma(s)E^{*}(\tau,s;a,\chi)$, *we have*

$$\boldsymbol{T}f(\,\cdot\,,s,\chi)=N^{-2s}\boldsymbol{A}\boldsymbol{T}f(\,\cdot\,,1-s,\bar{\chi}). \tag{12}$$

Proof. From (1′), we have

$$\boldsymbol{T}\Theta(t;\,\cdot\,,\chi)=\frac{1}{tN^{2}}\sum_{b \bmod N}\Big(\boldsymbol{T}e^{\frac{2\pi i}{N}<\cdot,b>}\Big)\Theta\Big(\frac{1}{tN^{2}};b,\bar{\chi}\Big).$$

If $b_i\equiv 0\ (p_i)$, we have

$$\boldsymbol{T}e^{\frac{2\pi i}{N}<\cdot,b>}=0 \qquad (\text{in case } n_i>r_i)$$

or

$$\Theta(t;b,\chi)=0 \qquad (\text{in case } n_i=r_i).$$

Hence,

$$\begin{aligned}\boldsymbol{T}\Theta(t;\,\cdot\,,\chi)&=\frac{1}{tN^{2}}\sum_{b\in I}\boldsymbol{T}\Big(\sum_{(u,N)=1}\overline{\chi(u)e^{\frac{2\pi i}{N}<\cdot,b>u}}\Big)\Theta\Big(\frac{1}{tN^{2}};b,\bar{\chi}\Big)\\&=\frac{1}{tN^{2}}\boldsymbol{T}\boldsymbol{A}\Theta\Big(\frac{1}{tN^{2}};\,\cdot\,,\bar{\chi}\Big).\end{aligned}$$

Therefore

$$\begin{aligned}\boldsymbol{T}f(\,\cdot\,,s,\chi)&=\int_0^\infty \boldsymbol{T}\Theta(t;\,\cdot\,,\chi)t^{s-1}dt\\&=\int_{1/N}^\infty \boldsymbol{T}\Theta(t;\,\cdot\,,\chi)t^{s-1}dt+N^{-2s}\boldsymbol{A}\int_{1/N}^\infty \boldsymbol{T}\Theta(t;\,\cdot\,,\bar{\chi})t^{-s}dt\\&=N^{-2s}\boldsymbol{A}\boldsymbol{T}f(\,\cdot\,,1-s,\bar{\chi}).\end{aligned}$$

Let

$$\begin{aligned}\boldsymbol{V}_i^{(k)}&=\{f\in\boldsymbol{V}_i;\ f(a')=f(a)\ \text{ if }\ a'\equiv a\ (p_i^k)\}\quad (r_i\leqq k\leqq n_i)\\&=\{0\}\qquad\qquad (k<r_i).\end{aligned}$$

We denote by $\boldsymbol{P}_k$ the projection operator on $\boldsymbol{V}_i^{(k)}\ominus\boldsymbol{V}_i^{(k-1)}$. Then, for $f\in\boldsymbol{V}$, we have

$$\sum_{k_i=r_i}^{n_i}\boldsymbol{P}_{k_1\cdots k_\lambda}f=f \tag{13}$$

where

$$\boldsymbol{P}_{k_1\cdots k_\lambda}=\boldsymbol{P}_{k_1}\otimes\cdots\otimes\boldsymbol{P}_{k_\lambda}$$

LEMMA.

$$\boldsymbol{P}_{k_1\cdots k_\lambda}E_N^{*}(\tau,s;\cdot,\chi)=N'N^{-1}\prod_{k_i=0}(p_i+1)^{-1}\Big(1-\frac{\chi(p_i)}{p_i^{2s}}\Big)\boldsymbol{T}^{(N')}E_{N'}^{*}(\tau,s;\cdot,\chi). \tag{14}$$

Proof. First we note that

$$(15)\qquad \boldsymbol{P}_k f(a) = \frac{1}{c_k}\sum_{a'\equiv a\ (p^k)} f(a') - \begin{cases} \dfrac{1}{c_{k-1}}\sum_{a'\equiv a\ (p^{k-1})} f(a') & \text{for}\quad k>r \\ 0 & \text{for}\quad k=r \end{cases}$$

$$= \frac{1}{c_k}\boldsymbol{T}^{(p^k)}\sum_{a'\equiv a\ (p^k)} f(a')$$

($c_k = p^{n-k}$ or $p^n + p^{n-1}$ according as $k>0$ or $k=0$).

We have only to prove the following.

$$1^\circ\qquad \sum_{\substack{a'\equiv a\ (N')\\ a'\in I}} E^*_N(\tau,s;a,\chi) = \Big(1-\frac{\chi(p)}{p^{2s}}\Big)E^*_{N'}(\tau,s;a,\chi)\quad \text{if}\quad (p,N)=1$$

$$2^\circ\qquad = E^*_{N'}(\tau,s;a,\chi)\qquad \text{if}\quad p\,|\,N.$$

In case $(p,N)=1$, we have

$$\sum_{a'\equiv a\ (N')} E^*_N(\tau,s;a,\chi) = \sum_{\substack{(u,N')=1\\ u \bmod N'}}\overline{\tilde{\chi}(u)}\sum_{\substack{\{m,n\}\equiv\{ua^{(1)},ua^{(2)}\}\ (N')\\ \not\equiv 0\qquad (p)}}\frac{y^s}{|m\tau+n|^{2s}}$$

$$= E^*_{N'}(\tau,s;a,\chi) - \sum_u \overline{\tilde{\chi}(u)}\sum_{\substack{\{m,n\}\equiv p\,cu\,\{a^{(1)},a^{(2)}\}(N)\\ (c\equiv p^{-1}\ (N'))}}\frac{y^s}{|m\tau+n|^{2s}}$$

$$= \Big(1-\frac{\overline{\tilde{\chi}(p)}}{p^{2s}}\Big)E^*_{N'}(\tau,s;a,\chi).$$

For the proof of 2°, it is sufficient to note that

$\{m,n\}\equiv\{ua^{(1)},\ ua^{(2)}\}\ (p^{k-1})$ if and only if $\{m,n\}\equiv\{uvb^{(1)},\ uvb^{(2)}\}\ (p^k)$ for some $b=\{b^{(1)},\ b^{(2)}\}\equiv a\ (p^{k-1})$ and $v\equiv 1\ (p^{k-1})$.

THEOREM. *$E(\tau,s;a,\chi)$ is a meromorphic function in the whole s-plane and satisfies the following functional equation.*

$$E(\tau,s;\cdot,\chi) = \frac{\sqrt{\pi}\,\Gamma\Big(s-\dfrac{1}{2}\Big)}{\Gamma(s)}\,\frac{L(2s-1,\bar{\tilde{\chi}})}{L(2s,\tilde{\chi})}\,\Phi^{(1)}\otimes\cdots\otimes\Phi^{(\lambda)}E(\tau,1-s;\cdot,\bar{\chi})$$

where, for $\quad i\leqq\mu,\ \Phi^{(i)}(a,b) = \chi_i\Big(\dfrac{\langle a,b\rangle}{p_i^k}\Big)p_i^{(1-2s)k-n+k}$ *if* $p_i^k\|a-b$

$$(0\leqq k\leqq n-r)$$

$= 0$ *otherwise*

for $\quad i>\mu,\ \Phi^{(i)}(a,b) = (\tilde{\chi}(p_i)p_i^{2s-1})^{1-n}\dfrac{p-1}{\tilde{\chi}(p_i)p_i^{2s}-1}$, *if* $a=b$

$$= p_i^{k-n+1} \frac{\tilde{\chi}(p_i)p_i^{2s-1} - 1}{\tilde{\chi}(p_i)p_i^{2s} - 1} (\tilde{\chi}(p_i)p_i^{2s-1})^{-k}$$

$$\text{if} \quad p_i^k \| a - b \qquad (0 \leqq k \leqq n-1).$$

Proof. We put $\gamma(s, \chi) = \pi^{-s}\Gamma(s)L(2s, \bar{\tilde{\chi}})$.

Since $L(s, \chi) = \prod_{\mu < i \leqq \lambda} \left(1 - \frac{\tilde{\chi}(p_i)}{p_i^{2s}}\right) L(s, \tilde{\chi})$, we have, from (12), (13) and (14),

$$\gamma(s, \chi)E(\tau, s;\ \cdot\ , \chi) \prod_{\mu < i \leqq \lambda} \left(1 - \frac{\overline{\tilde{\chi}(p_i)}}{p_i^{2s}}\right)$$

$$= \sum_{k_i = r_i}^{n_i} N^{-1}N' \prod_{k_i = 0} \left(1 - \frac{\overline{\tilde{\chi}(p_i)}}{p_i^{2s}}\right) (p_i + 1)^{-1} \boldsymbol{T}^{(N')} \{\pi^{-s}\Gamma(s)E_{N'}(\tau, s;\ \cdot\ , \chi)\}$$

$$= \sum_{k_i} \prod_{k_i = 0} \left(1 - \frac{\overline{\tilde{\chi}(p_i)}}{p_i^{2s}}\right)\left(1 - \frac{\tilde{\chi}(p_i)}{p_i^{2-2s}}\right)^{-1} N'^{-2s} \boldsymbol{A}^{(N')} \boldsymbol{P}_{k_1 \cdots k_\lambda} \{\pi^{s-1}\Gamma(1-s)E_N(\tau, 1-s;\ \cdot\ , \chi)\}$$

Since
$$\frac{\gamma(1-s, \bar{\chi})}{\gamma(s, \chi)} = \frac{\sqrt{\pi}\,\Gamma\left(s - \frac{1}{2}\right)}{\Gamma(s)} \frac{L(2s-1, \bar{\tilde{\chi}})}{L(2s, \bar{\tilde{\chi}})} S_\chi \tilde{N}^{2s-2},$$

we have

$$E(\tau, s;\ \cdot\ , \chi) = \frac{\sqrt{\pi}\,\Gamma\left(s - \frac{1}{2}\right)}{\Gamma(s)} \frac{L(2s-1, \bar{\tilde{\chi}})}{L(2s, \bar{\tilde{\chi}})} \Phi^{(1)} \otimes \cdots \otimes \Phi^{(\lambda)}\ E(\tau, 1-s;\ \cdot\ , \bar{\chi})$$

where
$$\Phi^{(i)} = \sum_{k = r_i}^{n_i} p_i^{(2s-2)r_i} p_i^{(1-2s)} \boldsymbol{A}^{(k)} \boldsymbol{P}_k \qquad (1 \leqq i \leqq \mu)$$

$$= \sum_{k=1}^{n_i} p_i^{(1-2s)k} \left(\frac{1 - \tilde{\chi}(p_i)p_i^{-2-2s}}{1 - \tilde{\chi}(p_i)p_i^{-2s}}\right) \overline{\tilde{\chi}(p_i)}^k \boldsymbol{P}_k + \boldsymbol{P}_0$$

$$(\mu < i \leqq \lambda).$$

By (11) and (15), $\Phi^{(i)}$ can be written as stated in our theorem. (cf. the proof of prop. 1, 2)

LITERATURE

[1] T. Kubota, Elementary theory of Eisenstein series (in Japanese), Tokyo University, 1968.

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