T. Zinno
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ON SOME PROPERTIES OF NORMAL MEROMORPHIC FUNCTIONS IN THE UNIT DISC

TOSHIKO ZINNO

1. We denote by D the unit disc $\{z; |z| < 1\}$ and by $\mathscr{S}$ the totality of one to one conformal mappings $z' = s(z)$ of D onto itself. A meromorphic function $f(z)$ in D is normal if and only if the family $\{f(s(z))\}_{s(z)\in\mathscr{S}}$ is a normal family in D in the sense of Montel. We denote by $\mathfrak{N}$ the totality of the normal meromorphic functions in D. Moreover, Noshiro introduced in [5] the notion of the normal meromorphic functions of the first category: $f(z)$ is a normal meromorphic function of the first category if and only if $f(z)$ belongs to $\mathfrak{N}$ and any sequence $\{f_n(z)\}$ obtained from the family $\{f(s(z))\}_{s(z)\in\mathscr{S}}$ can not admit a constant as a limiting function. We denote by $\mathfrak{N}_1$ the totality of the normal meromorphic functions of the first category. For instance, Schwarzian triangle functions belong to $\mathfrak{N}_1$. In §1, we shall give a necessary condition (Th. 1) and a sufficient condition (Th. 2) for a function to belong to $\mathfrak{N}_1$. Further we shall give some properties of a function of $\mathfrak{N}_1$. In these proofs the Hurwitz theorem will play an essential role.

In 1957, Lehto and Virtanen ([4]) showed that even if $f(z)$ and $g(z)$ belong to $\mathfrak{N}$, $f(z) \pm g(z)$ and $f(z)g(z)$ do not necessarily belong to $\mathfrak{N}$. Later Lappan ([2], [3]) gave sufficient conditions for $f(z) \pm g(z)$ and $f(z)g(z)$ to belong to $\mathfrak{N}$. In §2, we shall give a more general sufficient condition for $f(z)g(z)$ to belong to $\mathfrak{N}$ than that of Lappan.

§1. Normal meromorphic functions of the first category

2. We consider the hyperbolic distance

$$d(z_1, z_2) = \frac{1}{2}\log\frac{|1-\bar{z}_1 z_2| + |z_1 - z_2|}{|1-\bar{z}_1 z_2| - |z_1 - z_2|}$$

for z_1 and z_2 in D, and the chordal distance

$$\chi(\alpha, \beta) = \frac{|\alpha - \beta|}{\sqrt{1+|\alpha|^2}\sqrt{1+|\beta|^2}}$$

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and

$$\chi(\alpha, \infty) = \frac{1}{\sqrt{1 + |\alpha|^2}}$$

where α and β are complex values. Put

$$U(z, \delta) = \{\zeta;\ d(z, \zeta) < \delta\}.$$

We denote by $\bar{U}(z, \delta)$ the closure of $U(z, \delta)$.

LEMMA 1. *Let $f(z)$ be a function of $\mathfrak{N}_1$. Then so is*

$$\frac{af(z) + b}{cf(z) + d} \qquad (ad - bc \neq 0).$$

This follows immediately from the definition of $\mathfrak{N}_1$.

LEMMA 2 (*Noshiro* [5]). *Let $f(z)$ be a function of $\mathfrak{N}_1$. Then there exists a positive number ρ_0 such that for any point z in D, $f(z)$ takes every value at least once in $U(z, \rho_0)$.*

THEOREM 1. *If $f(z)$ belongs to $\mathfrak{N}_1$, then $f(z)$ has the following three properties:*

(i) *There exist a positive number ρ_0 and a positive integer q such that for any point z in D and every value α,*

$$1 \leqq q(z, \alpha) \leqq q,$$

where $q(z, \alpha)$ is the number of α-points of $f(z)$ in $U(z, \rho_0)$.

(ii) *For any two values α and β $(\alpha \neq \beta)$,*

$$\inf_{\substack{\nu = 1, 2, 3, \cdots \\ \mu = 1, 2, 3, \cdots}} d(z_\nu(\alpha), z_\mu(\beta)) > 0$$

where $z_\nu(\alpha)$ and $z_\mu(\beta)$ denote α-points and β-points of $f(z)$ respectively.

(iii) *For any value α and any positive number ρ, there exists a positive number $m_\rho (< 1)$ such that*

$$\chi(f(z), \alpha) > m_\rho \qquad in \qquad z \in D - \bigcup_{\nu=1}^{\infty} U(z_\nu(\alpha), \rho).$$

Proof of (i). Let ρ_0 be the same quantity in Lemma 2. Then $q(z, \alpha) \geqq 1$ for any point z in D and any value α. Suppose that the set

$\{q(z, \alpha);\ z \in D$ and α is an arbitrary value$\}$ is unbounded. There exist a sequence $\{z_n\}$ of points in D and a sequence $\{\alpha_n\}$ of values such that

(1. 1)
$$\lim_{n\to\infty} q(z_n, \alpha_n) = \infty.$$

Put

$$f_n(z) = f\left(\frac{z + z_n}{1 + \bar{z}_n z}\right).$$

Since $f(z)$ belongs to $\mathfrak{N}_1$, there exist subsequences $\{f_{n_k}(z)\}$ of $\{f_n(z)\}$ and $\{\alpha_{n_k}\}$ of $\{\alpha_n\}$, a non-constant function $f_0(z)$ and a value α_0 such that $\lim\limits_{k\to\infty} \alpha_{n_k} = \alpha_0$ and $\{f_{n_k}(z)\}$ converges uniformly to $f_0(z)$ on each compact subset of D. Put

$$g_k(z) = f_{n_k}(z) - \alpha_{n_k},\ g_0(z) = f_0(z) - \alpha_0,\ \text{if } \alpha_0 \neq \infty$$

or

$$g_k(z) = \frac{1}{f_{n_k}(z)} - \frac{1}{\alpha_{n_k}},\ g_0(z) = \frac{1}{f_0(z)},\ \text{if } \alpha_0 = \infty.$$

Then $\{g_k(z)\}$ converges uniformly to $g_0(z)$ on each compact subset of D. By the Hurwitz theorem, the number of zeros of $g_k(z)$ in $U(0, \rho_0)$ is not larger than that of $g_0(z)$ in $\bar{U}(0, \rho_0)$ for every sufficiently large k. On the other hand, since a transformation $s(z) \in \mathcal{S}$ preserves the hyperbolic distance, the former is equal to $q(z_{n_k}, \alpha_{n_k})$. This contradicts (1. 1).

Proof of (ii). Suppose that there exist two values α and β $(\alpha \neq \beta)$ such that $\inf\limits_{\substack{\nu=1,2,3,\cdots \\ \mu=1,2,3,\cdots}} d(z_\nu(\alpha), z_\mu(\beta)) = 0$. Then there exist subsequences $\{z'_n\}$ and $\{z''_n\}$ of $\{z_\nu(\alpha)\}$ and $\{z_\mu(\beta)\}$ such that

$$\lim_{n\to\infty} d(z'_n, z''_n) = 0.$$

Put $f_n(z) = f\left(\dfrac{z + z'_n}{1 + \bar{z}'_n z}\right)$, $\zeta_n = \dfrac{z''_n - z'_n}{1 - \bar{z}'_n z''_n}$ and $\zeta = s_n(z) = \dfrac{z + z'_n}{1 + \bar{z}'_n z}$. By $\zeta = s_n(z)$, 0 and ζ_n correspond to z'_n and z''_n respectively. Obviously $\lim\limits_{n\to\infty} d(0, \zeta_n) = \lim\limits_{n\to\infty} d(z'_n, z''_n) = 0$. Since $f(z)$ belongs to $\mathfrak{N}$, a subsequence $\{f_{n_k}(z)\}$ of $\{f_n(z)\}$ converges uniformly to a limiting function $f_0(z)$ on each compact subset of D. Therefore $\lim\limits_{k\to\infty} f_{n_k}(\zeta_{n_k}) = \lim\limits_{k\to\infty} f_{n_k}(0) = f_0(0)$. On the other hand, $f_{n_k}(\zeta_{n_k}) = f(z''_{n_k}) = \beta$ and $f_{n_k}(0) = f(z'_{n_k}) = \alpha$. Hence $\alpha = \beta$; this is a contradiction.

Remark. As we see above, we can derive (ii) under the weaker condition $f(z) \in \mathfrak{N}$ than the condition $f(z) \in \mathfrak{N}_1$.

Proof of (iii). By Lemma 1, we may assume without loss of generality that $\alpha = 0$. Let $\{a_\nu\}_{\nu=1}^{\infty}$ be all the zeros of $f(z)$ in D. Suppose that there exists a positive number ρ such that

$$\inf_{z \in D - \bigcup_{\nu=1}^{\infty} U(a_\nu, \rho)} \chi(f(z), 0) = 0.$$

Then there exists a sequence $\{z_n\}$ of points in $D - \bigcup_{\nu=1}^{\infty} U(a_\nu, \rho)$ such that $\lim_{n\to\infty} f(z_n) = 0$. Put $f_n(z) = f\left(\frac{z + z_n}{1 + \bar{z}_n z}\right)$. Since $f(z)$ belongs to $\mathfrak{N}_1$, there exists a subsequence $\{f_{n_k}(z)\}$ of $\{f_n(z)\}$ converging uniformly to a non-constant limiting function $f_0(z)$ on each compact subset of D. It holds

$$f_0(0) = \lim_{k\to\infty} f_{n_k}(0) = \lim_{k\to\infty} f(z_{n_k}) = 0.$$

Taking δ, $0 < \delta < \frac{\rho}{2}$, sufficiently small, $f_0(z)$ has only one zero in $U(0, \delta)$. Let m be its multiplicity. By the Hurwitz theorem, the number of zeros of $f_{n_k}(z)$ in $U(0, \delta)$ is equal to m for every sufficiently large k. Namely, that of $f(z)$ in $U(z_{n_k}, \delta)$ must be equal to m for every sufficiently large k. On the other hand, we took $\{z_{n_k}\}$ and δ such that $a_\nu \notin U(z_{n_k}, \delta)$ for all ν and all k. This is a contradiction. Thus the proof of Theorem 1 is complete.

3. The inverse of Theorem 1 also holds. In fact, we can give its proof assuming (i), (ii) and (iii) *only* for zeros and poles.

THEOREM 2. *Let $f(z)$ be meromorphic in D. Suppose that $f(z)$ satisfies the following three conditions:*

(i)′ *There exists a positive number ρ_0 such that $f(z)$ takes zero and ∞ at least once in $U(z, \rho_0)$ for any point z in D.*

(ii)′ *Let a_ν and b_μ be zeros and poles of $f(z)$ in D respectively, then* $\inf_{\substack{\nu=1,2,3,\cdots \\ \mu=1,2,3,\cdots}} d(a_\nu, b_\mu) > 0$.

(iii)′ *For any positive number ρ, there exists a positive number m_ρ such that*

$$|f(z)| < m_\rho \qquad in \quad z \in D - \bigcup_{\nu=1}^{\infty} U(b_\nu, \rho)$$

and

$$|f(z)| > \frac{1}{m_\rho} \qquad in \quad z \in D - \bigcup_{\nu=1}^{\infty} U(a_\nu, \rho).$$

Then $f(z)$ *belongs to* $\mathfrak{N}_1$.

Proof. Take any sequence $\{s_n(z)\}$ out of $\mathscr{S}$. Put $f_n(z) = f(s_n(z))$. For any fixed point z_0 in D, put $\zeta_n = s_n(z_0)$.

(a) If $\inf_{\substack{n=1,2,3,\cdots \\ \nu=1,2,3,\cdots}} d(\zeta_n, b_\nu) > 0$, then $U(\zeta_n, \delta_1) \subset D - \bigcup_{\nu=1}^{\infty} U(b_\nu, \delta_1)$, where $0 < \delta_1 < \frac{1}{2} \inf_{\substack{n=1,2,3,\cdots \\ \nu=1,2,3,\cdots}} d(\zeta_n, b_\nu)$. By the condition (iii)′, $f(z)$ is bounded in $U(\zeta_n, \delta_1)$ for $n = 1, 2, 3, \cdots$. Hence $f_n(z)$ is also bounded in $U(z_0, \delta_1)$ for $n = 1, 2, 3, \cdots$. Thus $\{f_n(z)\}$ is a normal family in $U(z_0, \delta_1)$.

(b) If $\inf_{\substack{n=1,2,3,\cdots \\ \nu=1,2,3,\cdots}} d(\zeta_n, b_\nu) = 0$, then there exist subsequences $\{\zeta_{n_k}\}$ and $\{b_{\nu_k}\}$ of $\{\zeta_n\}$ and $\{b_\nu\}$ such that $\lim_{k\to\infty} d(\zeta_{n_k}, b_{\nu_k}) = 0$. By the condition (ii)′,

$$\inf_{\substack{k=1,2,3,\cdots \\ \nu=1,2,3,\cdots}} d(\zeta_{n_k}, a_\nu) > 0.$$

It holds that $U(\zeta_{n_k}, \delta_2) \subset D - \bigcup_{\nu=1}^{\infty} U(a_\nu, \delta_2)$, where $0 < \delta_2 < \frac{1}{2} \inf_{\substack{k=1,2,3,\cdots \\ \nu=1,2,3,\cdots}} d(\zeta_{n_k}, a_\nu)$. By the condition (iii)′ there exists a positive number m such that

$$|f(z)| > \frac{1}{m} \text{ in } z \in U(z_{n_k}, \delta_2) \text{ for } k = 1, 2, 3, \cdots,$$

so that

$$|f_{n_k}(z)| > \frac{1}{m} \text{ in } z \in U(z_0, \delta_2) \text{ for } k = 1, 2, 3, \cdots$$

Thus, $\{f_{n_k}(z)\}$ is also a normal family in $U(z_0, \delta_2)$. Therefore, there exists a subsequence $\{f_{m_k}(z)\}$ of $\{f_n(z)\}$ such that $\{f_{m_k}(z)\}$ converges uniformly to a limiting function on each compact subset of D. Since a transformation $s(z) \in \mathscr{S}$ preserves the hyperbolic distance, it is easy to see by the condition (i)′ that any limiting function of the above normal family is non-constant. The proof is now complete.

4. Let $f(z)$ be meromorphic in D and let $n(r, \alpha)$ be the number of α-points of $f(z)$ in the domain $\{z; |z| < r\}$.

THEOREM 3. *If $f(z)$ belongs to $\mathfrak{N}_1$, then there exist two positive numbers A and B such that for every r, sufficiently near* 1,

$$\frac{B}{1-r} < n(r, \alpha) < \frac{A}{1-r}, \tag{1. 2}$$

where A and B are independent of the value α.

To get Theorem 3, we need the following

LEMMA 3. *For positive numbers r and ρ, with $0 < r < 1$ and $0 < \rho < d(0, r)$, let θ be the positive angle formed by the real axis and the line segment, starting from the origin, tangent to the circle $d(r, z) = \rho$.*

Then $$\sin\theta = \frac{(e^{4\rho} - 1)(1 - r^2)}{4e^{2\rho} r}.$$

This is obtained by an elementary calculation.

Proof of Theorem 3. We shall first prove the left inequality of (1. 2). Put

$$\zeta_n = \frac{e^{2n\rho_0} - 1}{e^{2n\rho_0} + 1}$$

and $R_n = \{z;\ (2n-1)\rho_0 \leqq d(0, z) < (2n+1)\rho_0\}$ for $n = 1, 2, 3, \cdots$, where ρ_0 is the same quantity in (i) of Theorem 1. Let $m_n(\alpha)$ be the number of α-points of $f(z)$ in R_n and let θ_n be the positive angle formed by the real axis and the line segment, starting from the origin, tangent to the circle $d(\zeta_{2n}, z) = \rho_0$. For any r, $\zeta_3 \leqq r < 1$, there exists a positive integer N such that

$$\zeta_{2N+1} \leqq r < \zeta_{2N+3}. \tag{1. 3}$$

Obviously

$$n(r, \alpha) > m_N(\alpha). \tag{1. 4}$$

In the ring domain R_N, we can take at least $\left[\frac{\pi}{\theta_N}\right]$ mutually disjoint open discs with a hyperbolic radius ρ_0, where [ ] denotes the Gauss sign. Therefore by (i) of Theorem 1, we have

(1. 5) $$m_N(\alpha) \geqq \left[\frac{\pi}{\theta_N}\right] > \frac{\pi}{\theta_N} - 1 > \frac{2}{\sin \theta_N} - 1,$$

and moreover, by Lemma 3

$$= C\frac{e^{8N\rho_0} - 1}{e^{4N\rho_0}} - 1, \text{ where } C = \frac{2e^{2\rho_0}}{e^{4\rho_0} - 1}$$

Thus by combining (1. 3), (1. 4) and (1. 5)

$$\begin{aligned}(1 - r)\, n(r, \alpha) &> (1 - \zeta_{2N+3}) m_N(\alpha) \\ &> \frac{2}{e^{(4N+6)\rho_0} + 1}\left(C\frac{e^{8N\rho_0} - 1}{e^{4N\rho_0}} - 1\right).\end{aligned}$$

It follows immediately that there exists a positive number B such that

$$n(r, \alpha) > \frac{B}{1 - r}$$

for every r, sufficiently near 1, and that B is independent of α.

We shall now prove the right inequality of (1. 2). Put $D(\rho) = \{z;\ d(0, z) < \rho\}$, $A(r) = \iint_{|z|<r} d\sigma(z) = \frac{\pi r^2}{1 - r^2}$ and $s_0 = \iint_{U(z,\rho_0)} d\sigma(z)$, where $d\sigma(z) = \frac{r\,dr\,d\theta}{(1 - r^2)^2}$. Since $d\sigma(z)$ is invariant by $s(z) \in \mathscr{S}$, s_0 is independent of z. Obviously, for any fixed value α,

$$D(d(0, r) + \rho_0) \supset \bigcup_{z_\nu(\alpha) \in D(d(0, r))} U(z_\nu(\alpha), \rho_0).$$

By (i) of Theorem 1, each point in the domain $D(d(0, r) + \rho_0)$ belongs to at most q-pieces of the open discs in $\{U(z_\nu(\alpha), \rho_0);\ z_\nu(\alpha) \in D(d(0, r))\}$. Hence it holds

$$q \iint_{z \in D(d(0, r) + \rho_0)} d\sigma(z) \geqq \sum_{z_\nu(\alpha) \in D(d(0, r))} \iint_{z \in U(z_\nu(\alpha), \rho_0)} d\sigma(z)$$

$$qA(r') \geqq n(r, \alpha) s_0,$$

where $r' = \dfrac{e^{2\rho_0} - 1 + r(e^{2\rho_0} + 1)}{e^{2\rho_0} + 1 + r(e^{2\rho_0} - 1)}$.

We get immediately that

$$\frac{A}{1 - r} \geqq n(r, \alpha),$$

where A is a constant which is independent of α. The proof of Theorem 3 is complete.

Let $T(r, f)$ be the characteristic function of $f(z)$ in the sense of Nevanlinna. By Theorem 3 and Lehto and Virtanen ([4], p. 58), we shall get the following

COROLLARY 1. *If $f(z)$ belongs to $\mathfrak{N}_1$, then there exist two positive numbers A' and B' such that*

$$B' \log \frac{1}{1-r} + O(1) < T(r, f) < A' \log \frac{1}{1-r} + O(1).$$

COROLLARY 2. *If $f(z)$ belongs to $\mathfrak{N}_1$, then for any value α,*

(1) $\sum_{\nu=1}^{\infty}(1 - |z_\nu(\alpha)|) = \infty$

and

(2) $\sum_{\nu=1}^{\infty}(1 - |z_\nu(\alpha)|)^{1+\lambda} < \infty$ *for any positive number λ.*

Proof of (1). For any value α,

$$\sum_{\nu=1}^{\infty}(1 - |z_\nu(\alpha)|) \geqq \sum_{n=1}^{\infty} \sum_{z_\nu(\alpha) \in R_n} (1 - |z_\nu(\alpha)|) \geqq \sum_{n=1}^{\infty} (1 - \zeta_{2n+1}) m_n(\alpha)$$

By (1. 5)

$$> \sum_{n=1}^{\infty} \frac{2}{e^{(4n+2)\rho_0} + 1} \left\{ C \frac{e^{8n\rho_0} - 1}{e^{4n\rho_0}} - 1 \right\} = \infty.$$

Proof of (2). For any positive number λ and any value α,

$$\sum_{|z_\nu(\alpha)| < r} (1 - |z_\nu(\alpha)|)^{1+\lambda} = \int_0^r (1-t)^{1+\lambda} dn(t, \alpha)$$

$$= (1-r)^{1+\lambda} n(r, \alpha) + (1+\lambda) \int_0^r (1-t)^\lambda n(t, \alpha) dt$$

By Theorem 3

$$\leqq A(1-r)^\lambda + A(1+\lambda) \int_0^r \frac{1}{(1-t)^{1-\lambda}} dt = O(1).$$

Hence $\sum_{\nu=1}^{\infty}(1 - |z_\nu(\alpha)|)^{1+\lambda} < \infty$.

§2. Products of normal meromorphic functions

5. THEOREM 4. *Let $f(z)$ and $g(z)$ be two functions of $\mathfrak{N}$. Let a_ν and*

a'_ν be zeros of $f(z)$ and $g(z)$ respectively and let b_ν and b'_ν be poles of $f(z)$ and $g(z)$ respectively. Suppose that

(1) $\inf\limits_{\substack{\nu=1,2,3,\cdots\\ \mu=1,2,3,\cdots}} d(a_\nu, b'_\mu)^* > 0$ *and* $\inf\limits_{\substack{\nu=1,2,3,\cdots\\ \mu=1,2,3,\cdots}} d(a'_\nu, b_\mu) > 0$

and

(2) *for any positive number ρ there exists a positive number m_ρ such that*

$$|f(z)| < m_\rho \quad in \quad z \in D - \bigcup_{\nu=1}^{\infty} U(b_\nu, \rho),$$

$$|g(z)| < m_\rho \quad in \quad z \in D - \bigcup_{\nu=1}^{\infty} U(b'_\nu, \rho),$$

$$|f(z)| > \frac{1}{m_\rho} \quad in \quad z \in D - \bigcup_{\nu=1}^{\infty} U(a_\nu, \rho)$$

and

$$|g(z)| > \frac{1}{m_\rho} \quad in \quad z \in D - \bigcup_{\nu=1}^{\infty} U(a'_\nu, \rho).$$

Then the product $f(z)g(z)$ belongs to $\mathfrak{N}$.

Proof. Take any sequence $\{s_n(z)\}$ out of $\mathcal{S}$. Put $f_n(z) = f(s_n(z))$ and $g_n(z) = g(s_n(z))$ for $n = 1, 2, 3, \cdots$. Since $f(z)$ and $g(z)$ belong to $\mathfrak{N}$, it may be assumed without loss of generality that two sequences $\{f_n(z)\}$ and $\{g_n(z)\}$ converge uniformly to limiting functions $f_0(z)$, $g_0(z)$ on each compact subset of D respectively. For any fixed point z_0 in D, put $\zeta_n = s_n(z_0)$. We denote by δ_1 the least value of $\inf\limits_{\substack{\nu=1,2,3,\cdots\\ n=1,2,3,\cdots}} d(a_\nu, \zeta_n)$, $\inf\limits_{\substack{\nu=1,2,3,\cdots\\ n=1,2,3,\cdots}} d(b_\nu, \zeta_n)$, $\inf\limits_{\substack{\nu=1,2,3,\cdots\\ n=1,2,3,\cdots}} d(a'_\nu, \zeta_n)$ and $\inf\limits_{\substack{\nu=1,2,3,\cdots\\ n=1,2,3,\cdots}} d(b'_\nu, \zeta_n)$.

(a) If $\delta_1 > 0$, then by the condition (2) there exists a positive number m such that

$$\frac{1}{m} < |f(z)| < m \text{ and } \frac{1}{m} < |g(z)| < m \quad \text{in} \quad z \in U\left(\zeta_n, \frac{\delta_1}{2}\right)$$

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*) For two sequences $\{z_n\}$ and $\{z'_m\}$ of points in D, we shall define $\inf\limits_{\substack{n=1,2,3,\cdots\\ m=1,2,3,\cdots}} d(z_n, z'_m) = \infty$, if $\{z_n\}$ or $\{z'_m\}$ is empty.

for $n = 1, 2, 3, \cdots$. Since $U\left(z_0, \frac{\delta_1}{2}\right)$ is mapped one to one comformally onto $U\left(\zeta_n, \frac{\delta_1}{2}\right)$ by $z' = s_n(z)$, it holds

$$\frac{1}{m^2} < |f_n(z)g_n(z)| < m^2 \quad \text{in} \quad z \in U\left(z_0, \frac{\delta_1}{2}\right).$$

Thus $\{f_n(z)g_n(z)\}$ is a normal family in $U\left(z_0, \frac{\delta_1}{2}\right)$.

(b) Suppose that $\delta_1 = 0$, say, $\inf\limits_{\substack{\nu=1,2,3,\cdots \\ n=1,2,3,\cdots}} d(a_\nu, \zeta_n) = 0$. There exist subsequences $\{a_{\nu_k}\}$ and $\{\zeta_{n_k}\}$ of $\{a_\nu\}$ and $\{\zeta_n\}$ such that

$$\lim_{k\to\infty} d(\zeta_{n_k}, a_{\nu_k}) = 0. \tag{2. 1}$$

By Condition (1), $\delta_2 = \inf\limits_{\substack{k=1,2,3,\cdots \\ \mu=1,2,3,\cdots}} d(a_{\nu_k}, b'_\mu) > 0$. By Condition (2), $g(z)$ is bounded in $U\left(\zeta_{n_k}, \frac{\delta_2}{2}\right)$, so that $g_{n_k}(z)$ is bounded in $U\left(z_0, \frac{\delta_2}{2}\right)$ for every sufficiently large k. On the other hand, by (2. 1)

$$\lim_{k\to\infty} f(\zeta_{n_k}) = \lim_{k\to\infty} f(a_{\nu_k}) = 0,$$

so that $\lim\limits_{k\to\infty} f_{n_k}(z_0) = \lim\limits_{k\to\infty} f(\zeta_{n_k}) = 0$. It follows that for every sufficiently large k, $f_{n_k}(z)$ is bounded in a neighborhood $U(z_0, \delta_3)$ of z_0. Put $\delta = \min\left(\frac{\delta_2}{2}, \delta_3\right)$. The product $f_{n_k}(z)g_{n_k}(z)$ is bounded in $U(z_0, \delta)$ for every sufficiently large k. Thus $\{f_{n_k}(z)g_{n_k}(z)\}$ is a normal family in $U(z_0, \delta)$. Therefore, there exists a subsequence $\{f_{m_k}(z)g_{m_k}(z)\}$ of $\{f_n(z)g_n(z)\}$ such that $\{f_{m_k}(z)g_{m_k}(z)\}$ converges uniformly to a limiting function on each compact subset of D. The proof is complete.

6. The following Examples 1 and 2 show that Theorem 4 fails to hold without Condition (1) or Condition (2).

EXAMPLE 1. There exist two normal meromorphic functions $T_1(z)$ and $T_2(z)$ such that $T_1(z)$ and $T_2(z)$ satisfy Condition (2) but not Condition (1) and $T_1(z)T_2(z)$ does not belong to $\mathfrak{N}$.

To give this example, we need the following

LEMMA 4. *Let d be an irrational number satisfying $0 < d < 1$. Then the set $\{nd - [nd]\}_{n=1}^{\infty}$ is dense on the closed interval $[0, 1]$, where* [ ] *in* { } *denotes*

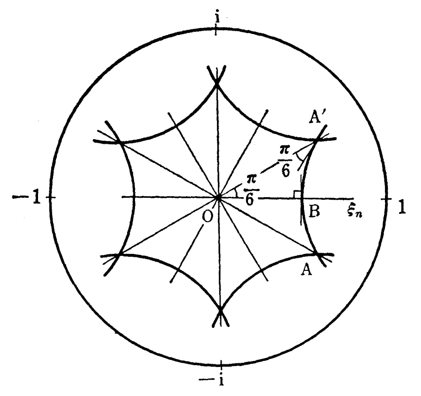

Fig. 1

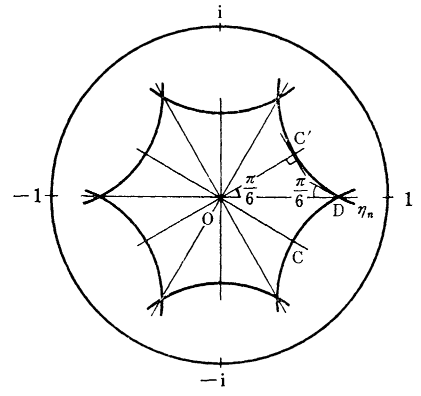

Fig. 2

the Gauss sign. (*see G.H. Hardy and E.M. Wright* [1], p. 155)

Let $T_1(z)$ and $T_2(z)$ be Schwarzian triangle functions whose fundamental triangles are shown in Figures 1 and 2 respectively. Let their system of triangles be those shown in [Fig. 1] for $T_1(z)$ and in [Fig. 2] for $T_2(z)$, where we assume $T_1(O)=0$, $T_1(A)=T_1(A')=\infty$, $T_1(B)=1$, $T_2(O)=\infty$, $T_2(C)=T_2(C')=0$ and $T_2(D)=1$. Then $T_1(z)$ and $T_2(z)$ belong to $\mathfrak{N}_1$, so that $T_1(z)$ and $T_2(z)$ satisfy Condition (2) by Theorem 1. Let ξ_n and η_n be zeros of $T_1(z)$ and poles of $T_2(z)$ on the segment $\{z=x+iy;\ 0\leqq x<1,\ y=0\}$ respectively. By an elementary calculation, we get $d(0,\xi_n)=n\log(\sqrt{2}+\sqrt{3})$ and $d(0,\eta_n)=2n\log(\sqrt{2}+1)$ for $n=1,2,3,\cdots$. Since $\dfrac{\log(\sqrt{2}+\sqrt{3})}{2\log(\sqrt{2}+1)}$ is a positive irrational number less than 1, it follows by Lemma 4 that the set

$$\left\{n\log(\sqrt{2}+\sqrt{3})-2\log(\sqrt{2}+1)\left[n\,\frac{\log(\sqrt{2}+\sqrt{3})}{2\log(\sqrt{2}+1)}\right]\right\}_{n=1}^{\infty}$$

is dense on the closed interval $[0,\ 2\log(\sqrt{2}+1)]$. Thus it is easy to see that there exist subsequences $\{\xi_{n_k}\}$ and $\{\eta_{n_k}\}$ of $\{\xi_n\}$ and $\{\eta_n\}$ such that

$$\lim_{k\to\infty} d(\xi_{n_k},\eta_{n_k})=0. \tag{2. 2}$$

Hence $T_1(z)$ and $T_2(z)$ do not satisfy Condition (1). The Product $\varphi(z)=T_1(z)T_2(z)$ does not belong to $\mathfrak{N}$. In fact, if $\varphi(z)$ belongs to $\mathfrak{N}$, then we must have by (2. 2)

$$\lim_{k\to\infty}\varphi(\xi_{n_k})=\lim_{k\to\infty}\varphi(\eta_{n_k}).$$

On the other hand, $\varphi(\xi_{n_k}) = 0$ and $\varphi(\eta_{n_k}) = \infty$. This is a contradiction.

Now we shall give our second example.

LEMMA 5 (*Lehto and Virtanen* [4]). *Let $f(z)$ be a function of $\mathfrak{N}$. If $f(z)$ has an asymptotic value α, then the value α is an angular limit of $f(z)$.*

EXAMPLE 2. Let $f(z)$ be an elliptic modular function and let $g(z)$ be a function of $\mathfrak{N}_1$. Then $f(z)$ and $g(z)$ satisfy Condition (1) because $f(z)\neq 0, 1$ and ∞. But the product $f(z)g(z)$ does not belong to $\mathfrak{N}$.

In fact, let $e^{i\theta_1}$ be a point at which $f(z)$ has an angular limit ∞, let a_ν be zeros of $g(z)$, and let ρ_0 and q the same quantities as those in Theorem 1. By Theorem 1, there exists a positive number M such that

$$|g(z)| > M \quad \text{in} \quad z \in D - \bigcup_{\nu=1}^{\infty} U\left(a_\nu, \frac{\rho_0}{3q}\right). \tag{2. 3}$$

Moreover, since the number of zeros of $g(z)$ in $U(z, \rho_0)$ is at most q for every point z in D, the point $e^{i\theta_1}$ is an accessible boundary point in the intersection $\tilde{\Delta}$ of the domain $D - \bigcup_{\nu=1}^{\infty} U\left(a_\nu, \frac{\rho_0}{3q}\right)$ and a Stolz domain Δ at $e^{i\theta_1}$. Hence there exists a path Γ ending at $e^{i\theta_1}$ in the domain $\tilde{\Delta}$, so that $\lim_{\substack{z\to e^{i\theta_1}\\ z\in\Gamma}} f(z) = \infty$. Therefore by (2. 3) $\lim_{\substack{z\to e^{i\theta_1}\\ z\in\Gamma}} f(z)g(z) = \infty$. If $f(z)g(z)$ belongs to $\mathfrak{N}$, then by Lemma 5 $f(z)g(z)$ must have an angular limit ∞ at $e^{i\theta_1}$. On the other hand, since $g(z)$ has infinitely many zeros in the intersection of every neighborhood of $e^{i\theta_1}$ and the Stolz domain Δ, $f(z)g(z)$ can not possess an angular limit at $e^{i\theta_1}$.

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Department of Mathematics,
Mie University.