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Nagoya Math. J.
Vol. 34 (1969), 121–127

A NOTE ON GALOIS COHOMOLOGY GROUPS OF ALGEBRAIC TORI

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§1. Introduction

Let k be a complete field of characteristic 0 whose topology is defined by a discrete valuation and let T be an algebraic torus of dimension d defined over k. As is well known, T has a splitting field K which is a finite Galois extension of k with Galois group $\mathfrak{G}$. For a ring R, denote by T_R the subgroup of R-rational points of T. Then T_K and $T_{\mathfrak{o}_K}$, $\mathfrak{o}_K$ being a valuation ring of K, become $\mathfrak{G}$-modules in the usual manner.

In the present paper, we shall show some properties of $\mathfrak{G}$-modules T_K and $T_{\mathfrak{o}_K}$. Namely, in Section 2, we shall obtain Theorem 1 as an analogy to the results as is well known in the local fields. In Section 3, we shall consider the Galois cohomology groups of T_K and $T_{\mathfrak{o}_K}$ as $\mathfrak{G}$-modules [Theorem 2]. Analogous results in the case of number fields were obtained in [11] and [15]. In Section 4, we shall obtain the explicit structure of the Galois cohomology groups of $T_{\mathfrak{o}_K}$ for the totally ramified extension of prime degree.

The auther wishes to express his heartfelt thanks to Prof. T. Kubota for his kind leading.

§2. Unramified extension

In this section, we suppose that the splitting field K is always an unramified extension of k. We denote by $\mathfrak{u}_K$ (resp. $\mathfrak{u}_k$) the group of units of K (resp. k). For a unique prime divisor $\mathfrak{P}$ (resp. $\mathfrak{p}$) of K, we set for the integer $r \geqq 0$

$$\mathfrak{u}_K^{(r)} = \{\alpha \in \mathfrak{u}_K,\ \alpha \equiv 1 \bmod. \mathfrak{P}^r\},\ \ \mathfrak{u}_K^{(0)} = \mathfrak{u}_K,$$
$$\mathfrak{u}_k^{(r)} = \{\alpha \in \mathfrak{u}_k,\ \alpha \equiv 1 \bmod. \mathfrak{p}^r\},\ \ \mathfrak{u}_k^{(0)} = \mathfrak{u}_k,$$

and define $T_{\mathfrak{o}_K}^{(r)}$ by

Received May 30, 1968

$$T^{(r)}_{\mathfrak{o}_K} = \mathrm{Hom}\,(\hat{T}, \mathfrak{u}^{(r)}_K) = \{x \in T_K,\ \xi(x) \in \mathfrak{u}^{(r)}_K \text{ for all } \xi \in \hat{T}\}$$

where $\hat{T}$ is the character module of T.

As is well known, T_k is the $\mathfrak{G}$-invariant subgroup of T_K. Hence, for the valuation ring $\mathfrak{o}_k$ of k, we set

$$T^{(r)}_{\mathfrak{o}_k} = \mathrm{Hom}_{\mathfrak{G}}(\hat{T},\ \mathfrak{u}^{(r)}_K) = \hat{T}^{(r)}_{\mathfrak{o}_K} \cap T_k.$$

LEMMA 1. *For all* $r \geqq 0$, *we have*

$$T^{(r)}_{\mathfrak{o}_K} = \{x \in T_k,\ \xi(x) \in \mathfrak{u}^{(r)}_k \textit{ for all } \xi \in (\hat{T})_k\}$$

Proof. Take $x \in T_k$ with $\xi(x) \in \mathfrak{u}^{(r)}_k$ for all $\xi \in (\hat{T})_k$. Then, for any $\eta \in \hat{T}$, we have $N_{K/k}(\eta(x)) \in \mathfrak{u}^{(r)}_k$ and hence $\eta(x) \in \mathfrak{u}^{(r)}_K$ from the theory of local fields. The converse is trivial.

We denote by N the norm mapping $T_K \longrightarrow T_k$ in the usual sense. Then, it is clear that N maps $T^{(r)}_{\mathfrak{o}_K}$ into $T^{(r)}_{\mathfrak{o}_k}$ for any r. Hence, passing to the quotient, we can define a mapping N_r

$$N_r:\ T^{(r)}_{\mathfrak{o}_K}/T^{(r+1)}_{\mathfrak{o}_K} \longrightarrow T^{(r+1)}_{\mathfrak{o}_k}/T^{(r+1)}_{\mathfrak{o}_k}.$$

LEMMA 2. *For all* $r \geqq 1$, N_r *is surjective.*

Proof. By a well known property of local fields, we have the exact sequence

$$0 \longrightarrow \mathfrak{u}^{(r+1)}_K \longrightarrow \mathfrak{u}^{(r)}_K \longrightarrow \bar{K} \longrightarrow 0 \qquad (r \geqq 1),$$

where $\bar{K}$ is the residue field of K.
Since $\hat{T}$ is a Z-free module, we obtain the exact sequence

$$0 \longrightarrow \mathrm{Hom}\,(\hat{T}, \mathfrak{u}^{(r+1)}_K) \longrightarrow \mathrm{Hom}\,(\hat{T}, \mathfrak{u}^{(r)}_K) \longrightarrow \mathrm{Hom}\,(\hat{T}, \bar{K}) \longrightarrow 0.$$

On the other hand, we have $\mathrm{Hom}\,(\hat{T}, \bar{K}) \cong (\hat{T})^* \otimes \bar{K}$, $(\hat{T})^*$ being the dual module of $\hat{T}$. Since $\bar{K}$ is a cohomologically trivial $\mathfrak{G}$-module, $\mathrm{Hom}\,(\hat{T}, \bar{K})$ is also cohomologically trivial.[1] Hence,

$$T^{(r)}_{\mathfrak{o}_k}/T^{(r+1)}_{\mathfrak{o}_k} = (T^{(r)}_{\mathfrak{o}_K}/T^{(r+1)}_{\mathfrak{o}_K})^{\mathfrak{G}} = N_r(T^{(r)}_{\mathfrak{o}_K}/T^{(r+1)}_{\mathfrak{o}_K}).$$

PROPOSITION 1. $T^{(r)}_{\mathfrak{o}_k} = N(T^{(r)}_{\mathfrak{o}_K})$, *for all* $r \geqq 1$.

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[1] Cf. [8], Theorem 2.

Proof. Since $T_{\mathfrak{o}_K}^{(r)} = \text{lim. proj. } T_{\mathfrak{o}_K}^{(r)}/T_{\mathfrak{o}_K}^{(n)}$ and $T_{\mathfrak{o}_k}^{(r)} = \text{lim. proj. } T_{\mathfrak{o}_k}^{(r)}/T_{\mathfrak{o}_k}^{(n)}$, our proposition follows from lemma 2 and [Bourbaki, Alg. comm. §2].

COROLLARY 1. *The* 0*-dimensional Galois cohomology groups* $\hat{H}^0(G, T_{\mathfrak{o}_K}^{(r)})$ *are trivial for all* $r \geqq 1$.

COROLLARY 2. *For every dimension* n, *the Galois cohomology groups* $\hat{H}^n(G, T_{\mathfrak{o}_K}^{(1)})$ *are trivial.*

Proof. Since $\mathfrak{u}_K^{(1)}$ is cohomologically trivial by virtue of unramifiedness, $T_{\mathfrak{o}_K}^{(1)} = \text{Hom}(\hat{T}, \mathfrak{u}_K^{(1)}) \cong (\hat{T})^* \otimes \mathfrak{u}_K^{(1)}$ is also cohomologically trivial.

THEOREM 1. *For an unramified extension* K/k, *the group* $T_{\mathfrak{o}_k}/N\, T_{\mathfrak{o}_K}$ *is isomorphic to the group* $T_{\bar{k}}^{(\mathfrak{p})}/N\, T_{\bar{K}}^{(\mathfrak{P})}$, *where* $T^{(\mathfrak{P})}$ (*resp.* $T^{(\mathfrak{p})}$) *is the reduction modulo* $\mathfrak{P}$ (*resp.* $\mathfrak{p}$) *of* T.[2]

Proof. By a well known property of local fields, we have the exact sequence

$$0 \longrightarrow \mathfrak{u}_K^{(1)} \longrightarrow \mathfrak{u}_K \longrightarrow \bar{K}^* \longrightarrow 0,$$

where $\bar{K}^*$ is the multiplicative group of non-zero elements of the residue field. Since $\hat{T}$ is a Z-free module, we obtain the exact sequence

$$0 \longrightarrow \text{Hom}(\hat{T}, \mathfrak{u}_K^{(1)}) \longrightarrow \text{Hom}(\hat{T}, \mathfrak{u}_K) \longrightarrow \text{Hom}(\hat{T}, \bar{K}^*) \longrightarrow 0\,.$$

Passing to cohomology groups, we have the exact sequence

$$\begin{aligned} 0 \longrightarrow \text{Hom}_{\mathfrak{G}}(\hat{T}, \mathfrak{u}_K^{(1)}) &\longrightarrow \text{Hom}_{\mathfrak{G}}(\hat{T}, \mathfrak{u}_K) \longrightarrow \text{Hom}_{\mathfrak{G}}(\hat{T}, \bar{K}^*) \\ &\longrightarrow H^1(\mathfrak{G}, \text{Hom}(\hat{T}, \mathfrak{u}_K^{(1)})) \longrightarrow \cdots\cdot \end{aligned}$$

on the other hand, we have $\text{Hom}(\hat{T}, \bar{K}^*) = T_{\bar{K}}^{(\mathfrak{P})}$, and, by virtue of the unramifiedness, $\text{Hom}_{\mathfrak{G}}(\hat{T}, \bar{K}^*) = T_{\bar{k}}^{(\mathfrak{p})}$. Hence our theorem follows from the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & T_{\mathfrak{o}_K}^{(1)} & \longrightarrow & T_{\mathfrak{o}_K} & \longrightarrow & T_{\bar{K}}^{(\mathfrak{P})} & \longrightarrow & 1 \\ & & {\scriptstyle N}\downarrow & & {\scriptstyle N}\downarrow & & {\scriptstyle N}\downarrow & & \\ 1 & \longrightarrow & T_{\mathfrak{o}_k}^{(1)} & \longrightarrow & T_{\mathfrak{o}_k} & \longrightarrow & T_{\bar{K}}^{(\mathfrak{p})} & \longrightarrow & 1, \end{array}$$

from proposition 1, and corollary 2.

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[2] Cf. [12], Chap. V. §2. Proposition 3. and [1], Chap. 11.

COROLLARY. *If k is a locally compact field, we have $T_{\mathfrak{o}_k} = N T_{\mathfrak{o}_K}$.*

Proof. By virtue of the Lang's theorem [7], 1-dimensional Galois cohomology groups of a connected algebraic group defined over a finite field is trivial. Hence our corollary follows from Theorem 2 in the next section.

Remark. If we take a 1-dimensional torus $T = G_m$, our theorem is a familiar result for the unit group of a local field.

§3. Cyclic extension

In this section, we suppose that k is a locally compact and the splitting field K is a cyclic extension of degree n of k.

LEMMA 3. (T. *Springer*[3]) *For an arbitrary extension K of k, the 1-dimensional Galois cohomology group $H^1(\mathfrak{G}, T_K)$ of T_K is finite.*

Proof. Let $(K: k) = n$. Then we have the exact sequence

$$1 \longrightarrow F \xrightarrow{i} T \xrightarrow{n} T \longrightarrow 1 \qquad (F\colon \text{finite}),$$

where n is n-th. power mapping from T to T. Passing to cohomology groups, we have the exact sequence

$$\cdots\cdots \longrightarrow H^1(k, F) \xrightarrow{i^*} H^1(k, T) \xrightarrow{n^*} H^1(k, T) \longrightarrow \cdots\cdots.$$

In $H^1(k, T) \cong H^1(\mathfrak{G}, T_K)$, the order of each elements divides n and hence i^* is surjective.

LEMMA 4. *For sufficiently large integers m, the Herbrand quotients $h(T^{(m)}_{\mathfrak{o}_K})$ of $T^{(m)}_{\mathfrak{o}_K}$ are trivial.*

Proof. We denote by e the ramification index in K/k and take $m = em'$. Then we have $\mathfrak{u}^{(m)}_K \cong \mathfrak{P}^m = \mathfrak{p}^{m'} \mathfrak{o}_K \cong \mathfrak{o}_K$ as $\mathfrak{G}$-modules and hence $\mathrm{Hom}(\hat{T}, \mathfrak{u}^{(m)}_K) \cong \mathrm{Hom}(\hat{T}, \mathfrak{o}_K)$. Denote by $\{\omega^\sigma\}_{\sigma\in\mathfrak{G}}$ the normal basis of K/k and set $M = \sum_{\sigma\in\mathfrak{G}} \mathfrak{o}_k\omega^\sigma$ (direct). Then we have the exact sequence

$$0 \longrightarrow M \longrightarrow \mathfrak{o}_K \longrightarrow \mathfrak{o}_K/M \longrightarrow 0 \qquad (\mathfrak{o}_K/M\colon \text{finite}).$$

Since $\hat{T}$ is Z-free, we obtain the exact sequence

$$0 \longrightarrow \mathrm{Hom}(\hat{T}, M) \longrightarrow \mathrm{Hom}(\hat{T}, \mathfrak{o}_K) \longrightarrow \mathrm{Hom}(\hat{T}, \mathfrak{o}_K/M) \longrightarrow 0.$$

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3) Cf. [14], Proof of Theorem 3.2.

On the other hand, M is a $\mathfrak{G}$-regular module and hence $h(\mathrm{Hom}(\hat{T}, M))=1$. Since $\mathrm{Hom}(\hat{T}, \mathfrak{o}_K/M) \cong (\mathfrak{o}_K/M)^d$ is finite, our lemma follows from the properties of Herbrand quotient.

THEOREM 2. *For a cyclic extension K/k, the Galois cohomology groups $\hat{H}^n(\mathfrak{G}, T_{\mathfrak{o}_K})$ of $T_{\mathfrak{o}_K}$ have the same order for all dimensions n*[4].

Proof. Our theorem follows from lemma 4, the exact sequence

$$0 \longrightarrow \mathrm{Hom}(\hat{T}, \mathfrak{u}_K^{(m)}) \longrightarrow \mathrm{Hom}(\hat{T}, \mathfrak{u}_K) \longrightarrow \mathrm{Hom}(\hat{T}, \mathfrak{u}_K/\mathfrak{u}_K^{(m)}) \longrightarrow 0,$$

and the properties of Herbrand quotient.

COROLLARY 1. *The Herbrand quotient $h(T_K)$ of T_K is n^d, where $d = \dim . T$.*

Proof. Let η_i, $1 \leqq i \leqq d$, be a basis of $\hat{T}$ and let ϕ be the map $T_K \longrightarrow \boldsymbol{Z}^d$ defined by

$$\phi(x) = (v_K(\eta_1(x)), \cdots\cdots, v_K(\eta_d(x))), \text{ for } x \in T_K$$

where v_K is the discrete valuation. Then we have the exact sequence

$$0 \longrightarrow T_{\mathfrak{o}_K} \longrightarrow T_K \longrightarrow \boldsymbol{Z}^d \longrightarrow 0.$$

Our corollary follows from lemma 4 and the properties of Herbrand quotient.

COROLLARY 2. *If K/k is an unramified extension, we have $H^1(\mathfrak{G}, T_K) = 0$.*

Proof. This follows from corollary 1 and corollary of theorem 1.

§4. Totally ramified extension.

In this section, we suppose that K is a totally ramified extension of prime degree q of a $\mathfrak{p}$-adic field k. From the theory of local fields, there exists an integer $t \geqq 0$ such that the Hasse map ψ is given by

$$\psi(x) = \begin{cases} x & , \text{ for } x \leqq t, \\ x + q(x-t), & \text{for } x \geqq t. \end{cases}$$

As is well known, $N_{K/k}(\mathfrak{u}_K^{(\psi(n))}) = \mathfrak{u}_k^{(n)}, (n > 0)$, and $N_{K/k}(\mathfrak{u}_K^{(\psi(n)+1)}) = \mathfrak{u}_k^{(n+1)}$, $(n \geqq 0)$. Hence we have

$$T_{\mathfrak{o}_k}^{(t)} = \{x \in T_k, \ \xi(x) \in \mathfrak{u}_k^{(t)} \text{ for all } \xi \in (\hat{T})_k\}$$

in the same way as in lemma 1.

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[4] Cf. [11], Theorem 2, and [15], Theorem 3.

Now, let η_i, $1 \leqq i \leqq d$, be a basis of $\hat{T}$ such that η_i, $1 \leqq i \leqq s$, is a basis of $(\hat{T})_k$, where $s = \text{rank}\,(\hat{T})_k$. Let Φ_K (resp. ϕ_k) be the map $T_K \longrightarrow (K^*)^d$, (resp. $T_k \longrightarrow (k^*)^s$), defined by

$$\Phi_K(x) = (\eta_1(x), \cdots\cdots, \eta_d(x)), \text{ for } x \in T_K,$$

$$\phi_k(x) = (\eta_1(x), \cdots\cdots, \eta_s(x)), \text{ for } x \in T_k.$$

Then Φ_K is an isomorphism and ϕ_k an injection.

LEMMA 5. *The norm map N: $T^{(t)}_{\mathfrak{o}_K} \longrightarrow T^{(t)}_{\mathfrak{o}_k}$ is surjective.*

Proof. This follows from $N_{K/k}(\mathfrak{u}^{(t)}_K) = \mathfrak{u}^{(t)}_k$ and the above property of ϕ_k.

Let N_0 be the mapping $T_{\mathfrak{o}_K}/T^{(1)}_{\mathfrak{o}_K} \longrightarrow T_{\mathfrak{o}_k}/T^{(1)}_{\mathfrak{o}_k}$ induced by the norm map N. Since

$$\phi_k(N(x)) = (\eta_1(N(x)), \cdots\cdots, \eta_s(N(x)))$$
$$= (N(\eta_1(x)), \cdots\cdots, N(\eta_s(x))), \text{ for } x \in T_K,$$

the image of N_0 is isomorphic to $\bar{K}^{*p} \times \cdots\cdots \times \bar{K}^{*p}$, where $\bar{K}^{*p}$ is the group of the n-th. powers of elements of $\bar{K}^*$. Since the group $T_{\mathfrak{o}_k}/T^{(1)}_{\mathfrak{o}_k}$ is a proper subgroup of $(\mathfrak{u}_k/\mathfrak{u}^{(1)}_k)^s = (\bar{K}^*)^s$, we have the following

PROPOSITION 2. *If the characteristic p of the residue field $\bar{k}$ is not equal to q, the cokernel of N_0 is trivial.*

Let now N_t be the mapping $T^{(t)}_{\mathfrak{o}_K}/T^{(t+1)}_{\mathfrak{o}_K} \longrightarrow T^{(t)}_{\mathfrak{o}_k}/T^{(t+1)}_{\mathfrak{o}_K}$ induced by the norm map N. Then the image of N_t is isomorphic to $\mathscr{P}(\bar{K}) \times \cdots\cdots \times \mathscr{P}(\bar{K})$, where $\mathscr{P}$ is Artin-Schreier map, i.e. $\mathscr{P}(x) = x^p - x$ for $x \in \bar{K}$. Since $T^{(t)}_{\mathfrak{o}_k}/T^{(t+1)}_{\mathfrak{o}_k}$ is a proper subgroup of $(\bar{K})^s$, we have

PROPOSITION 3. *If $p = q$, the cokernel of N_0 is trivial.*

THEOREM 3. *Let K be a totally ramified extension of prime degree q of k. Then, for every dimension $n \in Z$, the Galois cohomology groups $\hat{H}^n(\mathfrak{G}, T_{\mathfrak{o}_K})$ of $T_{\mathfrak{o}_K}$ are trivial.*

Proof. Our theorem follows from the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & T^{(t)}_{\mathfrak{o}_K} & \longrightarrow & T_{\mathfrak{o}_K} & \longrightarrow & T_{\mathfrak{o}_K}/T^{(t)}_{\mathfrak{o}_K} & \longrightarrow & 1 \\ & & {\scriptstyle N}\big\downarrow & & {\scriptstyle N}\big\downarrow & & {\scriptstyle N'}\big\downarrow & & \\ 1 & \longrightarrow & T^{(t)}_{\mathfrak{o}_k} & \longrightarrow & T_{\mathfrak{o}_k} & \longrightarrow & T_{\mathfrak{o}_k}/T^{(t)}_{\mathfrak{o}_k} & \longrightarrow & 1 \end{array}$$

lemma 5, proposition 2 and proposition 3.

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