M. Nisio
Nagoya Math. J.
Vol. 34 (1969), 89–104

ON THE CONTINUITY OF STATIONARY GAUSSIAN PROCESSES

MAKIKO NISIO

1. Introduction

Let us consider a stochastically continuous, separable and measurable stationary Gaussian process[1] $\boldsymbol{X} = \{X(t), \ -\infty < t < \infty\}$ with mean zero and with the covariance function $\rho(t) = EX(t+s)X(s)$. The conditions for continuity of paths have been studied by many authors from various viewpoints. For example, Dudley [3] studied from the viewpoint of ε-entropy and Kahane [5] showed the necessary and sufficient condition in some special case, using the rather neat method of Fourier series.

In this note we shall discuss the continuity of paths of $\boldsymbol{X}$, making use of the idea presented by Kahane. Our results are following: We express the covariance function ρ in the form

$$\rho(t) = \int_{-\infty}^{\infty} e^{it\lambda}\, dF(\lambda)$$

with a finite measure dF, symmetric with respect to origin.

Put $\qquad s_n = F(2^n, 2^{n+1}], \qquad n = 0, 1, 2, \cdots.$

THEOREM 1. *If* $E \sup\limits_{t\in[0,1]} |X(t)| < \infty$, *then* $\sum\limits_{n=0}^{\infty} \sqrt{s_n} < \infty$.

THEOREM 2. *Suppose that we can choose a decreasing sequence* $\{M_n\}$ *so that* $M_n \geq s_n$ *and* $\sum\limits_{n=0}^{\infty} \sqrt{M_n} < \infty$. *Then* $E \sup\limits_{t\in[0,1]} |X(t)| < \infty$.

THEOREM 3. *Suppose that* ρ *is convex on a small interval* $[0, \delta]$. *Then* $\sum\limits_{n=0}^{\infty} \sqrt{s_n} < \infty$, *if* $\boldsymbol{X}$ *has continuous paths.*

By virtue of Theorem 2, we can easily see

COROLLARY. *Suppose that* ρ *is convex on a small interval* $[0, \delta]$ *and* s_n *is*

Received May 24, 1968

[1] We mean a real valued process.

decreasing. Then $\boldsymbol{X}$ *has continuous paths if, and only if,* $E \sup_{t\in[0,1]} |X(t)| < \infty$, *which is equivalent to* $\sum_{n=0}^{\infty} \sqrt{s_n} < \infty$.

The author with to express her sincerely thanks to Professor T. Sirao for his valuable suggestions.

2. Lemmas

Let $\{T_j, j = 1, 2, \cdots\}$ be a sequence of increasing positive numbers such that $\sum_{j=1}^{\infty} \frac{1}{T_j} < \infty$. According to [5, p. 69], we shall define following functions,

$$\chi(x) = \max(1-|x|, 0), \qquad -\infty < x < \infty,$$

$$\theta_r(\lambda) = \prod_{j=r}^{\infty} \chi\left(\frac{\lambda}{T_j}\right), \qquad -\infty < \lambda < \infty,$$

$$K_r(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{it\lambda} \chi\left(\frac{\lambda}{T_r}\right) d\lambda$$

$$l_r(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{it\lambda} \theta_r(\lambda) d\lambda = \frac{1}{\sqrt{2\pi}} \int_{-T_r}^{T_r} e^{it\lambda} \theta_r(\lambda) d\lambda$$

$$l_r^*(t) = \frac{1}{\sqrt{2\pi}} \int_{-T_{r-1}}^{T_{r-1}} e^{it\lambda} \theta_r(\lambda) d\lambda.$$

As to these functions, we can easily see that θ_r is symmetric, non-negative and continuous, and l_r and l_r^* continuous. Since

$$K_r(t) = \frac{\sqrt{2}}{t^2 T_r \sqrt{\pi}} (1 - \cos T_r t) \geq 0, \tag{1}$$

l_r is non-negative as the convolution of K_n, $n \geq r$.
The following Lemma 1 is clear.

LEMMA 1.

$$l_r(t) = (l_{r+1}^* * K_r)(t) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} l_{r+1}^*(t-s) K_r(s) ds$$

$$l_r(t) = (l_{r+1} * K_r)(t) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} l_{r+1}(t-s) K_r(s) ds$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} l_r(t) dt = 1.$$

We express $\boldsymbol{X}$ in the form

$$X(t) = \int_{-\infty}^{\infty} e^{it\lambda} d\Phi(\lambda)$$

with a random measure $d\Phi$. Let $\boldsymbol{X}$ satisfy the condition of Theorem 1. We define stationary Gaussian processes $\boldsymbol{Y}_r$ and $\boldsymbol{Y}_r^*$ by

$$Y_r(t,\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} X(t-s,\omega)l_r(s)ds \tag{2}$$

and

$$Y_r^*(t,\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} X(t-s,\omega)l_r^*(s)ds \tag{3}$$

respectively. By virtue of the condition of Theorem 1, we can see that, for a. a. ω, the Lebesgue integral of the right side of (2), as well as (3), is a continuous function of t. Moreover, $\boldsymbol{Y}_r$ and $\boldsymbol{Y}_r^*$ are expressible in the form

$$Y_r(t) = \int_{-T_r}^{T_r} e^{it\lambda}\theta_r(\lambda)d\Phi(\lambda) \tag{4}$$

and

$$Y_r^*(t) = \int_{-T_{r-1}}^{T_{r-1}} e^{it\lambda}\theta_r(\lambda)d\Phi(\lambda). \tag{5}$$

As to the supremum value of these processes, we have Lemma 2,

LEMMA 2.

$$E\sup_{t\in[0,1]}|Y_r(t)| \leq a$$

$$E\sup_{t\in[0,1]}|Y_r^*(t)| \leq 2a$$

where $a = E\sup_{t\in[0,1]}|X(t)|$.

Proof. By Lemma 1, we have

$$E\sup_{t\in[0,1]}|Y_r(t)| < \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} E\sup_{t\in[0,1]}|X(t-s)|\,l_r(s)ds = a.$$

Put $Z_r(t) = Y_r(t) - Y_r^*(t)$. Then $\boldsymbol{Z}_r$ has continuous paths and is expressible in the form

$$Z_r(t) = \int_{T_{r-1}<|\lambda|\leq T_r} e^{it\lambda}\theta_r(\lambda)d\Phi(\lambda).$$

Therefore $\boldsymbol{Z}_r$ and $\boldsymbol{Y}_r^*$ are mutually independent. So, for any topological Borel set A of $C[0,1]$,

$$P(\widetilde{\boldsymbol{Y}}_r \in A) = \int_{C[0,1]} P(\widetilde{\boldsymbol{Y}}_r^* \in A - \xi)\, P(\widetilde{\boldsymbol{Z}}_r \in d\xi)$$

where $\tilde{f}$ stands for the restriction on $[0,1]$ of f. Hence, for $\varepsilon > 0$,

$$P(\sup_{t\in[0,1]} |Y_r(t)| < c) \leq \sup_{\xi \in C[0,1]} P(\sup_{t\in[0,1]} |Y_r^*(t) + \xi(t)| < c)$$

$$\leq P(\sup_{t\in[0,1]} |Y_r^*(t) + \xi_c(t)| < c) + \varepsilon$$

with $\xi_c \in C[0,1]$. On the other hand, by virtue of the symmetricty of $\boldsymbol{Y}_r^*$,

$$P(\sup_{t\in[0,1]} |Y_r^*(t) + \eta(t)| < c) = P(\sup_{t\in[0,1]} |Y_r^*(t) - \eta(t)| < c), \quad \eta \in C[0,1].$$

Therefore, we have

$$1 - (\sup_{t\in[0,1]} |Y_r^*(t)| \geq c) = P(2 \sup_{t\in[0,1]} |Y_r^*(t)| < 2c)$$

$$\geq P(\sup_{t\in[0,1]} |Y_r^*(t) + \xi_c(t)| + \sup_{t\in[0,1]} |Y_r^*(t) - \xi_c(t)| < 2c)$$

$$\geq P(\sup_{t\in[0,1]} |Y_r^*(t) + \xi_c(t)| < c,\ \sup_{t\in[0,1]} |Y_r^*(t) - \xi_c(t)| < c)$$

$$\geq 1 - 2P(\sup_{t\in[0,1]} |Y_r^*(t) + \xi_c(t)| \geq c)$$

$$\geq 2P(\sup_{t\in[0,1]} |Y_r(t)| < c) - 2\varepsilon - 1.$$

Tending ε to 0, we get

$$P(\sup_{t\in[0,1]} |Y_r^*(t)| \geq c) \leq 2P(\sup_{t\in[0,1]} |Y_r(t)| \geq c).$$

Hence

$$(6) \qquad \sum_{n=0}^{N} \frac{n}{2^k} P\Big(\frac{n}{2^k} \leq \sup_{t\in[0,1]} |Y_r^*(t)| < \frac{n+1}{2^k}\Big)$$

$$= \frac{1}{2^k} \sum_{n=1}^{N} P(\sup_{t\in[0,1]} |Y_r^*(t)| \geq \frac{n}{2^k}\Big) - \frac{N}{2^k} P\Big(\sup_{t\in[0,1]} |Y_r^*(t)| \geq \frac{N+1}{2^k}\Big)$$

$$\leq 2\sum_{n=0}^{N+1} \frac{n+1}{2^k} P\Big(\frac{n}{2^k} \leq \sup_{t\in[0,1]} |Y_r(t)| < \frac{n+1}{2^k}\Big) + \frac{N+1}{2^{k-1}} P\Big(\sup_{t\in[0,1]} |Y_r(t)| \geq \frac{N+1}{2^k}\Big).$$

Appealing to the former half of Lemma 2, we have $NP\Big(\sup_{t\in[0,1]} |Y_r(t)| \geq \frac{N+1}{2^k}\Big)$ tends to 0, as $N \uparrow \infty$. So, (6) implies the latter half of Lemma 2.

Define stationary Gaussian processes $\boldsymbol{V}_r$ and $\boldsymbol{V}_r^*$ by

$$V_r(t) = \frac{1}{\sqrt{2\pi}} \int_{|s| > \frac{1}{\sqrt{T_r}}} Y_{r+1}(t-s) K_r(s) ds$$

and

$$V_r^*(t) = \frac{1}{\sqrt{2\pi}} \int_{|s| > \frac{1}{\sqrt{T_r}}} Y_{r+1}^*(t-s) K_r(s) ds.$$

Then we can easily see, by Lemma 2,

LEMMA 3.

$$E \sup_{t \in [0,1]} |V_r(t)| \leq \frac{4\sqrt{2}\, a}{\sqrt{\pi}\, T_r}$$

$$E \sup_{t \in [0,1]} |V^*(t)| \leq \frac{8\sqrt{2}\, a}{\sqrt{\pi}\, T_r} .$$

3. Proof of Theorem 1.

To prove Theorem 1, we shall firstly show the following proposition,

PROPOSITION. *Let $\{T_r\}$ be a sequence of increasing positive numbers such that $\sum_{r=1}^{\infty} \frac{1}{\sqrt{T_r}} < \infty$. Then*

$$\sum_{j=1}^{\infty} \Big(\int_{T_j < |\lambda| \leq T_{j+1}} \prod_{k=j+1}^{\infty} \Big(1 - \frac{|\lambda|}{T_k}\Big)^2 dF(\lambda) \Big)^{\frac{1}{2}} < \infty.$$

Proof.

We define successively random variables S_j, S_j' and H_j, $j = 1, 2, \cdots$, as follows,

$$S_1(\omega) \equiv 0$$

$$H_1(\omega) \equiv Y_1(S_1(\omega), \omega)$$

$$S_1'(\omega) \equiv \begin{cases} \min \{t;\ |t| \leq \tau_1,\ Y_2^*(t,\omega) = \min\limits_{|s| \leqslant \tau_1} Y_2^*(s,\omega), & \text{if } H_1(\omega) < \min\limits_{|s| \leqslant \tau_1} Y_2^*(s,\omega) \\ \min \{t;\ |t| \leq \tau_1,\ Y_2^*(t,\omega) = \max\limits_{|s| \leqslant \tau_1} Y_2^*(s,\omega), & \text{if } H_1(\omega) > \max\limits_{|s| \leqslant \tau_1} Y_2^*(s,\omega) \\ \min \{t;\ |t| \leq \tau_1,\ Y_2^*(t,\omega) = H_1(\omega), & \text{otherwise} \end{cases}$$

where $\tau_1 = 1 + \frac{1}{\sqrt{T_1}}$. We can easily see that S_1' is measurable with respect to the Borel field, $\mathscr{B}_1$, spanned by $\{d\Phi(\lambda),\ |\lambda| \leq T_1\}$.

$$S_{j+1}(\omega) \equiv \begin{cases} S_j'(\omega), \text{ if } Y_{j+1}(S_j'(\omega), \omega) \geq H_j(\omega) \\ \min\{t\,;\,|t| \leq \tau_j,\ Y_{j+1}(t,\omega) = \max\limits_{|s|\leq \tau_j} Y_{j+1}(s,\omega), \text{ if } H_j(\omega) > \max\limits_{|s|\leq\tau_j} Y_{j+1}(s,\omega) \\ \min\{t\,;\,|t| \leq \tau_j,\ Y_{j+1}(t,\omega) = H_j(\omega), \text{ otherwise.} \end{cases}$$

$H_{j+1}(\omega) \equiv Y_{j+1}(S_{j+1}(\omega), \omega)$.

$$S_{j+1}'(\omega) \equiv \begin{cases} \min\{t\,;\,|t| \leq \tau_{j+1},\ Y_{j+2}^*(t,\omega) = \min\limits_{|s|\leq\tau_{j+1}} Y_{j+2}^*(s,\omega)\}, \\ \qquad\qquad\qquad \text{if } H_{j+1}(\omega) < \min\limits_{|s|\leq\tau_{j+1}} Y_{j+2}^*(s,\omega) \\ \min\{t\,;\,|t| \leq \tau_{j+1},\ Y_{j+2}^*(t,\omega) = \max\limits_{|s|\leq\tau_{j+1}} Y_{j+2}^*(t,\omega)\}, \\ \qquad\qquad\qquad \text{if } H_{j+1}(\omega) > \max\limits_{|s|\leq\tau_{j+1}} Y_{j+2}^*(s,\omega) \\ \min\{t\,;\,|t| \leq \tau_{j+1},\ Y_{j+2}^*(t,\omega) = H_{j+1}(\omega)\}, \text{ otherwise,} \end{cases}$$

where $\tau_j = 1 + \frac{1}{\sqrt{T_1}} + \cdots\cdots + \frac{1}{\sqrt{T_j}}$. Successively, we can prove that S_j and S_j' are measurable w. r. to the Borel field, $\mathscr{B}_j$, spanned by $\{d\Phi(\lambda), |\lambda| \leq T_j\}$.

We shall show the boundedness of H_j.

LEMMA 4.

$$\sup_{j=1,2,\cdots} |H_j(\omega)| < \infty, \qquad \text{a. a. } \omega.$$

Proof. By virtue of Lemma 3, we have

$$\sum_{r=1}^{\infty} E \sup_{|t|\leq\tau} |V_r(t)| < \infty, \tag{7}$$

where $\tau = \lim\limits_{j\to\infty} \tau_j$. On the other hand,

$$\begin{aligned} &\sup_{|t|\leq\tau} \left| \frac{1}{\sqrt{2\pi}} \int_{|s|\leq\frac{1}{\sqrt{T_r}}} X(t-s) l_r(s) ds \right| \\ &\leq \frac{1}{\sqrt{2\pi}} \int_{|s|\leq\frac{1}{\sqrt{T_r}}} \sup_{|u|\leq 2\tau} |X(u)| l_r(s) ds \\ &\leq \sup_{|u|\leq 2\tau} |X(u)| < \infty, \qquad \text{a. a. } \omega. \end{aligned} \tag{8}$$

Therefore, we see

$$\sup_{r=1,2,\cdots}\sup_{|t|\leqslant r}|Y_r(t)|<\infty. \qquad \text{a. a. } \omega.$$

Recalling the definition of H_j, we have Lemma 4.

(9) $$H_{j+1}(\omega)-H_j(\omega)$$
$$=\{(H_{j+1}(\omega)-H_j(\omega))\vee 0\}-\{(Y_j(S_j(\omega),\omega)-\sup_{|s|\leqslant\tau_j}Y_{j+1}(s,\omega))\vee 0\}^{2)}.$$

On the other hand, for $t\in[-\tau_{j-1},\tau_{j-1}]$,

$$Y_j(t)=\frac{1}{\sqrt{2\pi}}\int_{|s|\leq\frac{1}{\sqrt{T_j}}}Y_{j+1}(t-s)K_j(s)ds+\frac{1}{\sqrt{2\pi}}\int_{|s|>\frac{1}{\sqrt{T_j}}}Y_{j+1}(t-s)K_j(s)ds$$
$$\leq\sup_{|t|\leqslant\tau_j}Y_{j+1}(t)+\sup_{|t|\leqslant\tau_{j-1}}V_j(t).$$

So,

$$Y_j(t)-\sup_{|s|\leqslant\tau_j}Y_{j+1}(s)\leq\sup_{|s|\leqslant\tau_{j-1}}V_j(t), \qquad |t|\leq\tau_{j-1}.$$

Therefore,

$$(Y_j(S_j)-\sup_{|s|\leqslant\tau_j}Y_{j+1}(s))\vee 0\leq\sup_{|t|\leqslant\tau_{j-1}}|V_j(t)|.$$

Appealing to Lemma 3, we have

(10) $$\sum_{j=1}^{\infty}E\{(Y_j(S_j)-\sup_{|s|\leqslant\tau_j}Y_{j+1}(s))\vee 0\}<\infty.$$

As to the first term of the right side of (9),

$$\sum_{j=1}^{n}(H_{j+1}-H_j)\vee 0$$
$$=H_{n+1}-H_1+\sum_{j=1}^{n}(Y_j(S_j)-\sup_{|s|\leq\tau_j}Y_{j+1}(s))\vee 0.$$

Therefore, using Lemma 4 and (10), we get

(11) $$\sum_{j=1}^{\infty}(H_{j+1}-H_j)\vee 0<\infty, \qquad \text{a. a. } \omega.$$

On the other hand, recalling the definition of H_j, we see

(12) $$(H_{j+1}-H_j)\vee 0=(Y_{j+1}(S_j')-H_j)\vee 0$$

2) $a\vee b=\max(a,b)$.

$$\geq \{(Y_{j+1}(S'_j) - Y^*_{j+1}(S'_j)) \vee 0\} - \{(H_j - \sup_{|t| \leqslant \tau_j} Y^*_{j+1}(t)) \vee 0\}.$$

So, using the similar method as (10), we get

$$\sum_{j=1}^{\infty} E\{(H_j - \sup_{|t| \leqslant \tau_j} Y^*_{j+1}(t)) \vee 0\} < \infty. \tag{13}$$

Therefore, combining (11) and (13) to (12), we have

$$\sum_{j=1}^{\infty} (Y_{j+1}(S'_j) - Y^*_{j+1}(S'_j)) \vee 0 < \infty, \qquad \text{a. a. } \omega. \tag{14}$$

Put $\gamma_j = Y_{j+1}(S'_j) - Y^*_{j+1}(S'_j)$ and

$$v_j = \int_{T_j < |\lambda| \leqslant T_{j+1}} \prod_{k=j+1}^{\infty} \left(1 - \frac{|\lambda|}{T_k}\right)^2 dF(\lambda).$$

Then, we see, appealing to the independence of $d\Phi$,

$$P(\gamma_j \leq x / \mathscr{B}_j) = \frac{1}{\sqrt{2\pi v_j}} \int_{-\infty}^{x} e^{-\frac{y^2}{2v_j}} dy,$$

since S'_j is $\mathscr{B}_j$-measurable.
Hence

$$E(\gamma_j \vee 0) = \frac{\sqrt{v_j}}{\sqrt{2\pi}} \tag{15}$$

and

$$E(\gamma_j \vee 0)^2 = \frac{v_j}{2}\ . \tag{16}$$

Appealing to the following Lemma

LEMMA. [5, p. 64]. *If X is a non-negative random variable with mean finite, then*

$$P(X > \lambda E(X)) \geq (1-\lambda)^2 \frac{(EX)^2}{EX^2}\ , \qquad \forall \lambda \in (0,1),$$

we can derive

$$P\Big(\sum_{j=1}^{\infty}(\gamma_j \vee 0) > \frac{\sum_{j=1}^{n}\sqrt{v_j}}{2\sqrt{2\pi}}\Big) \geq P\Big(\sum_{j=1}^{n}(\gamma_j \vee 0) > \frac{\sum_{j=1}^{n}\sqrt{v_j}}{2\sqrt{2\pi}}\Big) \geq \frac{1}{4\pi}\ .$$

So,

$$P\Big(\sum_{j=1}^{\infty}(\gamma_j \vee 0) \geqq \frac{\sum_{j=1}^{\infty}\sqrt{v_j}}{2\sqrt{2\pi}}\Big) \geqq \frac{1}{4\pi}\ .$$

By virtue of (14), we conclude

$$\sum_{j=1}^{\infty}\sqrt{v_j} < \infty.$$

This completes the proof of Proposition.

Making use of Proposition, we can easily prove Theorem 1. Put $T_k = 2^k$ and $\alpha = \prod_{k=0}^{\infty}(1-3\cdot 2^{-k-2})^2$. Then we have

$$2\alpha F(2^j,\ 3\cdot 2^{j-1}] = \alpha\int_{2^j < |\lambda| \leqq \frac{3}{2}2^j} dF(\lambda) \leqq \int_{2^j < |\lambda| \leqq \frac{3}{2}2^j} \prod_{k=j+1}^{\infty}\Big(1-\frac{|\lambda|}{2^k}\Big)^2 dF(\lambda)$$

$$\leqq \int_{2^j < |\lambda| \leqq 2^{j+1}} \prod_{k=j+1}^{\infty}\Big(1-\frac{|\lambda|}{2^k}\Big)^2 dF(\lambda).$$

So, by Proposition,

$$\sum_{j=0}^{\infty} F(2^j,\ 3\cdot 2^{j-1}]^{\frac{1}{2}} < \infty. \tag{17}$$

Put $T_k = 3\cdot 2^{k-1}$ and $\alpha = \prod_{k=0}^{\infty}\Big(1-\frac{1}{3}2^{-k+1}\Big)^2$. Then we have

$$2\alpha F(3\cdot 2^{j-1},\ 2^{j+1}] \leqq \int_{3\cdot 2^{j-1} < |\lambda| \leqq 2^{j+1}} \prod_{k=j+1}^{\infty}\Big(1-\frac{|\lambda|}{3\cdot 2^{k-1}}\Big)^2 dF(\lambda)$$

$$\leqq \int_{3\cdot 2^{j-1} < |\lambda| \leqq 3\cdot 2^j} \prod_{k=j+1}^{\infty}\Big(1-\frac{|\lambda|}{3\cdot 2^{k-1}}\Big)^2 dF(\lambda).$$

So,

$$\sum_{j=0}^{\infty} F(3\cdot 2^{j-1},\ 2^{j+1}]^{\frac{1}{2}} < \infty. \tag{18}$$

By virtue of (17) and (18), we have Theorem 1.

4. Proof of Theorem 2

We shall first assume that s_n is decreasing and $\sum_{n=0}^{\infty}\sqrt{s_n} < \infty$. We put $c(j) = 2^{2^j}$ and define ξ_j and η_j by

$$\xi_j(t) = \int_{c(j-1)<|\lambda|\leq c(j)} e^{it\lambda}\, d\Phi(\lambda)$$

and

$$\eta_j = \max_{k=0,\cdots,c(j+1)} \left| \xi_j\left(\frac{k}{c(j+1)}\right)\right|, \quad j = 1, 2, \cdots$$

respectively. Then the process ξ_j has continuous paths. Appealing to the following Lemma,

LEMMA. [5. Proposition 2].

$$E\eta_j \leq h + \sum_{k=0}^{c(j+1)} \int_h^\infty |x|\, d\mu_{\xi_j\left(\frac{k}{c(j+1)}\right)}(x), \quad h > 0$$

where μ_ξ is the probability law of ξ, we have

$$E\eta_j \leq h + (c(j+1)+1)\sqrt{\frac{2\sigma_j}{\pi}}\, e^{-\frac{h^2}{2\sigma_j}}$$

where $\sigma_j = 2F(c(j-1),\ c(j)]$. Let $h = h(j) = \sqrt{2\sigma_j \log c(j+1)}$.

Then we see

$$E\eta_j \leq 2h(j). \tag{19}$$

Since $\sigma_j = \sum_{k=2^{j-1}}^{2^j-1} s_k$, we get

$$2^j\sigma_j = 2^j \sum_{k=2^{j-1}}^{2^j-1} s_k \leq 2^{2j} s_{2^{j-1}}.$$

Hence,

$$\sqrt{2^j\sigma_j} \leq 2^j\sqrt{s_{2^{j-1}}} \leq 4 \sum_{k=2^{j-2}}^{2^{j-1}} \sqrt{s_k}.$$

Consequently, by (19), we have

$$\sum_{j=1}^{\infty} E\eta_j < \infty. \tag{20}$$

Define ζ and θ by

$$\zeta(j,k,p,q,r) = \xi_j\left(\frac{k}{c(j+1)} + \frac{q}{c(j+1)c(p)} + \frac{r}{c(j+1)c(p+1)}\right)$$
$$- \xi_j\left(\frac{k}{c(j+1)} + \frac{q}{c(j+1)c(p)}\right),$$

$r=1,\ldots,c(p),\ q=1,\ldots,c(p),\ k=0,\ldots,c(j+1),\ p=1,2,\ldots,$ $j=1,2,\ldots,$ and

$$\theta(j,p)=\max_{k,q,r}|\zeta(j,k,p,q,r)|.$$

Then we see

$$E\zeta^2(j,k,p,q,r)=2\int_{c(j-1)<\lambda\leq c(j)}\left(1-\cos\frac{r}{c(j+1)c(p+1)}\lambda\right)dF(\lambda)$$

$$\leq\frac{1}{2}\frac{\sigma_j}{c^2(j)c^2(p)}.$$

Again, using the same Lemma, we have

$$E\theta(j,p)\leq 2\sqrt{\log c(p+1)\,c(j+1)}\,\frac{\sqrt{\sigma_j}}{c(j)c(p)}.$$

Therefore

$$\sum_{j=1}^{\infty}\sum_{p=1}^{\infty}E\theta(j,p)<\infty. \tag{21}$$

By virtue of the separability of X and ξ_j, we have

$$\sup_{t\in[0,1]}|X(t)|\leq\sum_{j=1}^{\infty}\sup_{t\in[0,1]}|\xi_j(t)|+|d\Phi(0)|,\qquad \text{a. a. }\omega,$$

and

$$\sup_{t\in[0,1]}|\xi_j(t)|\leq\eta_j+\sum_{p=1}^{\infty}\theta(j,p),\qquad \text{a. a. }\omega.$$

So, taking (20) and (21) into account, we complete the proof of Theorem 2 in the first case.

Define a symmetric finite measure G by

$$G(A)=F(A)+\sum_{n=0}^{\infty}(M_n-s_n)\delta_{2^{n+1}}(A),\qquad A\subset[0,\infty),$$

where δ_a is the delta measure concentrated at a. Let X_1 and X_2 be the mutually independent stationary Gaussian processes whose covariance function has the spectral measure F and $\sum_{n=0}^{\infty}(M_n-s_n)\delta_{2^n}(A)$, respectively. Then G is the spectral measure of the covariance function of X_1+X_2 and $G(2^n,2^{n+1}]=M_n$. So, using the result, we just proved,

$$E\sup_{t\in[0,1]}|X_1(t)+X_2(t)|<\infty.$$

Repeating the same method as Lemma 2, we have

$$E \sup_{t\in[0,1]} |X_1(t)| < \infty.$$

This completes the proof of Theorem 2.

5. Proof of Theorem 3

To prove Theorem 3, we shall first show the following Lemma,

LEMMA 5. *Assume that a symmetric, positive continuous function R is convex and decreasing on $[0,\pi]$. Then any Fourier coefficient a_n, i. e. $a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int}R(t)dt$, is non-negative. Moreover, $\sum_{h=-\infty}^{\infty} a_n = R(0)$.*

Proof. By symmetricity of R, for $n\geq 1$,

$$a_{-n} = a_n = \frac{1}{\pi}\int_0^{\pi} R(t)\cos nt\, dt = \frac{1}{n\pi}\int_0^{n\pi} R\Big(\frac{s}{n}\Big)\cos s\, ds. \tag{22}$$

$$\int_{2k\pi}^{2(k+1)\pi} R\Big(\frac{s}{n}\Big)\cos s\, ds$$

$$=\int_0^{\frac{\pi}{2}}\Big(R\Big(\frac{2k\pi+s}{n}\Big)-R\Big(\frac{2k\pi+\pi-s}{n}\Big)-R\Big(\frac{2k\pi+\pi+s}{n}\Big)+R\Big(\frac{2k\pi+2\pi-s}{n}\Big)\Big)\cos s\, ds.$$

By virtue of the convexity of R, the integrand is non-negative, and we have

$$\int_{2k\pi}^{2(k+1)\pi} R\Big(\frac{s}{n}\Big)\cos s\, ds\geq 0.$$

On the other hand, by the monotonicity of R,

$$\int_{2k\pi}^{2k\pi+\pi} R\Big(\frac{s}{n}\Big)\cos s\, ds = \int_0^{\frac{\pi}{2}}\Big(R\Big(\frac{2k\pi+s}{n}\Big)-R\Big(\frac{2k\pi+\pi-s}{n}\Big)\Big)\cos s\, ds\geq 0.$$

Therefore, appealing to (22), $a_n\geq 0$.

Since R is continuous and bounded variation, its Fourier series converges to R uniformly on any closed subset of $(-\pi,\pi)$. Hence $\sum_{n=-\infty}^{\infty} a_n = R(0)$.

LEMMA 6. *Let R be a continuous, symmetric and positive definite function on $(-\infty, \infty)$. Assume that each Fourier coefficient a_n, i.e. $a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int}R(t)dt$,* *is non-negative. Then, the spectral measure dG of R satisfies*

$$\sum_{n=0}^{\infty}\sqrt{G(2^n,\ 2^{n+1}]} < \infty$$

if $\displaystyle\sum_{n=0}^{\infty}\sqrt{\sum_{k=2^n+1}^{2^{n+1}} a_k} < \infty.$

Proof. By the symmetry of dG,

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int}\Big(\int_{-\infty}^{\infty} e^{it\lambda}\,dG(\lambda)\Big)\,dt \tag{23}$$

$$= \frac{1}{\pi}\int_0^{\infty}\frac{\sin(\lambda-n)\pi}{\lambda-n}\,dG(\lambda) + \frac{1}{\pi}\int_0^{\infty}\frac{\sin(\lambda+n)\pi}{\lambda+n}\,dG(\lambda),$$

where $\dfrac{\sin 0\pi}{0}$ is read as $\lim\limits_{x\to 0}\dfrac{\sin\pi x}{x} = \pi.$

Put $\displaystyle f(\lambda) = \sum_{n=2^k-4}^{2^{k+1}+3}\frac{\sin(\lambda-n)\pi}{\lambda-n}, \qquad \lambda\geqq 0, \quad (k = 4, 5, \dots).$

Then we have, for $m = 2^{k-1}+1, \dots, 2^k, \qquad \mu\in(0,1],$

$$f(2m-1+\mu) = \sum_{j=0}^{2m-2^k+3}\frac{\sin(j+\mu)\pi}{j+\mu} + \sum_{l=1}^{2^{k+1}+4-2m}\frac{\sin(l-\mu)\pi}{l-\mu}$$

$$\geqq\Big(\frac{1}{\mu}-\frac{1}{1+\mu}+\frac{1}{2+\mu}+\frac{1}{1-\mu}-\frac{1}{2-\mu}+\frac{1}{3-\mu}\Big)\sin\mu\pi \geqq \frac{7}{12}\,\frac{\sin\mu\pi}{\mu(1-\mu)},$$

and, by the same method,

$$f(2m+\mu)\geqq\frac{7}{12}\,\frac{\sin\mu\pi}{\mu(1-\mu)}, \quad \mu\in(0,1],\ \ m = 2^{k-1},\dots,2^k-1.$$

Therefore,

$$\int_{2^k+}^{2^{k+1}} f(\lambda)\,dG(\lambda)\geqq\frac{7}{12}\sum_{l=2^k}^{2^{k+1}-1}\int_{0+}^{1}\frac{\sin\pi\mu}{\mu(1-\mu)}\,dG(l+\mu). \tag{24}$$

On the other hand, we have the following inequalities,

$$f(\lambda)\geqq 0, \qquad \lambda\in[2^{k+1},\ 2^{k+1}+4]\cup[2^k-5,\ 2^k]. \tag{25}$$

$$f(2^{k+1}+4+\mu) = \frac{-\sin\mu\pi}{1+\mu}+\frac{\sin\mu\pi}{2+\mu}-\frac{\sin\mu\pi}{3+\mu}+\cdots+\frac{\sin\mu\pi}{2^k+8+\mu} \tag{26}$$

$$\geqq\Big(-\frac{1}{1+\mu}+\frac{1}{2+\mu}-\frac{1}{3+\mu}\Big)\sin\mu\pi\geqq -\frac{5}{6}\,\frac{\sin\mu\pi}{1+\mu}\geqq-\frac{5}{6}(3-\sqrt{8})\,\frac{\sin\mu\pi}{\mu(1-\mu)},$$

$$\mu\in[0,1].$$

$$f(2^{k+1}+j+\mu)\geqq-\frac{5}{6}(3-\sqrt{8})\frac{\sin\mu\pi}{\mu(1-\mu)}, \quad \mu\in[0,1],\ j = 5,\dots,2^{k+1}-1, \tag{27}$$

$$(28)\qquad f(2^k - j + \mu) \geq -\frac{5}{6}(3-\sqrt{8})\frac{\sin \mu\pi}{\mu(1-\mu)}, \quad \mu \in [0,1], \quad j = 6, \ldots, 2^{k-1}+1.$$

Hence, by (25) and (28),

$$(29)\qquad \int_{2^{k-1}+}^{2^k} f(\lambda) dG(\lambda) \geq -\frac{5}{6}(3-\sqrt{8}) \sum_{l=2^{k-1}}^{2^k-1} \int_{0+}^{1} \frac{\sin \mu\pi}{\mu(1-\mu)} dG(l+\mu)$$

and, by (25) and (27),

$$(30)\qquad \int_{2^{k+1}+}^{2^{k+2}} f(\lambda) dG(\lambda) \geq -\frac{5}{6}(3-\sqrt{8}) \sum_{l=2^{k+1}}^{2^{k+2}-1} \int_{0+}^{1} \frac{\sin \mu\pi}{\mu(1-\mu)} dG(l+\mu).$$

As to the value of integral of f on the remainder set of λ, we see

$$(31)\qquad \left|\int_0^{2^{k-1}} f(\lambda) dG(\lambda)\right| \leq \sum_{m=2^k-2}^{\infty} \int_0^{2^{k-1}} \frac{1}{(2m-\lambda)(2m+1-\lambda)} dG(\lambda)$$

$$\leq \sum_{m=2^k-2}^{\infty} \frac{G[0, 2^{k-1}]}{(2m-2^{k-1})^2} \leq \frac{R(0)}{2^{k-1}-5},$$

and, similarly

$$(32)\qquad \left|\int_{2^{k+2}}^{\infty} f(\lambda) dG(\lambda)\right| \leq \frac{R(0)}{2^{k+1}-5}$$

On the other hand,

$$(33)\qquad \left|\sum_{n=2^k-4}^{2^{k+1}+3} \int_0^{\infty} \frac{\sin(\lambda+n)\pi}{\lambda+n} dG(\lambda)\right|$$

$$\leq \sum_{m=2^{k-1}-2}^{2^k+2} \frac{R(0)}{(2m)^2} \leq \frac{R(0)}{2^k-5}$$

Consequently, taking (23) into account, we have

$$(34)\qquad \delta_k + \frac{3R(0)}{2^{k-1}-5} \geq \frac{7}{12}\Delta_k - \frac{5}{6}(3-\sqrt{8})(\Delta_{k+1}+\Delta_{k-1}),$$

where $\delta_k = \sum_{n=2^k-4}^{2^{k+1}+3} \frac{a_n}{\pi}$ and $\Delta_k = \sum_{l=2^k}^{2^{k+1}-1} \int_{0+}^{1} \frac{\sin \mu\pi}{\mu(1-\mu)} dG(l+\mu)$.

Since $\Delta_k \leq \pi \cdot G(2^k, 2^{k+1}]$, Δ_k tends to 0 as $n \uparrow \infty$. Therefore (34) implies

$$\sum_{k\geq 5}\sqrt{\delta_k} + \sum_{k\geq 5} \frac{\sqrt{3R(0)}}{\sqrt{2^{k-1}-5}} + \sqrt{\Delta_4} \geq \left(\sqrt{\frac{7}{12}} - 2\sqrt{\frac{5}{6}(3-\sqrt{8})}\right)\sum_{k\geq 5}\sqrt{\Delta_k} \geq \left(\sqrt{\frac{7}{12}} - \sqrt{\frac{6.88}{12}}\right)\sum_{k\geq 5}\sqrt{\Delta_k}.$$

By the assumption of Lemma 6, i.e., $\sum_k \sqrt{\delta_k} < \infty$, we have

$$\sum_{k=1}^{\infty} \sqrt{\Delta_k} < \infty. \tag{35}$$

Appealing to the following inequality

$$\frac{\sin \mu\pi}{\mu(1-\mu)} \geq 1, \quad \text{on } [0, 1],$$

we have $\Delta_k \geq G(2^k, 2^{k+1}]$ and, by (35), we complete the proof of Lemma 6.

Using Lemmas 5 and 6, we can easily prove Theorem 3. By the assumption of Theorem 3, we can choose a positive Δ, so that ρ is positive convex and decreasing on $[0, \Delta]$. Define a Gaussian process $\tilde{X}$ by $\tilde{X}(t) = X\left(\frac{\pi t}{\Delta}\right)$. Then the covariance function $\tilde{\rho}$ of $\tilde{X}$ is $\tilde{\rho}(t) = \rho\left(\frac{\pi t}{\Delta}\right)$, and its spectral measure $\tilde{F}$ is $\tilde{F}(A) = F\left(\frac{\Delta}{\pi} A\right)$ for any Borel set A. Since $\tilde{\rho}$ satisfies the condition of Lemma 5, we can construct a periodic covariance function R by

$$R(t) = \sum_{n=-\infty}^{\infty} a_n e^{int}, \quad -\infty < t < \infty,$$

where $a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{\rho}(t) e^{-int} dt$. Let Y be a stationary Gaussian process with mean zero and with the covariance function R. Since $R = \tilde{\rho}$ on $[-\pi, \pi]$, Y has the locally same probability law as $\tilde{X}$. So, Y has continuous paths. Hence Kahane's Theorem [5, p. 73], [3, p. 300] tells us that

$$\sum_{k=0}^{\infty} \sqrt{\sum_{n=2^k+1}^{2^{k+1}} a_n} < \infty.$$

Therefore, by Lemma 6, we have

$$\sum_{n=0}^{\infty} \sqrt{F\left(\frac{\Delta}{\pi} 2^n, \frac{\Delta}{\pi} 2^{n+1}\right]} < \infty.$$

This implies Theorem 3.

References

[1] Yu. K. Belayev, Continuity and Hölder's conditions for sample functions of stationary Gaussian processes, Fourth Berkeley Symposium on Mathematical Statistics and Probability, Vol. **2**, 23–33.

[2] J. Delporte, Extension des conditions suffisantes pour la construction de functions aléatoires normales, presque sûrement continues, possédant une covariance donnée, C. R. Acad. Sci. Paris **256** (1963), 3816–3819.

[3] R.M. Dudley, The sizes of compact subsets of Hilbert space and continuity of Gaussian processes, Jour. of Functional Analysis, **1** (1967), 290–330.
[4] X. Fernique, Continuité des processus Gaussiens, C. R. Acad. Sci, Paris 258 (1964), 6058–6060.
[5] J.P. Kahane, Series de Fourier aléatoires, Sém. Math. Supér. Univ. de Montréal. (1963).
[6] T. Sirao and H. Watanabe, On the Hölder continuity of stationary Gaussian processes, (to appear).

Kobe University