A REMARK ON PEIRCE'S RULE IN MANY-VALUED LOGICS

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Recently, S. Nagata gave an interesting series of rules beginning with Peirce's rule introduced in [3], each rule in the series being really stronger than its successor in the intuitionistic logics. (See [1] Nagata.) Namely, let p_0, $p_1, \ldots$ be any series of mutually distinct propositional variables. Let us define $\mathfrak{P}_0$ as denoting p_0. $\mathfrak{P}_1, \mathfrak{P}_2, \ldots$ be defined recursively by

$$\mathfrak{P}_{n+1} \equiv (((p_{n+1} \to \mathfrak{P}_n) \to p_{n+1}) \to p_{n+1}) .$$

Then, the series $\mathfrak{P}_1, \mathfrak{P}_2, \ldots$ is a series of above mentioned character. Nagata proved this by making use of the fact that the truth-value of $((p \to q) \to p) \to p$ is really smaller than the truth-value of q unless the truth-value of q is equal to 0 (TRUE) with respect to a certain truth-value evaluation of logics having a finite number of linearly ordered truth-values. In this short note, I will point out that this fact holds true for a vast class of truth-value evaluations of logics.

In my paper [2], I have given a condition which is satisfied by a vast class of evaluations of logics. Namely, let $\boldsymbol{D}$ be the domain of truth-values having the special truth-value 0 with respect to an evaluation of a logic having the logical constant $\to$ together with the usual inference rules for this logical constant. A combination of members of $\boldsymbol{D}$ which is denoted by the same symbol $\to$ is assumed to be defined in $\boldsymbol{D}$. Most of evaluations would satisfy the following conditions:

E1: $p \to 0 = 0$,

E2: $p \to p = 0$,

E3: $0 \to p = p$,

E4: $p \to (p \to q) = p \to q$,

E5: $p \to (q \to r) = q \to (p \to r)$,

E6: $p \to q = 0$ implies $(r \to p) \to (r \to q) = 0$.

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$\boldsymbol{D}$ can be regarded as almost partly ordered, if we assume these conditions and define $p \geq q$ by $p \rightarrow q = 0$. The relation $\geq$ can be proved reflexive and transitive. Unfortunately, $p = q$ can not be implied by $p \geq q$ and $q \geq p$. So, I will define $p > q$ in $\boldsymbol{D}$ by $p \geq q$ and $q \ngeq p$, not by $p \geq q$ and $p \neq q$.

Now, I will prove

THEOREM. *If the conditions* $\boldsymbol{E}1 - \boldsymbol{E}6$ *hold,*

$$q > ((p \rightarrow q) \rightarrow p) \rightarrow p$$

holds identically unless q *is equal to* 0.

Proof. The proposition $q \rightarrow (((p \rightarrow q) \rightarrow p) \rightarrow p)$ is provable in the sentential part $\boldsymbol{LOS}$ of the primitive logic. So, the truth-value expression $q \rightarrow (((p \rightarrow q) \rightarrow p) \rightarrow p)$ is identically equal to 0 according to Theorem 1 of my paper [2]. Hence, $q \geq ((p \rightarrow q) \rightarrow p) \rightarrow p$ by definition.

To show $q > ((p \rightarrow q) \rightarrow p) \rightarrow p$ unless q is equal to 0, let us assume $((p \rightarrow q) \rightarrow p) \rightarrow p \geq q$. Then, $p \geq q$ must hold true, because $p \geq ((p \rightarrow q) \rightarrow p) \rightarrow p$ can be easily proved. Hence,

$$0 = p \rightarrow p = (0 \rightarrow p) \rightarrow p = ((p \rightarrow q) \rightarrow p) \rightarrow p \geq q.$$

So, q must be equal to 0 according to $\boldsymbol{E}3$.

REFERENCES

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