A CORRECTION TO "THE SCHUR MULTIPLIERS OF THE MATHIEU GROUPS"

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In the paper [1] mentioned in the title, the authors attempted to determine the Schur multipliers of the five simple Mathieu groups. In rechecking the calculations, we find that an error was made, leading to incorrect results for M_{12} and M_{22}. Our purpose here is to compute again the multipliers of M_{12} and M_{22}, which turn out to be cyclic groups of orders 2 and 6 respectively. The multipliers of M_{11}, M_{23}, M_{24} were originally (and correctly) determined to be trivial.

The error in [1] is quite simple, and lies in the statements leading up to the formula (*) on page 738. We show in § 1 below that (*) is true with an additional condition. In all but two cases of [1] this condition is satisfied. The two exceptions occur in the calculations for the 2-part of the multiplier of M_{12} and M_{22}, and new calculations for these cases are given in § 2 and § 3.

In concluding this introduction, the authors wish to express their thanks to N. Ito for several helpful discussions.

§ 1

Let $\bar{G}$ be a proper covering of the finite group G with $\bar{G}/Z_m \simeq G$, where Z_m denotes the cyclic group of order m. Let $\{c_j\}$ denote those classes of G which do not split in $\bar{G}$. Suppose S is a subgroup of G whose inverse image in $\bar{G}$ is isomorphic to $S \times Z_m$. Let π denote the permutation character of G on the cosets of S. Furthermore, suppose that *if two elements of S of order not prime to m are conjugate in G, then they are already conjugate in S.* Then

$$(*) \qquad \sum \pi(c_j)^2/h_j = n \qquad \text{is integral,}$$

where h_j is the order of the centralizer of an element of c_j. The integer

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n is in fact $n_1 - n_2$, where n_1 is the number of irreducible constituents of φ^*, the character of G induced by a linear character φ of $S \times Z_m$ whose kernel is S. The additional condition mentioned in the introduction is in italics.

The above result is proved by computing the inner product of φ^* with itself. If $\bar{c}_j$ is any class of $\bar{G}$ in the inverse image of a non-splitting class c_j of G, then $\varphi^*(\bar{c}_j) = 0$ (see page 737 of [1]). For a splitting class d_j of G, we have $|\varphi^*(\bar{d}_j)|^2 = \pi(d_j)^2$, where $\bar{d}_j$ is any class of $\bar{G}$ in the inverse image of d_j. This follows directly from the assumptions if d_j consists of elements of order not prime to m. If d_j consists of elements of order prime to m, then d_j lifts to m classes of $\bar{G}$, of which only one class has elements of order prime to m. On that class $\varphi^*(\bar{d}_j) = \pi(d_j)$; on the remaining classes φ^* has value an $m^{\underline{th}}$-root of unity times $\pi(d_j)$.

In [1], it was shown independently of (*) that the 2-part of the multipliers of M_{11} and M_{23} were trivial, and that the 3-part of the multiplier of M_{22} is Z_3. The inducing argument with (*) was used to show that the 2-part of M_{24}, and the 3-part of $M_{11}, M_{12}, M_{23}, M_{24}$ were all trivial. A check of the relevant character tables shows that the additional condition is satisfied in each of these cases.

§ 2

In [2] Coxeter gave an explicit 6-dimensional projective representation of M_{12} over $GF(3)$. From the form of the matrices it is easily seen that this representation is a true projective representation. Since the center of $SL(6,3)$ is Z_2, it follows that M_{12} has a proper covering $\bar{M}_{12}$ with center Z_2.

The primes 5 and 11 divide $|\bar{M}_{12}|$ to the first power, so we can apply the theory of Brauer [3] to compute the degrees of the irreducible characters. We restrict ourselves to the projective characters of M_{12}, i.e. those characters of $\bar{M}_{12}$ which are faithful on the center Z_2. Now M_{11} has index 12 in M_{12}, and its covering in $\bar{M}_{12}$ must be $M_{11} \times Z_2$ by [1]. Induce the non-trivial linear character of $M_{11} \times Z_2$ up to $\bar{M}_{12}$. This induced character must be irreducible. For if not, its restriction to $M_{11} \times Z_2$ would show that its constituents can only have degrees 1 and 11. This is impossible, since these degrees must also be even. Thus $\bar{M}_{12}$ has an irreducible projective character of degree 12. $\bar{M}_{12}$ has a 5-block $B(5)$ of projective characters. $B(5)$ contains two exceptional characters of the same degree $\equiv \pm 1 \pmod 5$

and two non-exceptional characters of degrees $\equiv \pm 2 \pmod 5$, of which one is the 12. The remaining projective characters of $\bar{M}_{12}$ have degree $\equiv 0 \pmod 5$. $\bar{M}_{12}$ has an 11-block $B(11)$ of projective characters; $B(11)$ contains two exceptional characters of the same degree $\equiv \pm 5 \pmod{11}$, and five non-exceptional characters of degrees $\equiv \pm 1 \pmod{11}$. The projective characters of $\bar{M}_{12}$ not in $B(11)$ have degree $\equiv 0 \pmod{11}$.

Consider even positive divisors of $|M_{12}|$ less than $\sqrt{|M_{12}|}$. Those congruent to $1, -1, 5, -5 \pmod{11}$ are 12, 144; 10, 32, 54, 120; 16, 60, 192; 6, 72, 160 respectively. The degrees in $B(11)$ come from this list. Those divisors congruent to $0, \pm 1, \pm 5 \pmod{11}$, and moreover congruent to $1, -1, 2, -2 \pmod 5$ are 6, 16, 66, 176; 44, 54, 144, 264; 12, 22, 32, 72, 132, 192; 88, 198 respectively. The degrees in $B(5)$ come from this second list. Since 12 is in $B(5)$, there are only two possibilities for $B(5)$, $\{12, 132, 144, 144\}$ and $\{12, 32, 44, 44\}$. In the first case the two 144 characters are 5-conjugate and so take the same value on an element of order 3. Let $B(3)$ be the 3-block of $\bar{M}_{12}$ containing one of the 144. $B(3)$ has defect 1 because $144 = 9 \cdot 16$. Since $\bar{M}_{12}$ contains no elements of order 15, hence the block intersection argument [4], page 167, applied to $B(5) \cap B(3)$ gives a contradiction. Thus $B(5) = \{12, 32, 44, 44\}$, and $B(11)$ has a unique solution $\{12, 32, 10, 10, 120, 160, 160\}$. Now $|M_{12}| - \sum x_\mu^2 = 24,200$, where the sum is overall x_μ in $B(5) \cup B(11)$. Since the remaining degrees are $\equiv 0 \pmod{2.5.11}$, the only possibility is 110 twice. One can easily show that the two $110's$ and $10's$ are conjugate pairs by considering the restrictions to $M_{11} \times Z_2$. In summary, the projective degrees of M_{12} are $\overline{10}$, 12, 32, $\overline{44}$, $\overline{110}$, 120, $\overline{160}$, where the bar denotes a pair or conjugate characters.

Suppose M_{12} has a proper covering with center $Z_2 \times Z_2$. To each of the three cyclic subgroups Z_2 of the center corresponds a pair of conjugate projective characters 10_i, and $10'_i$, $i = 1, 2, 3$. Choose the notation so that the 10_i all coincide on restriction to M_{11}. The product $10_1 \times 10'_2$ is projective, and its irreducible constituents must have degrees in the above list. But $10_1 \times 10'_2$ restricted to M_{11} becomes $10 \times 10' = 1 + 44 + 55$ (see [1] for the character table of M_{11}). This is incompatible with the above list of degrees.

Suppose M_{12} has a proper covering with center Z_4. As before, there would exist a 4-fold irreducible projective character of degree 12. Repeating essentially identical numerical arguments we find in an 11-block of defect 1 a 4-fold projective character whose degree is not divisible by 4, which is a contradiction.

§ 3

In the following three lemmas we prove that the 2-part of the multiplier of M_{22} is cyclic of order two.

LEMMA 1. M_{22} has a proper covering $\bar{M}_{22}$ such that $\bar{M}_{22}/Z_2 \simeq M_{22}$.

Proof. M_{24} contains the holomorph of the elementary abelian group N of order 16 (Frobenius [5] and Witt [6]). This implies that M_{22} contains a subgroup H of index 77 where H is isomorphic to a split extension of N by A_6, the alternating group of degree 6. The representation of A_6 on N, considered as a 4 dimensional vector space over $GF(2)$, is irreducible, as follows by restriction from $GL(4, 2) \simeq A_8$. The character table of H is given in Table 1.

Table 1. The characters of H

(1)	360.16	1	5	5′	9	10	$\bar{8}$	15	15′	30	45	45′
$(2)^6$	32	1	1	1	1	−2	0	3	−1	2	−3	1
$(3)^4$	36	1	−1	2	0	1	−1	3	3	−3	0	0
$(3)^5$	9	1	2	−1	0	1	−1	0	0	0	0	0
$(5)^3_\pm$	5	1	0	0	−1	0	z	0	0	0	0	0
$(4)^3$	8	1	−1	−1	1	0	0	1	−1	0	1	−1
$(2)^8$	384	1	5	5	9	10	8	−1	−1	−2	−3	−3
$(4)^4$	16	1	1	1	1	−2	0	−1	−1	−2	1	1
$(4)^4$	32	1	1	1	1	−2	0	−1	3	2	1	−3
$(8)^2$	8	1	−1	−1	1	0	0	−1	1	0	−1	1
$(6)^2(2)^2$	12	1	−1	2	0	1	−1	−1	−1	1	0	0

H has a permutation representation of degree 16 on the cosets of A_6. The first column describes the conjugacy classes of H in terms of the cycle structure of their elements in this representation. The second column gives the order of the centralizer subgroups. There are two $(5)^3$ classes and two 8 dimensional characters. $z = \frac{1}{2}(1 \pm \sqrt{5})$.

The permutation character π of M_{22} on the cosets of H is $1 + 21 + 55$ and hence the double coset decomposition of M_{22} is

$$M_{22} = H + Hx_1H + Hx_2H.$$

Restricting π to H, we find $21_H = 1 + 5 + 15$ and $55_H = 1 + 9 + 15 + 30$. A well known result of Mackay states that these irreducible constituents must also occur in the permutation characters of H on the cosets of $H_1 = H \cap H^{x_1}$

and $H_2 = H \cap H^{x_2}$. From Table 1, the unique combinations are $1+15$ for H_1, and $1+5+9+15+30$ for H_2.

Thus H_1 has index 16. Since the 15 is faithful on H, we must have $H_1N = H, H_1 \cap N = 1$, and hence $H_1 \simeq A_6$.

H_2 is of index 60. Since the 5 and 9 have kernel N, H_2N has index 15 in H. The only subgroups of A_6 of this index are isomorphic to the symmetric group S_4. Put $N_2 = H_2 \cap N$, then $N_2 \simeq Z_2 \times Z_2$ and $H_2/N_2 \simeq S_4$. If E denotes the normal subgroup of H_2 corresponding to the extension of N_2 by the normal subgroup of order 4 in S_4, then $H_2/E \simeq S_3$. From Table 1 we note that H_2 contains no elements of order 8, and that H_2 intersects the $(3)^5$ class but not the $(3)^4$ class of H. Since the centralizer of an element of the $(3)^5$ class contains no involutions, $H_2/E \simeq S_3$ acts faithfully on N_2.

We now prove that (i) E is elementary abelian, (ii) H_2 splits over E and (iii) the representation of H_2/E on E has two irreducible constituents (each faithful of degree 2). Since S_3 acts faithfully on N_2, then $E \simeq Z_4 \times Z_4$ or $Z_2 \times Z_2 \times Z_2 \times Z_2$. If $E \simeq Z_4 \times Z_4$ the extension of N_2 by a Z_4 subgroup of the factor $S_4 \simeq H_2/N_2$ would produce a group containing elements of order 8. Thus $E \simeq Z_2 \times Z_2 \times Z_2 \times Z_2$. To prove (ii), consider the extension of N_2 by an S_3 subgroup of the factor S_4. Since S_3 acts faithfully on N_2 this extension must be isomorphic to S_4. An S_3 subgroup of this extension is a complement to E in H_2. The result (iii) is obvious.

If M_{22} has a proper 2-fold covering $\bar{M}_{22}$, then a proper 2-fold covering $\bar{H}$ is induced on H and the corresponding 2-cocycle is stable in the sense of Cartan and Eilenberg [7]. The converse is also true. Thus it is sufficient to produce a cocycle of H corresponding to a proper 2-fold covering which is stable in M_{22}.

Since the 2-Sylow subgroup of M_{22} is neither cyclic nor dihedral and since M_{22} contains a unique class of involutions, then these involutions must lift to involutions in $\bar{M}_{22}$. Thus the covering $\bar{N}$ induced on N is $\bar{N} \simeq N \times Z_2$. From the work of Schur [8], A_6 also has no proper 2-fold covering in which all involutions lift to involutions, so that $\bar{A}_6 \simeq A_6 \times Z_2$. Therefore, $\bar{H}$ splits over $\bar{N}$ with factor A_6. The representation of A_6 on $\bar{N}$ is 5 dimensional with irreducible constituents of degrees 1 and 4. Now either $\bar{H} \simeq H \times Z_2$ or $\bar{H}$ is a proper covering of H. Also $\bar{H} \simeq H \times Z_2$ if and only if the above 5 dimensional representation is decomposable. However,

A_6 has an indecomposable representation with these irreducible components. This follows from a result of Thompson [9], since one of the complex irreducible 5 dimensional characters of A_5 has modular irreducible constituents equal to precisely the above 1 and 4. Thus H has a proper covering $\bar{H}$. The explicit form of the corresponding 2-cocycle ω is not needed.

Since $\bar{H}_1 \simeq \bar{A}_6 \simeq A_6 \times Z_2$, ω is trivial on restriction to H_1. An argument similar to the one for $\bar{H}$ shows that $\bar{H}_2$ splits over $\bar{E} \simeq E \times Z_2$. The resulting 5 dimensional representation of S_3 has irreducible constituents of degrees 1, 2, 2. The representations of degree 2 are principal indecomposables and so must be direct summands. The representation is thus completely decomposable and so $\bar{H}_2 \simeq H_2 \times Z_2$ implying that ω is also trivial on H_2. By the stability criterion, $\bar{M}_{22}$ is a proper covering.

It is worth noting that part of the above proof can be repeated almost verbatim for M_{23}, if A_6 is replaced by A_7. The modular irreducible representation of degrees 1 and 4 lie in different blocks of A_7 and hence A_7 has no indecomposable 5 dimensional representation. This gives another proof that the 2-part of the multiplier of M_{23} is trivial.

LEMMA 2. The multiplier of M_{22} does not contain elements of order 4.

Proof. Suppose $\hat{M}_{22}$ is a proper 4-fold covering of M_{22} with $\hat{M}_{22}/Z_4 \simeq M_{22}$. Let H be the subgroup in lemma 1. H has odd index in M_{22} and hence its covering $\hat{H}$ is also proper. We will show that such a $\hat{H}$ cannot exist.

Note that A_7 occurs as a subgroup of M_{22}, see [1], page 734. The permutation character of M_{22} on the cosets of A_7 is $1+21+154$, (the possibility $1+21+55+99$ cannot be a permutation character; this follows by considering the restriction of the character 55 to the hypothetical subgroup). The coverings of A_7 with centre Z_4 are $A_7 \times Z_4$ and one other, which contains the proper 2-fold covering of A_7, see [8]. However, in this 2-fold covering all involutions of A_7 lift to elements of order 4 and, as previously noted, this cannot occur in M_{22}.

Use the inducing argument of §1 with $G = M_{22}$, $S = A_7$, $m = 4$, and $\pi = 1+21+154$. The only classes $\{c_j\}$ which need not split are $(2)^8$, $(6)^2(3)^2(2)^2$, and the $(4)^4(2)^2$ class with centralizer of order 16, (see [1] for the character table of M_{22}). The values of π on these classes are given in Table 2. From (*) the $(2)^8$ and $(6)^2(3)^2(2)^2$ class must split in $\hat{M}_{22}$.

In $\hat{H}$ the covering induced on A_6 must be $A_6 \times Z_4$. The argument is

Table 2.

c_j	h_j	$\pi(c_j)$
$(2)^8$	384	16
$(6)^2(3)^2(2)^2$	12	1
$(4)^4(2)^2$	16	4

the same as for A_7. Apply the inducing argument with $G = H$, $S = A_6$, $m = 4$, and $\pi = 1 + 15$. From above we know that the $(2)^6$ class splits. By (*) the $(4)^3(2)$ class also splits and thus φ^* contains 2 irreducible components. Their degrees must be either $4 + 12$ or $8 + 8$. Both cases lead to contradictions on restriction back to $A_6 \times Z_4$. Thus $\hat{H}$ does not exist.

The same calculations could be performed for the 2-fold covering $\bar{M}_{22}$. However, in the final step we could also have $\varphi^* = 6 + 10$, and, in fact, H does have projective characters with these degrees.

LEMMA 3. The 2 part of the multiplier of M_{22} is cyclic.

Proof. The primes 5, 7, 11 divide $|\bar{M}_{22}|$ to the first power. Restricting our attention to the projective characters of M_{22}, we find $\bar{M}_{22}$ has a 5-block $B(5)$ with five characters of degree $\equiv \pm 1 \pmod 5$, a 7-block $B(7)$ with 3 non-exceptional characters of degree $\equiv \pm 1 \pmod 7$ and 2 exceptional characters of degree $\equiv \pm 3 \pmod 7$, and an 11-block $B(11)$ with 5 non-exceptional characters of degree $\equiv \pm 1 \pmod{11}$ and 2 exceptional characters of degree $\equiv \pm 5 \pmod{11}$. Consider the even positive divisors of $|M_{22}|$ less than $\sqrt{|\bar{M}_{22}|}$. Those divisors congruent to $0, \pm 1 \pmod 5$, congruent to $0, \pm 1, \pm 3 \pmod 7$, and moreover, congruent to $1, -1, 5, -5 \pmod{11}$ are 56, 144, 210; 10, 120, 384, 560; 60, 126, 280; 6, 160, 336 respectively. The degrees in $B(11)$ come from this list.

A character of degree 60 would be exceptional for 7 and 11, and hence assume irrational values on elements of order 7 and 11. But then $\bar{M}_{22}$ would contain elements of order 77 by a theorem of Burnside, which is impossible. A character of degree 384 would be in a 2-block of $\bar{M}_{22}$ of defect 1, and thus M_{22} would also have an ordinary irreducible character of degree 384, which is impossible. $\bar{M}_{22}$ contains no elements of order 33. Thus, if $B(3)$ is a 3-block of defect 1 of projective characters of $\bar{M}_{22}$, the block intersection argument can be applied to $B(11) \cap B(3)$. In particular, this will show that characters of degree 6 or 336 do not occur, and that if a character of degree 120 or 210 appears, then a triple of degrees 120, 210, 330 must in fact occur. Such blocks $B(3)$ of defect 1 do exist by a result of Brauer, since they exist in the normalizer of a cyclic subgroup of order 3 in $\bar{M}_{22}$.

It is now fairly straightforward to show there exist unique solutions for the degrees in $B(5)$, $B(7)$, $B(11)$. We omit the details. There are 11 irreducible projective characters: their degrees are 440, 330, 210, 154, 154, $\overline{126}$, 120, 56, $\overline{10}$, where the bar denotes a pair of complex conjugate characters. Arguing as in the case of M_{12}, we can conclude that M_{22} has no proper covering over $Z_2 \times Z_2$. Indeed, the argument is simpler, since a 2-fold projective character of M_{22} of degree 100 must be a sum of 10 irreducible projective characters of degree 10.

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